Investigation of self-similar gravitational fields in space-time with plane symmetry

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For matter with equation of state $p = (\gamma - 1)e$, gravitational fields with plane symmetry that admit the conformal group with a dependence of the metric on $z = x^1/x^0$ (self-similar solutions in space-time with plane symmetry) are considered in the general theory of relatively. The investigation of the Einstein equations, which reduce to a system of ordinary differential equations, requires a qualitative investigation on a two-dimensional phase plane. Some exact solutions are given. The t region (z coordinate timelike) and the s region (z coordinate spacelike) are introduced. A special case of metric structure containing the flat Friedmann model is considered in detail. The construction of analytic solutions and solutions with weak and strong discontinuities (shock waves and contact discontinuities) is discussed. Solutions which can be joined to the flat Friedmann model through a weak discontinuity and through a shock wave are studied. The possibility of joining the flat Friedmann model to Minkowski space-time through a shock wave of limiting intensity is discussed.

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§1. INTRODUCTION

In the general theory of relativity, it is of interest to study gravitational fields with spherical, pseudospherical, and plane symmetry with metrics, respectively, of the form[1]

$$ds^2=T^2(t, x^1)(cdt)^2-X^2(t, x^1)(dx^1)^2-\mathcal{Y}^2(t, x^1)[(dx^2)^2+f^2(x^2)(dx^3)^2],$$
$$f(x^2)=\sin x^2,\ \mathrm{sh}\, x^2,\ 1. \quad (1.1)$$

The gravitational fields (1.1) admit of an exact analytic solution for dust [Tolman's solution for $f=\sin x^2$ (Ref. 2) and similar solutions for $f=\sinh x^2$ (Ref. 1)]; the worldlines of the matter are geodesics of the space-time.

When the source of the gravitational field in Einstein's equations (in the notation of Ref. 2)

$$R_i^k-{}^1/_2\delta_i^kR=(8\pi k/c^4)T_i^k \quad (1.2)$$

is the hydrodynamic energy—momentum tensor of matter with the equation of state (p is the pressure, and e the density of the internal energy)

$$T_{ik}=(e+p)u_iu_k-pg_{ik},\quad u^iu_i=1;\quad p=(\gamma-1)e,\quad 1\leqslant\gamma\leqslant2, \quad (1.3)$$

for $p\neq0$ ($1<\gamma\leqslant2$), the 4-acceleration $a_i\equiv u^k u_{i;k}$ is in general nonvanishing in accordance with the hydrodynamic equations[3]

$$(e+p)a_i=(\delta_i^k-u_iu^k)\partial p/\partial x^k,$$

and gas-dynamic effects are important; these effects are manifested in problems associated with the dynamics of gravitational collapse, as well as in cosmological models. In particular, study of the homogeneous anisotropic cosmological model of Bianchi type V with axial symmetry,[4] which is contained in (1.1) for $f(x^2)=1$ (Ref. 1) and for which the 4-normal n^i to the transitivity hypersurface V_3 of the group of motions G_4 does not coincide with the matter 4-velocity u^1, shows that in the region where n^i is spacelike the acoustic characteristics for $1<\gamma<2$ have an envelope (a limiting line). This makes the solution two-valued in the region of its existence ("subsonic" and "supersonic" regimes[5,4]) and leads to impossibility of extending a continuous solution to all values of the coordinates, so that it is necessary to consider solutions with discontinuity surfaces (shock waves).[4]

Below, we consider gravitational fields of the type (1.1) with $f=1$ for (1.3) which admit the group of conformal transformations (homothety group). The condition of conformal invariance[6-8] requires the presence of the generator $X_4=\xi^i_{(4)}\partial x^i$ with conformal Killing vector $\xi^i_{(4)}$ satisfying the equation

$$\xi_{i;k}+\xi_{k;i}=-2g_{ik}.$$

The conformal group is generated by X_4 and the generators $X_a=\xi^i_{(a)}\partial/\partial x^i$, $a=1, 2, 3$, where $\xi^i_{(a)}$ are the Killing vectors of the group of motions[1] which are the generators of the subgroup G_3 of the conformal group G_4. For (1.1) with $f=1$, we have[1,8-10]

$$\xi^i_{(1)}=x^3\delta_2^i-x^2\delta_3^i,\quad \xi^i_{(2)}=\delta_2^i,\quad \xi^i_{(3)}=\delta_3,$$
$$\xi^i_{(4)}=x^0\delta_0^i-x^1\delta_1^i+(1-\delta)(x^2\delta_2^i+x^3\delta_3^i),$$
$$\delta=\mathrm{const},\quad T=T(z),\quad X=X(z),\quad \mathcal{Y}=(x^1)^\delta Y(z),\quad z=x^1/x^0, \quad (1.4)$$

whereas for $f=\sin x^2$ and $\sinh x^2$ (Ref. 7)

$$\xi^i_{(4)}=x^0\delta_0^i-x^1\delta_1^i,\ T=T(z),\ X=X(z),\ \mathcal{Y}=x^1Y(z),\ z=x^1/x^0. \quad (1.5)$$

