# Nonlinear hydrodynamics of liquid crystals

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The Poisson-bracket method is used to construct nonlinear hydrodynamic equations for smectic *A* and *C* liquid crystals, for cholesterics, and also for exotic mesophases of disk-shaped molecules, biaxial nematics, and the blue phase of cholesterics.

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## I. INTRODUCTION

In recent years, much attention has been given to the derivation of nonlinear hydrodynamic equations for liquid crystals (LC) of various types.[1-2] This interest is due above all to the fact that LC are very "soft" systems, and in an experimental situation nonlinear effects are actually always important. For example, in electro- and thermoconvective instabilities the threshold (with respect to temperature or field) is significantly lower than in isotropic systems. Therefore the motion of LC becomes nonlinear even at rather small fields or temperature gradients. On the other hand, complete treatment of nonlinear effects in LC is quite complicated. The fact is that in LC the order parameter itself, as a rule, has many components, and in addition the interaction between different degrees of freedom is substantial.

Writing down the nonlinear equations of motion simply on the basis of symmetry considerations[1-2] is not rigorous, since in such a procedure some nonlinear terms are arbitrarily dropped. In the paper of Pleiner and Brand,[2] for example, only the nonlinear terms of third order are written in the free energy (the linear theory corresponds to allowance for quadratic terms); this in general is not sufficient to insure the stability of the system. Furthermore, in calculation of the subsequent nonlinear terms it is necessary each time to repeat anew the complicated procedure of derivation of the equations of hydrodynamics for the purpose of satisfying all the conservation laws.

In a paper of Dzyaloshinskiĭ and one of the authors,[3] the Poisson-bracket method was used to obtain the hydrodynamic equations. An advantage of this method consists in the fact that the equations of hydrodynamics need not be derived, since they are actually prescribed as Liouville equations for the hydrodynamic variables $x_i(\mathbf{r})$:

$$\frac{\partial x_i}{\partial t}=\{H,x_i\}, \qquad (1)$$

where $\{\ ,\ \}$ are Poisson brackets (PB). Because the PB between hydrodynamic variables are universal, since they depend only on the symmetry of the physical laws and are independent of the form of the Hamiltonian $H$, making the equations concrete requires expressing the energy $H$ in terms of the hydrodynamic variables. In the expression for the energy, we may restrict ourselves to those terms of the expansion, with respect to the gradients of hydrodynamic variables, which are required for solution of the given actual problem. Then (1) yields approximate equations of reversible hydrodynamics that automatically satisfy all the conservation laws. If it is required to take account of higher-order terms in the equations of hydrodynamics, it is sufficient to extend the expansion of the energy. In allowance for dissipation, the same applies to the dissipation function $R$, which can be prescribed with the required degree of accuracy.

As an illustration of the PB method, nematic LC were considered.[3] It is appropriate to investigate also other types of LC. This problem is also timely because in recent years new mesophases have been discovered, whose hydrodynamics in general has not been investigated. Below, this program will be carried out for the smectic *A* and *C* phases, cholesterics, a mesophase of "liquid columns" (discotic LC), the quite recently discovered biaxial nematic LC,[4] and the blue phase of cholesterics.

## II. SMECTIC LC OF *A* TYPE

In the method developed earlier,[3] the equations of motion are determined by the PB of the Hamiltonian $H$ of the system with the hydrodynamic variables $x_i$. Such variables are quantities satisfying the usual conservation laws: the density $\rho$, the momentum density $\mathbf{j}$, the entropy density $s$, etc.; and also the parameters that describe the symmetry breaking due to formation of an ordered system (in the case of nematic LC, these are the two independent components of the unit vector of the director $\mathbf{n}$; in the case of crystals, the displacement vector $\mathbf{u}$; etc.).

