Interaction of quantum nonlinear resonances

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The interaction between two quantum nonlinear resonances is analyzed under conditions when the interaction is weak and in the case when the resonances overlap. It is shown that the overlap of resonances that span a sufficiently large number of levels increase substantially the rate of growth of the system energy. The results are compared with those for a corresponding classical system.

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1. INTRODUCTION

Two limiting cases of resonance of light with atoms and molecules can be distinguished, depending on the structure of the energy spectrum. The first takes place when the frequencies of the transitions between the levels located in the vicinity of the resonant transition differ greatly from the frequency of this transition. The principal role in the resonant interaction is played then by transitions between two or several closely lying levels. The problem of the interaction of two- or three-level systems with an external field has been sufficiently well investigated (see, e.g., Ref. 1).

In the second limiting case the difference between the distances between the nearest levels is small (weak anharmonicity), so that a large number of transitions are immediately at resonance with the external field. This phenomenon, sometimes called quantum nonlinear resonance, was investigated in Ref. 2. Some of its aspect were considered also in Refs. 3 and 4.

Fundamentally new effects can appear when the external field contains at least two frequencies, for each of which there is a large number of near-resonant transitions. We shall explain this situation in greater detail.

Each nonlinear resonance is characterized by a certain width that can be determined by the number of transitions that are close to the resonant one. In the presence of two nonlinear resonances, the dynamics of the system is determined by the distance between them. It is well known that in classical dynamics the presence of two nonlinear resonances in a system can lead to the onset of a random (stochastic) motion if these resonances overlap.[5] It is not clear at all, however, what should take place when two nonlinear resonances overlap. Despite the fact that in this case a very strong interaction occurs effectively between the resonances, the result is far from obvious. The reason is that the quantum character of the object can lead to substantial changes in the dynamics of a system that is stochastic in the classical limit.[6]

We present here an analytic and numerical investigations of the features of a resonant interaction between a perturbation that contains one or two frequencies and a multilevel system, under conditions when a large number of transitions in the system participate in the resonance. The main result of the paper relates to an analysis of the case of resonance overlap, wherein all the small parameters of the problem vanish and the investigation is carried out numerically. The overlap of the resonances causes the correlator of the wave-function amplitudes to attenuate rapidly (within a finite time) and remain small during the problem calculation time. It follows hence also that the off-diagonal elements of the density matrix attenuate substantially. This makes it possible to describe (with a certain degree of accuracy) the dynamic picture with the aid of statistical methods. In particular, the overlap of the resonance and the onset of a stochastic (with a certain degree of accuracy) change of the phases of the wave-function amplitudes are accompanied by a rapid excitation of the system. The latter can also be of interest in connection with investigations of the mechanisms of collisionless dissociation of molecules or of ionization of atoms.

2. DESCRIPTION OF MODEL. DERIVATION OF ABBREVIATED EQUATIONS

To study the interaction between a nonlinear quantum system (molecule) and an external field containing two frequencies, we consider the following simplified model:

$$\begin{gathered} H=H_0+V_{int}(t), \\ H_0=\hbar\omega a^+a+\hbar^2\gamma(a^+a)^2, \quad V_{int}(t)=\hbar^{1/2}f(t)(a^++a), \\ f(t)=f_1\cos\Omega_1 t+f_2\cos\Omega_2 t, \end{gathered} \tag{2.1}$$

where ω and γ are respectively the frequency of the linear oscillations and the nonlinearity parameter; Ω_1, f_1 and Ω_2, f_2 are the frequencies and amplitudes of the external field. Dipole interaction of the field with the molecule is assumed. We assume hereafter for simplicity that $\gamma>0$ and $\Omega_1>\Omega_2$.

We represent the solution of the Schrödinger equation with a Hamiltonian H in the form

$$\psi(t)=\sum_{n=0}^{\infty}c_n(t)|n\rangle, \quad H_0|n\rangle=E_n|n\rangle, \quad E_n=\hbar\omega n+\hbar^2\gamma n^2. \tag{2.2}$$

The equations for $c_n(t)$ are of the form

$$i\hbar\dot{c}_n=E_nc_n+f(t)n^{1/2}(n^{1/2}c_{n-1}+(n+1)^{1/2}c_{n+1}). \tag{2.3}$$

We consider the case when resonances are possible in the system between the external field and the transition frequencies in the unperturbed system. We represent the conditions for the existence of resonances in the form

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$$\frac{1}{\hbar}\frac{\partial E_n}{\partial n}\Big|_{n=n_1} \approx \Omega_1, \tag{2.4}$$

$$\frac{1}{\hbar}\frac{\partial E_n}{\partial n}\Big|_{n=n_2} \approx \Omega_2. \tag{2.5}$$

Expressions (2.4) and (2.5) determine the numbers n_1 and n_2 of the levels near which the transitions are close to resonant. We assume hereafter $n_1, n_2 \gg 1$.