By analogy with classical gas dynamics, solutions of this type are said to be self-similar.[11,12] Self-similar solutions in general relativity, for the case of spherical symmetry ($f=\sin x^2$) with (1.5) (Refs. 13 and 14), are described by a system of ordinary differential equations which requires (as in the case of the Newtonian gas dynamics of a self-gravitating gas[11]) qualitative analysis of the integral curves in a three-dimensional space.[14] In the present paper, we consider the general case of plane symmetry with $f=1$ in (1.1) and (1.4); then the qualitative analysis is on a two-dimensional plane. The case $\delta=0$ was discussed earlier in Ref. 15 and considered in detail by the present author in Ref. 8. Below, we give the general equations for arbitrary value of the constant δ and

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find some exact solutions (Sec. 2). We discuss in detail (Sec. 3) the case $\delta=2/3\gamma$, which includes the flat Friedmann model among the solutions.[1)] In Sec. 4, we consider solutions that can be joined to the Friedmann solution through a weak discontinuity and through a shock wave.

Gravitational fields with conformal group on V_3 were considered in connection with cosmological models in Ref. 16.

§2. GRAVITATIONAL EQUATIONS

For the gravitational fields considered below with plane symmetry with metric in the form

$$ds^2=T^2(z)(dx^0)^2-X^2(z)(dx^1)^2-(x^1)^{2\delta}Y^2(z)[(dx^2)^2+(dx^3)^2],\quad z=x^1/x^0 \tag{2.1}$$

it follows[17] from the Einstein equations (1.2) that the components $u^0(z)$ and $u^1(z)$ of the 4-velocity are nonzero, and the energy density in (1.3) has the structure

$$e=E(z)/(x^0)^2. \tag{2.2}$$

By a coordinate transformation[8] the system (2.1) can be transformed to a comoving system with metric again of the type (2.1). We shall assume that the original system with (2.1) is comoving with $u^i=\delta_0^i/T$.

The variable z can have different orientations in space-time. Denoting

$$\mu=(zX/T)^2 \tag{2.3}$$

and calculating the 4-normal N_i to the hypersurface $z=\text{const}$, $f\equiv x^1-zx^0=0$, we find that $N_i=\partial f/\partial x^i$ is timelike (t region) for $\mu>1$, spacelike (s region)[2)] for $\mu<1$, and isotropic for $\mu=1$. The condition $\mu=1$ determines the light cone at each point. In the t region ($\mu>1$), the coordinate transformation

$$x^0=y^1F_0(y^0),\quad x^1=y^1F_1(y^0),\quad x^{2,3}=y^{2,3},$$
$$d\ln F_0=-\frac{\mu}{\mu-1}d\ln z,\quad F_1=zF_0$$

reduces (2.1) to a t system in the form ($\mu\geqslant 1$)

$$ds^2=(y^1)^2R_0^2(y^0)(dy^0)^2-R_1^2(y^0)(dy^1)^2-(y^1)^{2\delta}R_2^2(y^0)[(dy^2)^2+(dy^3)^2],$$
$$R_0^2=(TF_0)^2\frac{\mu}{\mu-1}\left(\frac{d\ln z}{dy^0}\right)^2,\quad R_1^2=(TF_0)^2(\mu-1),\quad z=f_1(y^0). \tag{2.4a}$$

In (2.4a), we have for the 4-velocity and the 3-velocity v of the matter the expressions

$$u_1=TF_0,\quad (v/c)^2=1/\mu. \tag{2.4b}$$

In the s region ($0<\mu<1$), the coordinate transformation

$$x^0=y^0\Phi_0(y^1),\quad x^1=y^0\Phi_1(y^1),\quad d\ln\Phi_0=-\frac{\mu}{\mu-1}d\ln z,\quad \Phi_1=z\Phi_0 \tag{2.5a}$$

reduces (2.1) to the s system in the form ($\mu\leqslant 1$)

$$ds^2=R_0^2(y^1)(dy^0)^2-(y^0)^2R_1^2(y^1)(dy^1)^2-(y^0)^{2\delta}R_2^2(y^1)[(dy^2)^2+(dy^3)^2],$$
$$R_0^2=(T\Phi_0)^2(1-\mu),\quad R_1^2=(T\Phi_0)^2\frac{\mu}{1-\mu}\left(\frac{d\ln z}{dy^1}\right)^2, \tag{2.5b}$$
$$R_2^2=\Phi_1^{2\delta}Y^2,\quad z=f_2(y^1).$$

In the s system, the matter 3-velocity v is

$$(v/c)^2=\mu. \tag{2.5c}$$

Equations (2.4b) and (2.5c) elucidate the physical meaning of the variable μ.