In the smectic *A* phase, the hydrodynamic variables are the quantities $\rho$, $\mathbf{j}$, and $s$ already mentioned above, and also the displacement $u_z$ of the smectic layer in the direction perpendicular to it. In order to obtain the nonlinear equations, which must take account also of the change in space of the direction of the normal to the layers, it is necessary to introduce, instead of $u_z$, another variable, which describes the deformation of the layers and which has the simplest possible transformation properties. This variable is a function $\Phi(\mathbf{r})$ such that the equations $\Phi(\mathbf{r})=$ const give a system of smectic layers (compare the variables $X^1, X^2, X^3$ in Ref. 3, which gave three systems of deformed crystal planes). We denote by $\mathbf{W}$ the gradient of this function:

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$$\mathbf{W}=\nabla\Phi. \tag{2}$$

In absolute equilibrium, this vector is constant in space, thereby prescribing a common direction of the normal for all layers and insuring constancy of the distance between layers. In a deviation from equilibrium, W gives the local direction of the normal, and its modulus describes small changes of the distance between layers. In the linear theory,

$$W_k=\delta_{kz}-\nabla_k u_z. \tag{3}$$

The equations of hydrodynamics may be written in the general form (1), where $x_i=\rho, \mathbf{j}, s, \mathbf{W}$, and where $H$ is the Hamiltonian of the system,

$$H=\int d^3r\,\varepsilon(\rho,\mathbf{j},s,\mathbf{W}), \tag{4}$$

to the right side of equation (1), dissipative terms are added in the standard way, by use of a dissipation function $R$. The equations for $\rho$ and $s$ have the form common to all liquid crystals:

$$\frac{\partial\rho}{\partial t}+\nabla(\rho\mathbf{v})=\nabla\frac{\partial R}{\partial\nabla\mu},\quad \mathbf{v}=\frac{\delta H}{\delta\mathbf{j}},\quad \mu=\frac{\delta H}{\delta\rho}, \tag{5}$$

$$\frac{\partial s}{\partial t}+\nabla(s\mathbf{v})=\nabla\frac{\partial R}{\partial\nabla T}+\frac{2R}{T},\quad T=\frac{\delta H}{\delta s}. \tag{6}$$

A difference in these equations can be due only to a different form of the dissipation function $R$ in different types of LC. In the method being used, this is due simply to the fact that the PB of $\rho$ and $s$ with the liquid-crystalline hydrodynamic variables (n, W, etc.) are zero.

In order to write the two remaining equations for j and W, it is necessary to evaluate the PB between them. It is simpler first to calculate $\{\mathbf{j},\Phi\}$. It was shown in Ref. 3 that one can find the PB if one knows the transformation properties of the variables in transformations of the group corresponding to the broken symmetry of the system. In the present case this is the translation group

$$x^i\to x^i+u^i$$

(smectic order corresponds only to translations $u^z$, but they need not be separated out in the nonlinear theory).

In these transformations, we have

$$\delta\Phi=-u^k\nabla_k\Phi,$$

whence follows[3]

$$\{j_{k1},\Phi_2\}=\frac{\delta\Phi_2}{\delta u_1{}^k}=-W_k\delta(1-2). \tag{7}$$

Finally, we have from (1) and (7)

$$\frac{\partial j_k}{\partial t}+\nabla_l(v_l j_k)+j_l\nabla_k v_l+\rho\nabla_k\mu+s\nabla_k T+W_k\nabla_l\sigma_l=\nabla_l\frac{\partial R}{\partial\nabla_l v_k} \tag{8}$$

$$\frac{\partial W_k}{\partial t}+\nabla_k(v_l W_l)=\nabla_k\frac{\partial R}{\partial\nabla_l\sigma_l}, \tag{9}$$

where $\sigma_k$ is the stress conjugate to the strain $W_k$,

$$\sigma_k=\frac{\delta H}{\delta W_k}. \tag{10}$$

The current equation (8), although not expressed in the usual way in terms of the momentum-flow tensor $\Pi_{kl}$, nevertheless satisfies the law of conservation of momentum. This is easily shown by integrating the equation over a volume with allowance for the translational invariance of the energy $H$. In order to write this equation in terms of $\Pi_{kl}$, it is necessary to know the structure of the energy density $\varepsilon$ and, specifically, on what gradients of hydrodynamic variables it depends.