We obtain from (2.3) the abbreviated equations that describe the motion of the system in the vicinity of the resonances (2.4) and (2.5). Expanding for this purpose the function E_n in a series in the vicinity of the point n_0, defined by the inequality

$$\frac{1}{\hbar}\frac{\partial E_n}{\partial n}\Big|_{n=n_0} \approx \Omega = \frac{\Omega_1+\Omega_2}{2}, \tag{2.6}$$

we have

$$E_n = E_{n_0} + \frac{\partial E_n}{\partial n}\Big|_{n=n_0}(n-n_0) + \frac{1}{2}\frac{\partial^2 E_n}{\partial n^2}\Big|_{n=n_0}(n-n_0)^2. \tag{2.7}$$

We represent $c_n(t)$ in (2.3) in the form

$$c_n(t) = A_m(t)\exp\left\{-\frac{i}{\hbar}\left[E_{n_0} + \frac{\partial E_n}{\partial n}\Big|_{n_0} m\right]t\right\} \quad (m \equiv n-n_0), \tag{2.8}$$

where we have introduced new amplitudes $A_m(t)$. Substituting (2.8) in (2.3) and excluding the rapidly oscillating terms, we obtain an equation for $A_m(t)$:

$$i\frac{\partial A_m}{\partial \tau} = \mu m^2 A_m + \frac{1}{2}V_1(e^{-i\nu\tau}A_{m+1} + e^{i\nu\tau}A_{m-1}) + \frac{1}{2}V_2(e^{i\nu\tau}A_{m+1} + e^{-i\nu\tau}A_{m-1}),$$
$$\nu = (\Omega_1-\Omega_2)/2\Omega, \quad \mu = \hbar\gamma/\Omega, \tag{2.9}$$
$$V_1 = f_1(\hbar n_0)^{1/2}/\hbar\Omega, \quad V_2 = f_2(\hbar n_0)^{1/2}/\hbar\Omega.$$

We have introduced in (2.9) the dimensionless time $\tau = \Omega t$. In the derivation of (2.9) we used a condition that allows us to neglect the fast oscillations. This condition can be represented in the form

$$V_1 \ll 1, \quad V_2 \ll 1,$$

$$\nu = \frac{\Omega_1-\Omega_2}{2\Omega} = \frac{\partial^2 E_n}{\partial n^2}\Big|_{n_0}(n_1-n_2)/2\frac{\partial E_n}{\partial n}\Big|_{n_0} = \frac{\gamma\hbar(n_1-n_2)}{E_{n_0}'} \ll 1. \tag{2.10}$$

In addition, we used the approximation $n,\ n+1 \approx n_0$, which is permissible subject to satisfaction of the inequality

$$\delta n/n_0 \ll 1, \tag{2.11}$$

where δn is the effective number of levels that take part in the dynamics of the system. It will be shown below that the conditions (2.10) and (2.11) correspond to the classical condition of moderate nonlinearity.[5]

We introduce the function $\varphi(\theta, \tau)$:

$$\varphi(\theta,\tau) = \sum_{m=-n_0}^{\infty} A_m(\tau)e^{im\theta}. \tag{2.12}$$

Using (2.9) and (2.12) and taking the condition (2.11) into account, we obtain an equation for $\varphi(\theta, \tau)$:

$$i\partial\varphi/\partial\tau = \mathcal{H}(\theta,\tau)\varphi, \tag{2.13}$$

$$\mathcal{H}(\theta,\tau) = -\mu\frac{\partial^2}{\partial\theta^2} + V_1\cos(\theta+\nu\tau) + V_2\cos(\theta-\nu\tau). \tag{2.14}$$

Equation (2.13) with the Hamiltonian (2.14) describes the interaction of two resonances: the term with V_1 corresponds to the resonance (2.4), and the term with V_2 to (2.5). The quantity 2ν characterizes the distance between the resonances.