We consider the gravitational equations (1.2) and (1.3) in the comoving system (2.1) with (2.2), using the variable (2.3) and also the variable

$$\eta=(d\ln Y/d\ln z)^{-1}, \tag{2.6}$$

which characterizes the shift along the x^2 and x^3 axes.[8] From the conservation laws $T^k_{i;k}=0$ with $i=0$ and 1 there follow the relations[8]

$$E=\text{const}/T^{\gamma/(\gamma-1)}=\text{const}/(XY^2)^\gamma z^2, \tag{2.7}$$
$$T=(XY^2z^{2/\gamma})^{\gamma-1}. \tag{2.8}$$

From the expressions (2.8) and (2.3), we obtain

$$X=\mu^{1/2(2-\gamma)}|z|^{-1/\gamma}Y^{2(\gamma-1)/(2-\gamma)}. \tag{2.9}$$

From the Einstein equations (1.2), (1.3) the components $T_0^1=0$ and $(\gamma-1)T_0^0+T_1^1=0$ lead[17] after long calculations, in which (2.8) is used, to the system

$$\begin{aligned}&\eta(\delta\eta+\gamma)\{\mu(2-\gamma)(2\eta+3\gamma)-\eta^2\delta[2(\gamma-1)(2-\gamma)+\delta\gamma(3\gamma-2)]\\&-2\eta[(\gamma-1)(2-\gamma)+\delta\gamma(5\gamma-4)]-\gamma(7\gamma-6)\}d\mu=2\mu\{\mu[2(2-\gamma)\delta\eta^2\\&+2\gamma(4-3\gamma)\delta\eta-\gamma^2(3\gamma-2)]-\eta^2(2-\gamma)\delta[2(\gamma-1)^2-\delta\gamma(3\gamma-2)]\\&-2\delta\gamma(2\gamma^2-7\gamma+4)\eta+\gamma^2(3\gamma-2)\}d\eta,\end{aligned} \tag{2.10}$$

$$\begin{aligned}2z\gamma[\mu-(\gamma-1)]d\eta/dz=&\mu(2-\gamma)(2\eta+3\gamma)-(\eta\delta+1)\{\eta[2(\gamma-1)(2-\gamma)\\&+\delta\gamma(3\gamma-2)]+\gamma(7\gamma-6)\}.\end{aligned} \tag{2.11}$$

The component of (1.2) with $i=k=0$ leads to an expression in (2.2) in the form

$$\frac{8\pi k}{c^4}\mu T^2E(z)=\frac{2-3\delta}{\eta(\delta\eta+\gamma)}[(1+\delta\eta)^2-\mu]. \tag{2.12}$$

Equations (2.8) and (2.10)–(2.12) (for $e\neq 0$, $\gamma\neq 2$) constitute[8] the entire system of gravitational equations. For $\gamma=2$, we must use instead of (2.9) the equation $R_{01}=0$, which can be written in the form[17]

$$z\frac{d\ln X}{dz}(\delta\eta+\gamma)\eta\gamma+z\frac{d\ln\eta}{dz}\eta\gamma+2(\gamma-1)(\eta+\gamma)-\gamma(\delta\eta+1)=0. \tag{2.9a}$$

Equation (2.10) can be investigated qualitatively on the plane (η, μ); Eq. (2.11) determines the variation of z along the integral curves on the plane (η, μ).

For all values of δ in (2.1), Eq. (2.10) has the solution

$$\mu=(1+\delta\eta)^2,\quad zX=T(1+\delta\eta), \tag{2.13}$$

for which in accordance with (2.12) we have $e=p=0$. Calculation of the Riemann tensor for (2.13) with allowance for (2.8) and (2.9) shows that this tensor vanishes; the solution (2.13) corresponds to Minkowski space-time. One must here consider test matter with the equation of state (1.3) moving against the background of the Minkowski space with metric in the comoving system in the form (2.1). The formulas of the transformation from the comoving coordinates to Galilean coordinates gives the law of motion of the test matter in Lagrangian coordinates. The corresponding formulas for $\delta=0$ are given in Ref. 8.