Although formally all the nonlinear hydrodynamics of smectics $A$ is described by the system of equations (5), (6), (8), and (9), for actual use of these equations it is necessary to have explicit expressions for the energy $\varepsilon$ and the dissipation function $R$. The simplest expression for the energy $\varepsilon$ that satisfies the symmetry requirements and insures constancy of the vector W in equilibrium has the form

$$\varepsilon=\mathbf{j}^2/2\rho+\varepsilon_0(\rho,s)+{}^1/_2\alpha(\mathbf{W}^2-1)^2+{}^1/_2\beta(\nabla\mathbf{W})^2 \tag{11}$$

with constant coefficients $\alpha$ and $\beta$. This expression in case of necessity may be made more complicated by introducing both later terms in the gradients and interaction between the liquid-crystalline variable W and the variables $\rho$ and j of an ordinary isotropic liquid. With this choice of $\varepsilon$, the stress $\sigma$ is

$$\sigma=2\alpha\mathbf{W}(\mathbf{W}^2-1)-\beta\nabla(\nabla\mathbf{W}). \tag{12}$$

In the linear approximation (3), the liquid-crystalline part of the energy (11) has the form

$$2\alpha\left(\frac{\partial u_z}{\partial z}\right)^2+\frac{1}{2}\beta\left(\frac{\partial^2u_z}{\partial x^2}+\frac{\partial^2u_z}{\partial y^2}\right)^2.$$

Equation (8) for the current, with the energy choice (11), can be transformed in the standard manner (see Ref. 3), by using the homogeneity of the energy, to the following form, which explicitly expresses the law of conservation of current:

$$\frac{\partial j_k}{\partial t}+\nabla_l\left(P\delta_{lk}+j_k v_l+W_k\sigma_l+\frac{\partial\varepsilon}{\partial\nabla\mathbf{W}}\nabla_k W_l\right)=\nabla_l\frac{\partial R}{\partial\nabla_l v_k},$$

where the pressure is

$$P=-\varepsilon+\mu\rho+Ts+\mathbf{vj}.$$

In the quadratic approximation, the simplest form of the dissipation function $R$ that insures its positive definiteness and its vanishing in equilibrium is the following:

$$R=\eta_{iklm}A_{ik}A_{lm}+\varkappa_{ij}\nabla_i T\nabla_j T+\Lambda(\nabla\sigma)^2+\xi_i\nabla_i T\nabla\sigma, \qquad A_{ik}={}^1/_2(\nabla_i v_k+\nabla_k v_i). \tag{13}$$

The tensor quantities that occur in (13) must be written with the aid of the normal unit vector $\nu=\mathbf{W}/|\mathbf{W}|$ and the unit tensor $\delta_{ij}$, for example:

$$\xi_i=\xi\nu_i,\qquad \varkappa_{ij}=\varkappa_0\delta_{ij}+\varkappa_1\nu_i\nu_j$$

etc.; for positive definiteness of the quadratic form (13), the conditions

$$\varkappa_0>0,\quad \varkappa_0+\varkappa_1>0,\quad \Lambda>0,\quad \xi^2<4\Lambda(\varkappa_0+\varkappa_1).$$

must be satisfied. In the linear theory, with W of the form (3), the system (5), (6), (8), (9) reduces to the well-known equations of the linear hydrodynamics of smectics $A$.[5]

We note that the unlinearized equations in the form (5), (6), (8), (9) describe the dynamics of the system even when the equilibrium layers are not planes but, for example, are rolled up into coaxial cylinders and the like. In this case, the vector W in equilibrium is not constant but is determined by the equation

$$\nabla\sigma=0.$$

The system of equations (5), (6), (8), (9) can if necessary be generalized by introducing the nematic variable **n**, attached to the axes of the molecules. The necessity for such generalization arises, for example, if the interaction of **n** with W is comparatively weak, so that the mode connected with oscillation of **n** with respect to W softens and **n** may be treated as an independent quasihydrodynamic variable. The necessity for introducing a separate variable n arises also in investigation of the dynamics of smectics *C*, to which we shall be turning immediately. The equations obtained in the next section may be formally applied also to a smectic *A* with a quasihydrodynamic variable **n**.

## III. SMECTIC LC OF TYPE *C*

In this case, in equilibrium the molecules are inclined to the smectic layer by some angle $\theta$ (the angle between the director n and the normal $\nu$ to the layer). But the intermolecular forces fix only the polar angle of the inclination. The azimuthal angle still remains arbitrary. Therefore we have an additional, as compared with smectic *A*, hydrodynamic variable n × W, corresponding to the projection of the director n on the plane of the smectic layer. Since the equation for n × W can be obtained from the equations for W and n, we shall work with the variables W and n.