We consider now in greater detail the conditions (2.10) and (2.11), which were used in the derivation of Eqs. (2.9) and (2.13). The characteristic number δn of the levels spanned by the resonance can be estimated from Eqs. (2.13) and (2.14) by putting $|\partial^2/\partial\theta^2| \sim (\delta n)^2$. This yields

$$\delta n \sim (V/\mu)^{1/2}, \quad V \sim V_1, V_2. \tag{2.15}$$

We introduce the dimensionless parameters of the perturbations ε_1 and ε_2 the nonlinearity α:

$$\varepsilon = (\hbar n_0)^{1/2}f_1/n_0E_{n_0}' = V_1/n_0, \quad \varepsilon_2 = (\hbar n_0)^{1/2}f_2/n_0E_{n_0}' = V_2/n_0,$$
$$\alpha = n_0E_{n_0}''/E_{n_0}' = 2n_0\mu. \tag{2.16}$$

Taking (2.15) into account, the inequality (2.11) takes the form

$$\delta n/n_0 \sim (\varepsilon/\alpha)^{1/2} \ll 1. \tag{2.17}$$

Putting $n_2 - n_1 \sim \delta n$ in (2.10) and using the definition (2.16), we get

$$(\varepsilon\alpha)^{1/2} \ll 1. \tag{2.18}$$

Combining the inequalities (2.17) and (2.18) we obtain for the nonlinearity α the condition

$$\varepsilon^{1/2} \ll \alpha^{1/2} \ll \varepsilon^{-1/2}, \tag{2.19}$$

at which the abbreviated equations (2.13) and (2.14) are valid. The system of inequalities (2.19) corresponds to the classical condition of moderate nonlinearity.[5]

3. ISOLATED QUANTUM NONLINEAR RESONANCE

To compare certain analytic results with the numerical analysis, we consider the particular case of one nonlinear resonance. It can be obtained from the expressions of the preceding section by putting $V_2 = 0$ and $V_1 \equiv V$. It is also convenient to expand (2.7) near the point $n_0 = n_1$. Equations (2.9) and (2.13) then take the form

$$i\frac{\partial A_m}{\partial\tau} = \mu m^2 A_m + \frac{1}{2}V(e^{-i\nu\tau}A_{m+1} + e^{i\nu\tau}A_{m-1}), \tag{3.1}$$

$$i\frac{\partial\varphi}{\partial\tau} = -\mu\frac{\partial^2\varphi}{\partial\theta^2} + V\cos(\theta+\nu\tau)\varphi, \tag{3.2}$$

$$\nu = \left(\Omega_1 - \frac{1}{\hbar}\frac{\partial E_n}{\partial n}\Big|_{n=n_1}\right)\Big/\Omega_1.$$

Expression (3.2) coincides with the equation for quantum nonlinear resonance, which was obtained earlier.[2] The general solution of (3.2) at $\nu = 0$ can be represented by an expansion in Mathieu periodic functions that satisfy the boundary conditions[1)]

$$\varphi(\theta,\tau) = \varphi(\theta+2\pi,\tau).$$

Using the properties of Mathieu functions, we easily obtain an estimate for the number δn of the levels captured in the resonance:

$$\delta n = 4(V/\mu)^{1/2}. \tag{3.3}$$

For the numerical calculations of the dynamics of the system under the condition that one quantum nonlinear

resonance is present, we used Eqs. (3.1). The accuracy of the calculation was monitored against the normalization condition

$$\sum_m |A_m(\tau)|^2=1. \tag{3.4}$$

In all cases the error did not exceed 10^{-4}. The main feature of the quantum nonlinear resonance is its saturation, which manifests itself in the following. If the initial conditions correspond to population of several levels near the resonant transition, then the nearest levels become populated in the course of time and the wave packet is spread out. This process continues up to a certain time τ^*, after which the spreading of the packet stops and the subsequent dynamics is connected with transition between the levels captured into the resonance. We present one typical variant of the numerical analysis: $V=5\times10^{-2}$, $\mu=10^{-3}$, and $A_m(0)=\delta_{m0}$. The frequency Ω_1 is chosen to be resonant to the $0\rightarrow1$ transition; this corresponds to $\nu=10^{-3}$. Numerical calculation yields $\tau^*=200$ and $\delta n\approx22$. At these values of the parameters μ and V we obtain from the analytic estimate (3.3) $\delta n\sim28$.