For $\delta=2/3$ in (2.1) we have $e=0$ in accordance with (2.12) for all the solutions of (2.10) except in the case

$2\eta+3\gamma=0$. All these solutions, except for (2.13), correspond to vacuum solutions with nonzero Riemann tensor; the presence of test matter with (1.3) moving on the background of the corresponding Einstein space is implied. For $\delta=2/3$ and $2\eta+3\gamma=0$, we have in accordance with the gravitational equations an exact solution in the form

$$\delta={}^2/_3,\quad 2\eta+3\gamma=0,\quad Y=z^{-2/3\gamma},\quad \mathscr{Y}=[(x^1)^{\gamma-1}x^0]^{2/3\gamma},\quad T=(Xz^{2/3\gamma})^{\gamma-1},$$

$$e=\frac{c^4}{18\pi k\gamma^2(x^0)^2\mu T^2}\left\{\mu+3(\gamma-1)-3\gamma z\frac{d\ln X}{dz}[\mu-(\gamma-1)]\right\},\qquad(2.14)$$

where the function $\mu(z)$ is determined by the equation obtained from $R_2^2-R/2=R_3^3-R/2$ in the form

$$\begin{gathered}18\gamma^2\mu(\mu-\gamma+1)\beta d\beta/d\mu=16[(2-3\gamma)\mu+(\gamma-1)(3\gamma^2-8\gamma-4)]\\ +6\gamma\beta[(3\gamma+2)\mu+(\gamma-1)(10+\gamma)]-9\gamma^2\beta^2(\mu+\gamma-1),\\ \beta\equiv d\ln\mu/d\ln z,\quad \mu=X^{2(2-\gamma)}z^{2(2+\gamma)/3\gamma}.\end{gathered}\qquad(2.14a)$$

For dust and for matter with maximally hard equation of state (velocity of sound equal to the velocity of light) $e=p$ there exist exact solutions of the equations of general relativity for arbitrary δ in (2.1). For dust ($\gamma=1$), we obtain in accordance with Ref. 1 two types of solution in parametric form with parameter ζ in the form

$$\begin{gathered}\frac{1}{z}-\frac{1}{z_0}=k_1(\pm\operatorname{sh}\zeta-\zeta),\quad Y=k_1(\pm\operatorname{ch}\zeta-1),\\ X=\frac{k_1^2}{Y}\left[(\delta-1)\operatorname{ch}^2\zeta+\delta+1\mp2\delta\operatorname{ch}\zeta\mp\frac{1}{z_0k_1}\operatorname{sh}\zeta\pm\zeta\operatorname{sh}\zeta\right],\\ e=\frac{c^4}{4\pi k}\frac{3\delta-2}{XY^2z^2(x^0)^2},\quad z_0=\text{const},\quad k_1=\text{const}.\end{gathered}\qquad(2.15)$$

For $e=p$ ($\gamma=2$), we have in accordance with (2.10), (2.11), and (2.9)

$$Y=\mu^{-1/4},$$

and for the functions $\mu(\eta)$ and $z(\eta)$ we obtain expressions of two types[3]:

$$\begin{gathered}\mu=\frac{(1+d_1^2)(1+\delta\eta)^2}{d_1^2+(1+\delta\eta)^2},\quad\left(\frac{z}{z_0}\right)^{4\delta}=\frac{(1+\delta\eta)^2}{d_1^2+(1+\delta\eta)^2}\\ \times\exp\left[-2d_1\operatorname{arctg}\frac{1+\delta\eta}{d_1}\right],\quad d_1=\text{const},\\ \mu=\frac{(1-d_2^2)(1+\delta\eta)^2}{(1+\delta\eta)^2-d_2^2},\quad\left(\frac{z}{z_0}\right)^{4\delta}=(1+\delta\eta-d_2)^{d_2-1}\\ \times(1+\delta\eta+d_2)^{-d_2-1}(1+\delta\eta)^2,\quad d_2=\text{const}.\end{gathered}$$