Since the Poisson bracket

$$\{W_{kl}, \mathbf{n}_s\}=0,$$

no "mixing" of W and n occurs, and the equations of motion in this case are made up additively from parts corresponding to a smectic *A* and to a nematic. The PB necessary for the "nematic" part were calculated earlier.[3] Thus we have instead of (8) and (9)

$$\frac{\partial j_k}{\partial t}+\nabla_l(v_l j_k)+j_l\nabla_k v_l+\rho\nabla_k\mu+s\nabla_k T+W_k\nabla_l\sigma_l+\mathbf{h}\nabla_k\mathbf{n}=\nabla_l\frac{\partial R}{\partial\nabla_l v_k} \tag{14}$$

$$\frac{\partial W_k}{\partial t}+\nabla_k(v_l W_l)=\nabla_k\frac{\partial R}{\partial\nabla_l\sigma_l}, \tag{15}$$

$$\frac{\partial\mathbf{n}}{\partial t}+v_k\nabla_k\mathbf{n}-[\omega\times\mathbf{n}]=\frac{\partial R}{\partial\mathbf{h}}, \tag{16}$$

$$[\mathbf{h}\times\mathbf{n}]=-\frac{\partial R}{\partial\omega}+\nabla_k\frac{\partial R}{\partial\nabla_k\omega}. \tag{17}$$

Here $\mathbf{h}=-\delta H/\delta\mathbf{n}$ is the molecular field, and $\omega$ is the angular velocity of the molecules.

Equation (17) is the equation of motion for an auxiliary variable L, which is the internal angular momentum of the molecules. In this equation L has been set equal to zero, because it contains an extraneous power of the time derivative of n, and in low-frequency dynamics, which is of interest to us, the value of L may be neglected. The situation may change if a spontaneous angular momentum of the molecules originates in the LC; but no such effect has so far been observed in actual LC, and we shall not discuss it. Equation (17) enables us to express the value of $\omega$ in terms of the remaining hydrodynamic variables. As a result there remains the system of equations (5), (6), (14)–(16) for $\rho$, **j**, *s*, **n**, and **W**.

We shall now discuss the possible form of the dissipation function and the energy. To the dissipative function (13) for a smectic *A* it is necessary to add the nematic terms

$$R'={}^1/_2\varphi_{ij}h_ih_j+{}^1/_2\rho_{ij}[\omega_i-{}^1/_2(\operatorname{rot}\mathbf{v})_i-\lambda_{imn}A_{mn}][\omega_j-{}^1/_2(\operatorname{rot}\mathbf{v})_j-\lambda_{jpq}A_{pq}]. \tag{18}$$

Here $\varphi_{ij}=\varphi_{ji}, \rho_{ij}=\rho_{ji}$. The tensors $\varphi_{ij}$, $\rho_{ij}$, and $\lambda_{imn}$ satisfy the transversality condition

$$\varphi_{ij}n_i=\rho_{ij}n_i=\lambda_{imn}n_i=0.$$

In a pure nematic, they have the following structure:

$$\varphi_{ij}=\varphi(\delta_{ij}-n_in_j),\quad \rho_{ij}=\rho_1(\delta_{ij}-n_in_j),\quad \lambda_{imn}=\lambda e_{ipm}n_pn_n,$$

where $\lambda$ is a reactive parameter that occurs in the equations of the dynamics of a nematic (see Ref. 6), and where $\varphi$ and $\rho_1$ are connected with the coefficient of rotational viscosity by the relation

$$1/\gamma_1=\varphi+1/\rho_1.$$

In a smectic *C* these tensors, as well as $\eta_{ijkl}$, $\varkappa_{ij}$, and $\xi_i$, is made up from the vectors n and $\nu$; and it must be taken into account that the state of a smectic *C* is not changed by the substitution n → –n and $\Phi\to-\Phi$ (or $\nu\to-\nu$). Therefore the tensors $\eta_{ijkl}$, $\varkappa_{ij}$, $\rho_{ij}$, $\varphi_{ij}$, and $\lambda_{imn}$ must be even functions both of n and of $\nu$, while the vector $\xi_i$ must be even in n and odd in $\nu$.

Besides the terms (13) and (18), the dissipation function *R* may contain also mixed terms, for example

$$R''=\gamma_{ij}(\nabla\sigma)\nabla_ih_j, \tag{19}$$

here $\gamma_{ij}$ is odd both in n and in $\nu$, and $\gamma_{ij}n_j=0$.