We note also that in the case of the classical treatment of a system with a Hamiltonian corresponding to the Schrödinger equation (3.2), the frequency of the phase oscillations is given by $\Omega_{ph}=(2\mu V)^{1/2}$, and the action return time τ_r at an initial population $I(0)=0$ is

$$\tau_r=\pi/\Omega_{ph}=\pi/(2\mu V)^{1/2}. \tag{3.5}$$

Expression (3.5) determines the characteristic return time of a quantum packet if the initial population of the null level is $A_m(0)=\delta_{m0}$. For the parameters cited above, the estimate (3.5) yields $\tau_r=314$, in good agreement with the $\tau_r\approx360$ obtained by numerical analysis.

In the classical case, the width of a nonlinear resonance is usually taken to mean the size of the action region bounded by the separatrix. To determine the "separatrix" of a quantum nonlinear resonance, numerical calculations were made at fixed parameters ν, μ, and V and with variation of the initial population of the system. The calculation results are shown in Fig. 1. The ordinates are the time-averaged values $\langle|Am(\tau)|^2\rangle$ of the level populations. The averaging time is $T=720$ and corresponds approximately to one period of the phase oscillations.

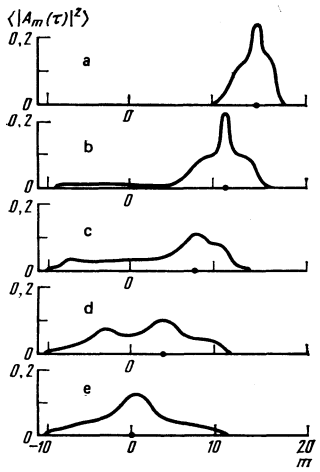

FIG. 1. Distribution function of the population over the levels vs. the initial conditions at $V=5\times10^{-2}$ and $\mu=10^{-3}$. •) Single populated level at $\tau=0$. a) $m=16$, b) $m=12$, c) $m=8$, d) $m=4$, e) $m=0$.

When the number m of the initial population increases to a certain critical value m^*, a slight restructuring of the motion takes place. The time needed to establish a quasistationary distribution increases with increasing m. The center of the packet shifts towards the region of the initial population, and the entire packet is located in the region of the resonant action of the external field. At $m\geq m^*$, a radical restructuring of the system motion takes place. The wave packet does not span the broad region of the resonant influence of the external field, and the number of spanned levels and the time of establishment of the quasistationary state decrease strongly.

The quantity $2m^*$ can be called, by analogy with the classical case, the action width of the quantum nonlinear resonance. The quantity m^* itself corresponds to the separatrix of the quantum nonlinear resonance. It corresponds to the energy boundary that separates the region of capture into the nonlinear resonance from the region of the nonresonant interaction.

By analogy with the classical case, we introduce the concept of the frequency width of the nonlinear resonance. To this end we define the frequency detuning from resonance $m\rightarrow m+1$ for an arbitrary level m:

$$\Delta\omega(m)=[(E_{m+1}-E_m)/\hbar-\Omega_1]/\Omega_1. \tag{3.6}$$

The quantity $\Delta\omega^*=\Delta\omega(m^*)$ characterizes the anharmonicity at which the systems goes off resonance, and determines the frequency half-width of the quantum nonlinear resonance. At the chosen values of the parameters μ, ν, and V a numerical analysis yields $m^*=12$ and $\Delta\omega^*=0.024$. To obtain analytic estimates of m^* and $\Delta\omega^*$ we use Eq. (3.3) for δn. We obtain

$$m^*\approx\delta n/2\approx14, \quad \Delta\omega^*=2\mu m^*=4(\mu V)^{1/2}=0.028, \tag{3.7}$$

in good agreement with the numerical results.

The results above (both analytic and numerical) show that in the quantum case the nonlinear resonance, just as in the classical limit, has a distinct boundary of the capture in energy. This provides the basis for posing the question considered below: what is the dynamics of a quantum system if several (say, two) resonances overlap?

4. INTERACTION BETWEEN TWO QUANTUM NONLINEAR RESONANCES

The interaction between two quantum nonlinear resonances is described by the Schrödinger equation (2.13) with the Hamiltonian (2.14). In the classical case, the Hamiltonian (2.14) is of the form

$$\mathcal{H}(\theta, J, \tau)=\mu J^2+V_1\cos(\theta+\nu\tau)+V_2\cos(\theta-\nu\tau), \tag{4.1}$$

where J and θ are the canonically conjugate dimensionless action and phase.