From (2.9a), we obtain the dependences $X(\eta)$ in the form

$$\begin{gathered}X=\eta^{-1/2}(1+\delta\eta)^{\alpha_1}(2+\delta\eta)^{\alpha_2}\\ \times[(1+\delta\eta)^2+d_1^2]^{\alpha_3}\exp\left[\frac{d_1(1-2\delta)}{2\delta}\operatorname{arctg}\left(\frac{1+\delta\eta}{d_1}\right)\right],\\ \alpha_1=\frac{2\delta-1}{2\delta},\quad\alpha_2=\frac{\delta+2d_1^2(1-\delta)}{2\delta(1+d_1^2)},\quad\alpha_3=\frac{1-2\delta+d_1^2(\delta-1)}{4\delta(1+d_1^2)};\\ X=\eta^{-1/2}(1+\delta\eta)^{\beta_1}(2+\delta\eta)^{\beta_2}(1+\delta\eta+d_2)^{\beta_3}(1+\delta\eta-d_2)^{\beta_4},\\ \beta_1=\frac{2\delta-1}{2\delta},\quad\beta_2=\frac{\delta-2d_2^2(1-\delta)}{2\delta(1-d_2^2)},\\ \beta_3=\frac{d_2^2(2-3\delta)}{4\delta(1-d_2^2)},\quad\beta_4=\frac{2(1-2\delta)+d_2^2\delta}{4\delta(1-d_2^2)}.\end{gathered}$$

§3. ANALYSIS OF THE GRAVITATIONAL EQUATIONS FOR $\delta=2/3\gamma$

We consider in detail the solutions of (2.10) for $\delta=2/3\gamma$. In this case, (2.10) has a solution corresponding to the flat Friedmann model:

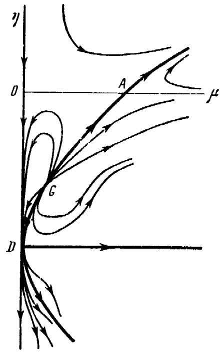

FIG. 1. Field of the integral curves of Eq. (3.2) for $\gamma=1$.

$$2\eta+3\gamma=0,\quad T=1,\quad X=Y=z^{-2/3\gamma},\quad\mu=z^{2(3\gamma-2)/3\gamma},\quad e=c^4/6\pi k\gamma^2(x^0)^2.\qquad(3.1)$$

Equations (2.10)–(2.12) for $\delta=2/3\gamma$ take the form

$$\begin{gathered}\eta(2\eta+3\gamma)(2\eta+3\gamma^2)[9\gamma(2-\gamma)\mu+2(3\gamma^2-12\gamma+8)\eta-3\gamma(7\gamma-6)]d\mu\\ =6\gamma\mu\{\mu[12(2-\gamma)\eta^2+12\gamma(4-3\gamma)\eta-9\gamma^3(3\gamma-2)]-\eta^2(2-\gamma)4(3\gamma^2-9\gamma+5)\\ -12\gamma(2\gamma^2-7\gamma+4)\eta+9\gamma^3(3\gamma-2)\}d\eta,\end{gathered}\qquad(3.2)$$

$$\begin{gathered}18\gamma^2[\mu-(\gamma-1)]zd\eta/dz=(2\eta+3\gamma)[9\gamma(2-\gamma)\mu+2(3\gamma^2-12\gamma+8)\eta\\ -3\gamma(7\gamma-6)],\end{gathered}\qquad(3.3)$$

$$\frac{8\pi k}{c^4}\mu T^2E(z)=\frac{2(\gamma-1)}{3\gamma^2\eta(2\eta+3\gamma^2)}[(2\eta+3\gamma)^2-9\gamma^2\mu].\qquad(3.4)$$

The analysis of the solutions reduces to study of the fields of the integral curves and singular points of Eq. (3.2), which are represented[4] for different intervals of γ in Figs. 1–4. The arrows indicate in accordance with (3.3) the direction of increasing $|z|$ along the integral curves.

For dust ($\gamma=1$), the solution with nonzero energy density e is represented in the (η,μ) plane by the straight line $2\eta+3\gamma=0$ (Fig. 1) and is expressed by Eqs. (2.14). After one integration, Eq. (2.14a) reduces to the form

$$(3\xi+2)^2(3\xi-1)=\text{const}\,X^{-3},\quad\xi=\frac{zd\ln X}{dz}=\frac{\beta}{2}-1.\qquad(3.5)$$

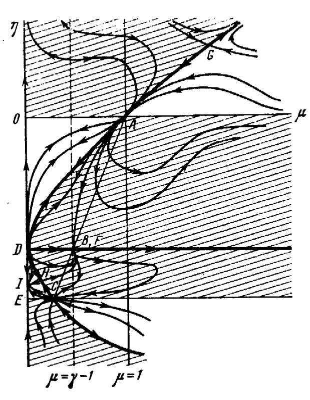

FIG. 2. Ultrarelativistic equation of state $\gamma=4/3$. The straight line AC (4.3) is shown.