The simplest form of expression for the energy that takes account of the nematic contribution and of the interaction of **n** with **W** is the following:

$$\begin{aligned}\varepsilon=\mathbf{j}^2/2\rho+\varepsilon_0(\rho, s)+{}^1/_2\alpha(\mathbf{W}^2-1)^2+{}^1/_2\beta(\nabla\mathbf{W})^2\\+{}^1/_2\gamma([\mathbf{n}\times\mathbf{W}]^2-\sin^2\theta)^2+{}^1/_2K_{ijkl}\nabla_in_k\nabla_jn_l.\end{aligned} \tag{20}$$

The tensor $K_{ijkl}$ is even in **n** and $\nu$. We shall not write out all the equations (14)–(16) with the concrete coefficients that follow from the form of *R* and $\varepsilon$. We give only the equation for **n**:

$$\frac{\partial n_i}{\partial t}+\mathbf{v}\nabla n_i+{}^1/_2[\mathbf{n}\operatorname{rot}\mathbf{v}]_i+e_{ipq}\lambda_{qmn}n_pA_{mn}=[\varphi_{ij}+(\rho^{-1})_{ij}]h_j-\gamma_{ji}\nabla_j\nabla\sigma. \tag{21}$$

## IV. DISCOTIC LC

We consider discotic liquid crystals.[7] The additional hydrodynamic variables here are connected with displacements of the two-dimensional lattice of liquid columns. Accordingly we have, instead of a single variable $\Phi$, two variables $\Phi^\alpha$ ($\alpha=1,2$), describing surfaces whose intersection forms a liquid column. The deformation of these surfaces is given by the gradients $W_k^\alpha=\nabla_k\Phi_\alpha$.

The role of director in these liquid crystals is played by the normal n to the plane of preferred orientation of the disk-shaped molecules. If the direction of this vector is rigidly fixed by the intermolecular forces, then n is parallel to $\mathbf{W}^1\times\mathbf{W}^2$, and the nematic hydrodynamic equations (connected with n) need not be considered. In this case we get simply equations (5), (6), (8), (9), but with an additional index $\alpha$ that takes into account the two-dimensional character of the lattice. But if the direction of n with respect to the lattice is not rigidly fixed, i.e., if the interaction of n with $W^\alpha$ is comparatively small, then n becomes a quasihydrodynamic variable, and equations (5), (6), (8), and (9) must be supplemented by equations (16) and (17) for n. Then the energy contains a term of interaction of the director n with $W^\alpha$, of the type $(\mathbf{n}\cdot\mathbf{W}^\alpha)^2$, and nematic energy of distortion of the n field. The dissipation function $R$ is also modified in this case, as for smectic $C$.

The equation of motion for n must be supplemented also in the case when the interaction of n with $W^\alpha$ fixes a definite nonzero angle of inclination of n to the line of the liquid column. In this case, as for smectic $C$, the axial angle of the vector n is a soft hydrodynamic variable, describing a Goldstone mode.

## V. BIAXIAL NEMATICS AND THE BLUE PHASE OF CHOLESTERICS

To describe the hydrodynamics of biaxial nematics, it is necessary to introduce, instead of the vector n, which prescribes the direction of the anisotropy axis in a uniaxial nematic, two orthogonal vectors $\mathbf{n}^{(1)}$ and $\mathbf{n}^{(2)}$, or three Euler angles describing their orientation. Thus three soft hydrodynamic modes are formed. In terms of $\mathbf{n}^{(1)}$ and $\mathbf{n}^{(2)}$, the equations of hydrodynamics are constructed in the same way as for ordinary nematics.[3] For this purpose we introduce an additional variable L, the density of internal angular momentum of the molecules, which may be set equal to zero in the final equations. The nonvanishing PB of the liquid-crystalline variables, with each other and with the remaining variables, have the form

$$\begin{gathered}\{L_1{}^\alpha, n_2{}^{(1)\beta}\}=-e^{\alpha\beta\gamma}n_1{}^{(1)\gamma}\,\delta(1-2),\quad \{L_1{}^\alpha, n_2{}^{(2)\beta}\}=-e^{\alpha\beta\gamma}n_1{}^{(2)\gamma}\,\delta(1-2),\\ \{L_1{}^\alpha, L_2{}^\beta\}=-e^{\alpha\beta\gamma}L_1{}^\gamma\delta(1-2),\quad \{j_{1k}, \mathbf{L}_2\}=\mathbf{L}_1\nabla_{1k}\delta(1-2),\\ \{j_{1k}, \mathbf{n}_2{}^{(1)}\}=-(\nabla_k\mathbf{n}^{(1)})\,\delta(1-2);\quad \{j_{1k}, \mathbf{n}_2{}^{(2)}\}=-(\nabla_k\mathbf{n}^{(2)})\,\delta(1-2).\end{gathered}\tag{22}$$