For a classical system with a Hamiltonian (4.1), the conditions for the resonances are

$$\omega(J_1)+\nu=0, \quad \omega(J_2)-\nu=0, \tag{4.2}$$

where $\omega(J)=2\mu J=\dot{\theta}$. It is known that the behavior of the system (4.1) depends essentially on the value of the resonance-overlap parameter K (Ref. 5)

$$K=\Delta\omega/\nu, \tag{4.3}$$

where $\Delta\omega$ is the frequency half-width of the isolated resonance, and $2\nu=\omega(J_1)-\omega(J_2)$ defines the distance between the resonances (4.2). For $\Delta\omega$ we have from (4.1) and (4.2)

$$\Delta\omega=2\mu\Delta J=2(2\mu V)^{1/2}, \tag{4.4}$$

where ΔJ is the half-width of the action resonance [the estimate (4.4) was obtained under the condition $V_1=V_2=V$]. Using (4.4) we obtain the explicit form of the resonance-overlap parameter (4.3):

$$K=2(2\mu V)^{1/2}/\nu. \tag{4.5}$$

In the classical case the interaction of two resonances was investigated in detail in Refs. 7–9. At $K\ll 1$ the resonances (4.2) are separated in frequency, and the interaction between them can be determined by the usual perturbation theory. In this case the motion of the system is quasi-periodic. At $K\gtrsim 1$ the resonances (4.2) overlap, and the behavior of the phase θ becomes close to random, so that the slow action variable J varies in diffuse fashion.

To study the features of the interaction between resonances in the quantum case, we integrated numerically the system (2.9). Just as in the case of an isolated resonance, the accuracy was monitored against the wave-function normalization condition (3.3). The program provided for an automatic choice of the number m of levels in (2.9) for a given calculation accuracy (10^{-4}). The maximum number of levels used in the investigation of the interaction between resonances was 200.

The system (2.9) was numerically analyzed under the following conditions: the initial population was assumed constant, i.e., $A_m(0)=\delta_{m0}$, while the distance 2ν between resonances was decreased until the resonances "touched" and subsequently overlapped. Figure 2 shows the calculated time-averaged level-population distribution function. The ordinates are the time-averaged populations $\langle|A_m(\tau)|^2\rangle$ as functions of the level m. The points mark the positions of the resonance frequencies Ω_1 and Ω_2 corresponding to the chosen values of the parameters. The parameters chosen were $V_1=V_2=0.05$, $\mu=10^{-3}$, and the averaging time $T=720$.

It is seen from Fig. 2 that at an overlap parameter $K<1$ the capture of the levels into the resonances (2.4) and (2.5) does not take place. The distribution function of the level populations differs from zero only near the initially populated level $m=0$. At $\nu=\nu^*=0.021$, the resonances come in contact. The population distribution function is then greatly altered, the number of populated levels increases, and the distribution function itself becomes close to homogeneous.

Using the numerical values for m^* and ν^* we can obtain, in accord with (4.3), the quantum resonance overlap parameter K^*:

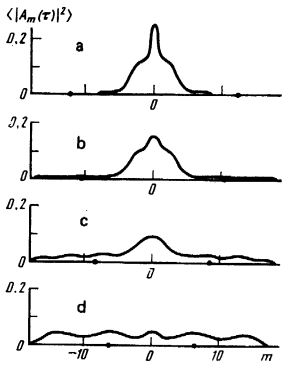

FIG. 2. Distribution function of the populations over the levels at various distances between the resonance, and at differently populated levels with $m=0$ ($V_1=V_2=0.05$; $\mu=10^{-3}$). •) Position of levels with transitions resonant to the frequencies Ω_1 and Ω_2. The values of ν are: a) 0.025, b) 0.021, c) 0.017, and d) 0.013.

$$K^*=\Delta\omega^*/\nu^*=2\mu m^*/\nu^*\approx 1.3,$$

which is in good agreement with the classical estimate that follows from (4.5), $K\approx 1$. With further decrease of ν the parameter K increases, but the significant interaction between the resonances takes place in the region $\nu^*\geq\nu>\nu_{\min}$. At $\nu<\nu_{\min}$ the motion, just as in the classical case, degenerates into an isolated resonance.