For $X=Y$, the considered solution reduces to the flat Friedmann model. All the remaining integral curves in Fig. 1 correspond to solutions that have the form (2.15), for which $e=0$.

The "vacuum" parabola (2.13) corresponds to the solution

$$9\mu=(2\eta+3)^2,\quad z=z_0(1+\eta),\quad Y=\frac{z_0-z}{z_0z},\quad X=-\frac{2z+z_0}{3z_0z},\quad T=1. \tag{3.6}$$

For this solution, the coordinate transformation

$$\eta^0-\eta^1=x^0(x^1)^{1/3}+\frac{1}{2z_0}(x^1)^{4/3}+(x^1)^{2/3}Y[(x^2)^2+(x^3)^2],$$
$$\eta^0+\eta^1=(x^1)^{2/3}Y,\qquad \eta^{2,3}=(x^1)^{2/3}Yx^{2,3}$$

reduces (2.1) to the Galilean form of Minkowski space.

For $\gamma>1$, the energy density is in accordance with (3.4) nonzero for all solutions except the vacuum parabola (2.13),

$$9\gamma^2\mu=(2\eta+3\gamma)^2=4(\eta-\eta_D)^2,\qquad \eta_D=-3\gamma/2, \tag{3.7}$$

which corresponds to the solution $(1<\gamma<2)$

$$z=z_0(\eta-\eta_D)^{\alpha_1}(\eta-\eta_H)^{\alpha_2}(\eta-\eta_G)^{\alpha_3},\qquad Y=\eta^{\beta_0}(\eta-\eta_D)^{\beta_1}(\eta-\eta_H)^{\beta_2}(\eta-\eta_G)^{\beta_3},$$

$$\alpha_1=\beta_1\eta_D=-\frac{9\gamma^3(\gamma-1)}{4(2-\gamma)(\eta_D-\eta_G)(\eta_D-\eta_H)}>0,$$
$$\alpha_2=\beta_2\eta_H=\frac{9\gamma^3[\mu_H-(\gamma-1)]}{4(2-\gamma)(\eta_H-\eta_D)(\eta_H-\eta_G)}<0, \tag{3.7a}$$
$$\alpha_3=\beta_3\eta_G=\frac{9\gamma^3[\mu_G-(\gamma-1)]}{4(2-\gamma)(\eta_G-\eta_D)(\eta_G-\eta_H)}>0,\quad \beta_0=-(\beta_1+\beta_2+\beta_3)=\frac{2-\gamma}{6-5\gamma}.$$

There are also the solutions

$$\eta=\eta_G,\quad \mu=\mu_G;\quad \eta=\eta_H,\quad \mu=\mu_H. \tag{3.7b}$$

The coordinates of the points G and H are determined by the equation $(\eta_G>\eta_H)$

$$2(2-\gamma)\eta^2+\gamma(3\gamma^2-18\gamma+20)\eta+3\gamma^2(6-5\gamma)=0,\qquad \mu=(2\eta+3\gamma)^2/9\gamma^2. \tag{3.8}$$

For (3.7a) in the limit $\eta\to\eta_D$ we have $z\to 0$; for $\eta\to\eta_H$ we have $z\to\infty$; and for $\eta\to\eta_G$ we have $z\to 0$. A continuous vacuum solution for all values of z is realized for (3.7b) and for the piece (3.7) between the singular points D and H.

The region $e>0$, which is determined by (3.4), is hatched in Figs. 2–4. In the s region, the direction

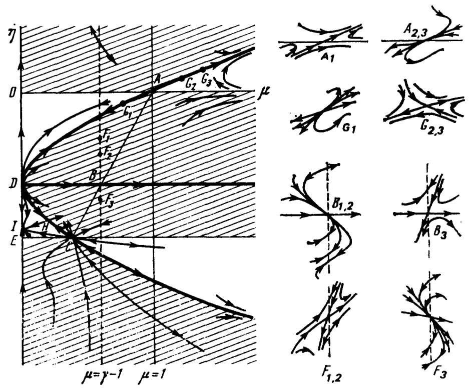

FIG. 4. Fields of the integral curves of Eq. (3.2) for different γ in the interval $1<\gamma<2$. The subscript 1 refers to the interval $1<\gamma<6/5$, the subscript 2 to the interval $6/5<\gamma<4/3$, and the subscript 3 to the interval $4/3<\gamma<2$. We have $\eta_B=\eta_D=-3\gamma/2$, $\eta_C=\eta_E=-3\gamma^2/2$, and $\eta_I=3\gamma(7\gamma-6)/2(3\gamma^2-12\gamma+8)$. At the points $\eta=\infty$ and $\mu=\mu_0$ we have $d\mu/d\eta=0$ and $z=z_1$.