These PB, along with the remaining PB for a normal liquid,[3] give the complete system of equations of the hydrodynamics of biaxial nematics, with a suitable choice of the energy $H$ and the dissipation function $R$:

$$\begin{gathered}\frac{\partial j_k}{\partial t}+\nabla_l(v_lj_k)+j_l\nabla_kv_l+\rho\nabla_k\mu+s\nabla_kT+\mathbf{h}^{(1)}\nabla_k\mathbf{n}^{(1)}+\mathbf{h}^{(2)}\nabla_k\mathbf{n}^{(2)}=\nabla_l\frac{\partial R}{\partial\nabla_lv_k},\\ \frac{\partial\mathbf{n}^{(1)}}{\partial t}+(\mathbf{v}\nabla)\mathbf{n}^{(1)}+[\overset{(1)}{\mathbf{n}}\times\boldsymbol{\omega}]=\frac{\partial R}{\partial\mathbf{h}^{(1)}},\quad \frac{\partial\mathbf{n}^{(2)}}{\partial t}+(\mathbf{v}\nabla)\mathbf{n}^{(2)}+[\overset{(2)}{\mathbf{n}}\times\boldsymbol{\omega}]=\frac{\partial R}{\partial\mathbf{h}^{(2)}}.\\ [\overset{(1)}{\mathbf{h}}\times\mathbf{n}^{(1)}]+[\overset{(2)}{\mathbf{h}}\times\mathbf{n}^{(2)}]=-\frac{\partial R}{\partial\boldsymbol{\omega}},\quad \mathbf{h}^{(1)}=-\frac{\delta H}{\delta\mathbf{n}^{(1)}},\quad \mathbf{h}^{(2)}=-\frac{\delta H}{\delta\mathbf{n}^{(2)}}.\end{gathered}\tag{23}$$

The orthogonality condition $\mathbf{n}^{(1)}\cdot\mathbf{n}^{(2)}=0$ leads to the following restriction on the dissipation function $R$:

$$\mathbf{n}^{(2)}\frac{\partial R}{\partial\mathbf{h}^{(1)}}+\mathbf{n}^{(1)}\frac{\partial R}{\partial\mathbf{h}^{(2)}}=0,$$

this must be taken into account in writing down $R$, along with the conditions

$$\mathbf{n}^{(1)}\frac{\partial R}{\partial\mathbf{h}^{(1)}}=\mathbf{n}^{(2)}\frac{\partial R}{\partial\mathbf{h}^{(2)}}=0,$$

that insure that the vectors $\mathbf{n}^{(1)}$ and $\mathbf{n}^{(2)}$ shall be unit vectors.

In some cases, it is necessary to describe the dynamics in terms of a complete five-component symmetric, traceless tensor $Q_{\alpha\beta}$. This is necessary, for example, if the coefficient $B$ of the cubic invariant in the Ginzburg-Landau expansion in nematics is anomalously small. In this case, non-Goldstone modes, i.e. modes not connected with oscillations of n in a uniaxial nematic or with oscillations of $\mathbf{n}^{(1)}$ and $\mathbf{n}^{(2)}$ in a biaxial nematic, become soft. Therefore non-Goldstone components of the tensor $Q_{\alpha\beta}$ become quasihydrodynamic variables, for which it is necessary to write the equations of hydrodynamics. All the components of $Q_{\alpha\beta}$ must be taken into account also in describing the dynamics of the blue phase of cholesterics (see, for example, Ref. 8), because it occurs within a narrow range near the transition, where non-Goldstone modes are softened. The equations of hydrodynamics are the same in all these cases, differing only with respect to the dependence of the energy $\varepsilon$ on $Q_{\alpha\beta}$; this dependence insures at equilibrium a minimum corresponding to one of the phases. These same equations may be applied also to the dynamics that describes transitions between phases, for example the formation of a nucleus of one phase within another.