To assess the degree of proximity of a system of two interacting quantum nonlinear resonances to the stochastic motion regime we calculated the correlation function $R_m(\tau)$ of the amplitudes $A_m(\tau)$ at various values of the parameters V and μ:

$$R_m(\tau)=\mathrm{Re}\frac{\langle A_m^*(\tau')A_m(\tau'+\tau)\rangle-|\langle A_m(\tau')\rangle|^2}{\langle|A_m(\tau')|^2\rangle-|\langle A_m(\tau')\rangle|^2}, \tag{4.6}$$

where $\langle\ldots\rangle$ means averaging over the time τ'. The quantity ν was chosen here constant, and the values of V and μ were varied in such a way that their product remained constant. In this case the resonance parameter K is fixed, and the number of captured levels δn can vary. We chose the parameters $\nu=0.015$ and $V\mu=5\times10^{-5}$, which corresponds to a resonance overlap ($K=1.6$). The averaging time T was chosen in the calculation of $R_m(\tau)$ such that the coordination function was determined at the specified degree of accuracy.

Figures 3a and 3b show the correlation function $R_0(\tau)$ calculated from Eq. (4.6) using different numbers of captured levels δn. The parameters for Fig. 3a are $V_1=V_2=0.05$ and $\mu=10^{-3}$, and those for Fig. 3b are $V_1=V_2=0.2$ and $\mu=2.5\times10^{-5}$. The corresponding numbers of the captured levels are $\delta n\approx 40$ and $\delta n\approx 175$, respectively. It is seen from these figures that the

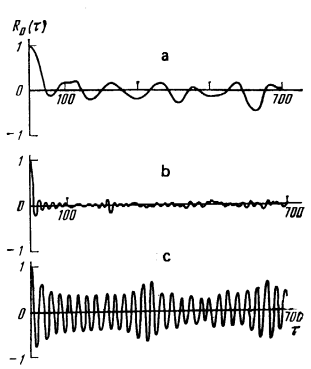

FIG. 3. Correlation function $R_0(\tau)$ for two interacting resonances: a) $K\approx 1.6$, $\delta n\approx 40$, $\varkappa\approx 28$; b) $K\approx 1.6$, $\delta n\approx 175$, $\varkappa\approx 112$; c) the function $R_0(\tau)$ for an isolated resonance: $\varkappa\approx 56$, $\delta n\approx 100$.

increase of the number δn of the captured levels that participate in the dynamics of the interacting resonances alters substantially the form of the correlation function, namely, the attenuation of the correlations becomes more intense with increasing δn.

Numerical calculations were made also of the function $R_m(\tau)$ at $m \neq 0$ and at different values of the parameters V and μ. The behavior of $R_m(\tau)$ as a function of δn remains qualitatively the same as before. The initial stage of the attenuation of the correlations at large δn is quite similar to their exponential attenuation in the classical case. Figure 3c shows for comparison the form of the function $R_0(\tau)$ for the case of an isolated resonance: $\nu = 0$, $V = 0.2$, $\mu = 2.5 \times 10^{-5}$, and the number of captured levels is $\delta n \approx 100$. It is seen that $R_0(\tau)$ is a quasi-periodic function and an increase of δn leads in this case only to an increase of the oscillation frequency.

The qualitative features of the results call for a certain discussion. It is necessary to note, first, that the system (2.9) with an infinite number of levels is replaced in the numerical analysis by a system with a finite number (~200) of levels with periodic coefficients. The motion of such a system is therefore quasi-periodic. In this sense, there is no "genuine" stochasticity (one of the conditions for which is the presence of a continuous frequency spectrum) in such systems. The results for the correlation function $R_0(\tau)$, which are shown in Figs. 3a and 3b, mean therefore that over a certain finite time the motion of the system can be regarded as random. On the other hand, the question of the properties of the system over long times calls for a more accurate investigation. In addition, one must consider also the question of the "degree of proximity" of the dynamics of a system with a finite number of resonances to a random law. The latter includes also an analysis of the degree of proximity of the spectrum of the frequencies of the wave-function amplitudes to a continuous spectrum.