of increasing z is reversed on going through the straight line $\mu=\gamma-1$ (except for the singular points B and F). In accordance with the definition[3] of the velocity of sound ω, we have $(\omega/c)^2=dp/de=\gamma-1$. In the s system (2.5b) for $0<\mu<\gamma-1$ we have in accordance with (2.5c) the relation $v<\omega$ (subsonic regime); for $\gamma-1<\mu<1$ we have $w<v<c$ (supersonic regime), and for $\mu=\gamma-1$ $(z=z_0)$ we have $v=\omega$.[5,8] Becuase of this nature of the variation of z, a continuous solution in the system (2.1) in the s region is two-valued (either subsonic or supersonic in the s system) and defined either for $z>z_0$ or for $z<z_0$; the construction of a solution for all values of z requires the introduction of discontinuous solutions.[5]

In accordance with (3.3) and (3.2), we have exact solutions corresponding to the singular points B and F on $\mu=\gamma-1$; using (2.9) and (2.3), we can write them in the form $(\gamma<2)$

$$\eta=\eta_B=-3\gamma/2,\quad \mu=\gamma-1,\quad T=z^{\alpha_0},\quad X=z^{\alpha_1},\quad Y=z^{\alpha_2}, \tag{3.9}$$
$$\alpha_0=-(3\gamma-2)(\gamma-1)/3\gamma(2-\gamma),\quad \alpha_1=-(2+\gamma)/3\gamma(2-\gamma),\quad \alpha_2=-2/3\gamma.$$

$$\eta=\eta_F=\frac{3\gamma^2(3\gamma-2)}{2(3\gamma^2-12\gamma+8)},\quad \mu=\gamma-1,\quad T=z^{\beta_0},\quad X=z^{\beta_1},\quad Y=z^{\beta_2}, \tag{3.10}$$
$$\beta_0=1+\beta_1,\quad \beta_1=(21\gamma^3-84\gamma^2+92\gamma-32)/3\gamma^2(2-\gamma)(3\gamma-2),\quad \beta_2=1/\eta_F.$$

For the solutions (3.9) and (3.10) the matter velocity of each value of z in the s system (2.5b) is in accordance with (2.5c) equal to the velocity of sound ω, and the lines $z=\text{const}$ are characteristics (as in the case of a centered simple wave in Newtonian gas dynamics without gravitation[3]).

When the evolution in time is studied as x^0 varies from 0 to ∞ at a fixed point x^1 the variable z changes from ∞ to 0. When the behavior with respect to x^1 is studied with x^1 varied from $-\infty$ to $+\infty$ at a fixed time x^0 the variable z changes from $-\infty$ to $+\infty$. At the point $x^0=0$, $x^1=0$ $(x^1=zx^0\to 0)$ the solution becomes multivalued in general; at the same time, $e=\infty$ in accordance with (2.2).

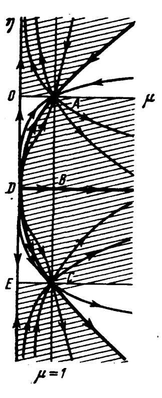

FIG. 3. $\gamma=2$. At $\mu=\infty$ and $\eta=\eta_0$ we have $z=\infty$ for $\eta_0<0$ and $z=0$ for $\eta_0>0$.

In general, the solutions (3.3) are nonsymmetric functions of z; at the same time, in accordance with (2.11), to each solution $\eta=\eta_1(z)$ there corresponds the solution $\eta=\eta_2=\eta_1(-z)$. In accordance with Fig. 4, the analytic solutions $\mu(\eta)$ in the range of z from 0 to ∞ correspond to the Friedmann solution (3.1) and to the curves from $D(z=0)$ to $I(|z|\to\infty)$ which lie entirely in the region $e<0$. For $1<\gamma<6/5$, the analytic solutions also correspond to curves drawn from $G(z=0)$ to the singular point $\eta=0$, $1/\mu=0$ (singular state) with asymptotic behavior ($\gamma<2$)