Thus the equations contain the hydrodynamic variables j, $\rho$, $s$, and $Q_{\alpha\beta}$ and the auxiliary variable L, the internal angular momentum, which is set equal to zero in the final equations. The equations of hydrodynamics are again constructed as for nematics,[3] with $Q_{\alpha\beta}$ instead of n. The nonvanishing PB between $Q_{\alpha\beta}$ and other variables are

$$\begin{gathered}\{j_{k1}, Q_2{}^{\alpha\beta}\}=-(\nabla_kQ^{\alpha\beta})\,\delta(1-2),\\ \{L_1{}^\alpha, Q_2{}^{\beta\gamma}\}=-(e^{\alpha\beta\nu}Q^{\nu\gamma}+e^{\alpha\gamma\nu}Q^{\beta\nu})\,\delta(1-2),\end{gathered}\tag{24}$$

whence are obtained the following equations:

$$\begin{gathered}\frac{\partial j_k}{\partial t}+\nabla_l(v_lj_k)+j_l\nabla_kv_l+\rho\nabla_k\mu+s\nabla_kT+h^{\alpha\beta}\nabla_kQ^{\alpha\beta}=\nabla_l\frac{\partial R}{\partial\nabla_lv_k},\\ \frac{\partial Q^{\alpha\beta}}{\partial t}+(v\nabla)Q^{\alpha\beta}-e^{\alpha\mu\nu}\omega^\mu Q^{\nu\beta}-e^{\beta\mu\nu}\omega^\mu Q^{\alpha\nu}=\frac{\partial R}{\partial h^{\alpha\beta}},\end{gathered}\tag{25}$$

$$e^{\alpha\mu\nu}Q^{\mu\beta}h^{\nu\beta}=-\frac{\partial R}{\partial\omega^\alpha},\quad h^{\alpha\beta}=\frac{\delta H}{\delta Q^{\alpha\beta}},$$

where $R$ satisfies the conditions

$$\mathrm{Sp}\frac{\partial R}{\partial\hat{h}}=0,\quad e^{\alpha\beta\gamma}\frac{\partial R}{\partial h^{\alpha\beta}}=0.$$

The fundamental problem is to write the terms that are necessary for the concrete problem in the expansion of the energy in $Q_{\alpha\beta}$ and its gradients. Since it is at present still not known how these new phases (biaxial nematics and the blue phase of cholesterics) are constructed, we shall not discuss this question in greater detail.

## VI. CHOLESTERIC LC

A cholesteric is a spiral structure with the following field distribution of the director n:

$$\mathbf{n}(\mathbf{r})=\mathbf{n}^{(1)}\cos\Phi+\mathbf{n}^{(2)}\sin\Phi,\tag{26}$$

where $\mathbf{n}^{(1)}$ and $\mathbf{n}^{(2)}$ are orthogonal unit vectors. In equilibrium, $\nabla\Phi=q_0\mathbf{l}$, where $\mathbf{l}=\mathbf{n}^{(1)}\times\mathbf{n}^{(2)}$ is the direction of the axis of the spiral, and where $q_0$ is its reciprocal pitch. In a choice of the dynamic variables, we must take into account their invariance with respect to a change of $\Phi$ by $\delta\Phi$ in a uniform rotation of $\mathbf{n}^{(1)}$ and $\mathbf{n}^{(2)}$ about $\mathbf{l}$ through an angle $\delta\Phi$. Such variables are $\mathbf{l}$ and a distortion $\mathbf{v}_s$ that vanishes in equilibrium (see Ref. 9):

$$\mathbf{v}_s=q_0\mathbf{l}-\mathbf{W},\quad \mathbf{W}=\nabla\Phi-n_i^{(1)}\nabla n_i^{(2)}. \tag{27}$$

The vector W has the same meaning as in smectics. In the modified form (27), it is invariant with respect to the indicated transformation. As a result, its curl is nonzero and is connected with the gradients of $\mathbf{l}$ by the Mermin-Ho relation[10]

$$(\mathrm{rot}\,\mathbf{W})_i=-{}^1/_2 e_{ijk}\mathbf{l}\,[\nabla_j\mathbf{l},\times\nabla_k\mathbf{l}]. \tag{28}$$