When comparing the quantum and classical treatments of systems with finite numbers of interacting resonances, it is natural to introduce in the quantum case a parameter that characterizes the degree of the proximity of the system to its classical analog. We choose this dimensionless parameter $\varkappa$ to be the ratio of the characteristic values of the change of the action of the system, as a result of the perturbation, to the quantity $\hbar$:

$$\varkappa = \Delta I / \hbar. \tag{4.7}$$

In the case of an isolated resonance, ΔI is of the order of the action width of the resonance. In the case of two interacting quantum nonlinear systems ΔI is of the order of double the action width of the isolated resonance. The parameter $\varkappa$ determines in fact the order of magnitude of the number of captured levels participating in the dynamics of the system. At $\varkappa \gtrsim 1$, when the number of the captured levels is small, one should expect a substantial quantum motion of the system. In the case of two interacting quantum nonlinear resonances, a motion regime close to stochastic (in the sense indicated above) should be expected at $\varkappa \gg 1$ and under the condition that the resonances overlap: $K \gtrsim 1$. We represent for this case the parameter $\varkappa$ in explicit form. To estimate ΔI we have from (2.1) and (2.6)

$$\Delta I \approx 4[f(\hbar n_0)^{1/2}/\gamma]^{1/2}, \tag{4.8}$$

where we chose $f_1 = f_2 = f$. From (4.8), taking the notation in (2.9) into account, we obtain for $\varkappa$:

$$\varkappa \approx 4(V/\gamma)^{1/2}, \tag{4.9}$$

which, as expected, is of the same order as the estimated number of levels captured in the resonances.

It is seen from Fig. 3 that the case a corresponds to essentially quantum motion ($\varkappa = 28$). The correlation function attenuates (within the considered times) insignificantly, and stochasticity effects, if they exist at all, are very weak. In case b, conversely, $\varkappa = 112$ and the correlation function (within the times considered) attenuates in a nearly exponential fashion.

One of the important manifestations of the dynamics of a classical system in the case of overlap of the resonances is the rapid growth of the average energy of the system with time.[8,9] This singularity is a direct consequence of the onset of the stochasticity of the phases in the system when the resonances overlap. A similar singularity is possessed also by the quantum system (2.9).

We consider now the energy of the system (2.9)

$$E(\tau) = \mu \sum_m |A_m(\tau)|^2$$

as a function of the parameter $\varkappa$ under conditions of resonance overlap ($K > 1$). The result of the numerical analysis are shown in Fig. 4, where the quantity $V\mu = 5 \times 10^{-5}$, which determines the frequency of the phase oscillations, was chosen to be the same in both cases. It is seen that an increase of the parameter $\varkappa$ leads to an increase of the growth rate of the system energy $E(\tau)$ with time.

It follows thus from the foregoing analysis that the system can behave quite differently, depending on the degree of interaction of the quantum nonlinear resonances and on the number of levels captured in each of the resonances. Thus, if the resonances overlap, an increase in the number of captured levels lead to the damping, characteristic of a random process, of the correlation function of the density-matrix phases. The correlation damping itself can here be arbitrarily regarded as going through two time stages: 1) rapid initial damping, analogous to the exponential damping in classical systems; 2) establishment of a residual correlation level as a result of quantum effects. This

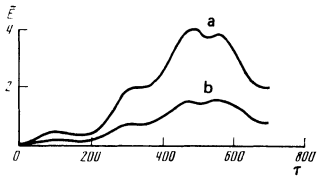

FIG. 4. Time dependence of the system energy for two interacting resonances at $K = 1.5$: a) $\mu = 4 \times 10^{-4}$, $V = 0.125$, $\varkappa = 70$; b) $m = 10^{-3}$, $V = 0.05$, $\varkappa \approx 22$.

behavior of the density-matrix phase correlations allows us to describe the motion of the system, over finite times, by statistical methods followed by allowance for the contribution of the residual correlations.

The interaction of quantum nonlinear resonances can be observed, for example, in the interaction between coherent laser radiation and polyatomic molecules. In this case the spectral properties of the resultant radiation depend substantially on the parameters $\varkappa$ and K. The region $\varkappa \gtrsim 1$ at arbitrary values of the parameter K corresponds to quasi-periodic motions. At $K \gtrsim 1$ and $\varkappa \gg 1$ the correlation properties of the radiation are close to random. In addition, in the latter case the rate of increase of the average molecule energy increases greatly, and this may be of interest in connection with research into the phenomena of collisionless ionization and dissociation of polyatomic molecules interacting with a laser-radiation field.

[1)]In the classical case the action $I = \hbar n_1$ is a continuous function, so that the parameter ν can be made to vanish by a suitable choice of I_1. In the quantum case, according to (2.4), ν is close to zero. In the numerical analysis that follows, however, we use the exact equations (3.1).

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