The PB of the variables $\mathbf{n}^{(1)}$, $\mathbf{n}^{(2)}$, and $\Phi$ with the current j and with the auxiliary variable, the angular momentum L, are given by formulas (7) and (22), whence are obtained the following PB containing W and $\mathbf{l}$:

$$\begin{gathered}\{L_1^i,\mathbf{W}_2\}=l_2^i\nabla_2\delta(1-2),\quad \{L_1^i,l_2^k\}=-e^{ikl}l^l\delta(1-2),\\ \{j_{i1},W_{k2}\}=\delta(1-2)(\nabla_k W_i-\nabla_i W_k)+W_{i1}\nabla_{k1}\delta(1-2),\\ \{j_{i1},\mathbf{l}_2\}=-\delta(1-2)\nabla_i\mathbf{l}.\end{gathered} \tag{29}$$

Using these PB and introducing the notation

$$\mathbf{h}=\frac{\delta H}{\delta\mathbf{l}}=\left.\frac{\delta H}{\delta\mathbf{l}}\right|_W+\frac{\delta H}{\delta W_i}\frac{\delta W_i}{\delta\mathbf{l}}=\tilde{\mathbf{h}}+[\mathbf{l},\times(\boldsymbol{\sigma}\nabla)\mathbf{l}], \tag{30}$$

we get the following system of nonlinear equations of dynamics:

$$\begin{gathered}\frac{\partial j_k}{\partial t}+\nabla_l(v_l j_k)+j_l\nabla_k v_l+\rho\nabla_k\mu+s\nabla_k T+\\ +W_k\nabla\boldsymbol{\sigma}-[\boldsymbol{\sigma}\times\mathrm{rot}\,\mathbf{W}]_k-\tilde{\mathbf{h}}\nabla_k\mathbf{n}=\nabla_l\frac{\partial R}{\partial\nabla_l v_k},\\ \frac{\partial\mathbf{W}}{\partial t}+\nabla(\mathbf{v}\mathbf{W}-\boldsymbol{\omega}\mathbf{l})+[\mathbf{v}\times\mathrm{rot}\,\mathbf{W}]+\omega_i\nabla l_i=\nabla\frac{\partial R}{\partial\nabla\boldsymbol{\sigma}}-e_{ikl}\frac{\partial R}{\partial h_i}l_k\nabla l_l,\\ \frac{\partial\mathbf{l}}{\partial t}+(\mathbf{v}\nabla)\mathbf{l}-[\boldsymbol{\omega}\mathbf{l}]=-\frac{\partial R}{\partial\mathbf{h}},\quad [\mathbf{l}\times\mathbf{h}]=-\frac{\partial R}{\partial\boldsymbol{\omega}}.\end{gathered} \tag{31}$$

The dependence of the energy on $\mathbf{v}_s=q_0\mathbf{l}-\mathbf{W}$ and $\mathbf{l}$ can be found by averaging the Frank energy. A more accurate averaging procedure than that carried out earlier[9] gives

$$\varepsilon(\mathbf{v}_s,\mathbf{l})={}^1/_2K_2(\mathbf{l}\mathbf{v}_s+{}^1/_2\mathbf{l}\,\mathrm{rot}\,\mathbf{l})^2+{}^1/_8K_3(2[\mathbf{l}\times\mathbf{v}_s]^2+{}^3/_2(\nabla\mathbf{l})^2+{}^1/_2(\mathbf{l}\,\mathrm{rot}\,\mathbf{l})^2). \tag{32}$$

The stress $\sigma$ and the molecular field $\tilde{h}$ are expressed in terms of $\varepsilon$ in the form

$$\boldsymbol{\sigma}=-\frac{\partial\varepsilon}{\partial\mathbf{v}_s},\quad \tilde{\mathbf{h}}=-q_0\boldsymbol{\sigma}+\frac{\partial\varepsilon}{\partial\mathbf{l}}-\nabla_i\frac{\partial\varepsilon}{\partial\nabla_i\mathbf{l}}.$$

We note that equations (31) are compatible with the condition (28). In writing the dissipation function $R$, which depends on $\nabla\sigma$, h, $A_{ik}$, $\omega-\frac{1}{2}\,\mathrm{curl}\,\mathbf{v}$, and $\nabla T$ with tensor coefficients made up from $\mathbf{l}$, it is necessary to take into account the invariance of the states with respect to a uniform substitution $\mathbf{l}\to-\mathbf{l}$, $\mathbf{W}\to-\mathbf{W}$. In other words, a uniform change of sign of $\mathbf{l}$, h, and $\sigma$ must not change $R$.

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