$$\begin{gathered}\mu\approx\text{const}\,\eta^{\beta},\quad \beta=-\frac{2(3\gamma-2)}{3(2-\gamma)},\quad \eta\approx\frac{3(2-\gamma)}{2}\frac{z-z_0}{z_0},\quad z\to z_0,\\ T\sim(z-z_0)^{\alpha_0},\quad X\sim(z-z_0)^{\alpha_1},\quad Y\sim(z-z_0)^{\alpha_2},\quad E\sim(z-z_0)^{\alpha},\\ \alpha_0=\frac{\gamma-1}{2-\gamma},\quad \alpha_2=-2\alpha_1=\frac{2}{3(2-\gamma)},\quad \alpha=-\frac{\gamma}{2-\gamma}\end{gathered}\tag{3.11}$$

and then in the region $e<0$ to the singular point $1/\eta=0$, $1/\mu=0$ with the asymptotic behavior ($\gamma<2$)

$$\begin{gathered}\mu\approx\text{const}\,\eta^{2},\quad \eta\sim z^{(2-\gamma)/\gamma},\quad z\to\infty,\\ Y\to Y_0,\quad X\sim z^{1/\gamma}\to\infty,\quad T\sim z^{(2\gamma-1)/\gamma}\to\infty,\quad E\sim z^{-(2\gamma-1)/(\gamma-1)}\to 0;\\ \mu\approx\frac{2(3\gamma^2-12\gamma+8)}{9\gamma(2-\gamma)}\eta,\quad \eta\sim z^{2(2-\gamma)/\gamma}\ (\eta<0).\end{gathered}\tag{3.12}$$

For $6/5<\gamma<2$, the analytic solutions correspond to curves from the singular point $A(\eta=0,\ \mu=1;\ z=z_0)$ with asymptotic behavior

$$\begin{gathered}\mu\approx 1+\frac{4}{3\gamma}\eta+\text{const}\,\eta^{-(3\gamma-2)/(6-5\gamma)},\quad z\approx z_0\left(1+\frac{2-\gamma}{6-5\gamma}\eta\right)\to z_0,\\ Y\sim(z-z_0)^{(2-\gamma)/(6-5\gamma)}\to\infty,\quad T\sim X\sim Y^{2(\gamma-1)/(2-\gamma)},\quad E\sim T^{-\gamma/(\gamma-1)}\to 0\end{gathered}\tag{3.13}$$

in the region $e>0$ to $\eta\to 0$, $\mu\to\infty$ (singular state, $z=z_1$) with (3.11) and then in the region $e<0$ again returning to the point $A(z=z_2)$ with subsequent repetition.[6] In accordance with (2.11), these last solutions are periodic in ln z.

§4. SOLUTIONS WITH WEAK DISCONTINUITIES AND SHOCK WAVES

Solutions containing discontinuty surfaces are of interest.[7] Piecewise analytic solutions can be constructed by joining solutions at the singular point $D(z=0)$ with the asymptotic behavior

$$\begin{gathered}\gamma\neq 1,\quad \mu\approx k_1(\eta-\eta_D)^2,\quad k_1=\text{const},\quad \eta-\eta_D\approx(z/z_0)^{(3\gamma-2)/3\gamma},\\ T=1,\quad X=Y\approx z^{-2/3\gamma},\quad E(z)=(9\gamma^2k_1-4)c^4/54\pi k k_1\gamma^4,\end{gathered}$$

at the singular point $B(z=z_1)$ with the asymptotic behavior

$$\eta_B=-\frac{3\gamma}{2},\quad \mu_B=\gamma-1,$$

$$1<\gamma<\frac{4}{3},\quad \mu-\mu_B\approx\frac{2(2-\gamma)(3\gamma-4)}{9\gamma(\gamma-1)}(\eta-\eta_B)+\text{const}(\eta-\eta_B)^{(3\gamma-2)/(4-3\gamma)},$$

$$\eta-\eta_B\approx\frac{3(\gamma-1)}{2-\gamma}\frac{z_0-z}{z_0},\quad T\approx T_0,\quad X\approx X_0,\quad Y\approx Y_0,\quad E\approx\text{const};$$

$$\gamma=\frac{4}{3},\quad \mu-\mu_B\approx\exp\left(\frac{1}{\eta-\eta_B}\right)\left[\text{const}+\frac{2}{3}\int^{\eta-\eta_B}\exp\left(-\frac{1}{\xi}\right)d\xi\right]$$

and at the singular point $G(z=0)$ for $1<\gamma<6/5$ and the point $A(z=z_2)$ [which lie on (3.7)] with the asymptotic behavior (3.13) for $6/5<\gamma<2$. At the point of joining, there is a weak discontinuity, through which the g_{ik}, their first derivatives, the components of the 4-velocity, the internal energy e, and the components of the Ricci tensor pass without breaks.

For $1<\gamma<6/5$, one can construct a solution (Fig. 4) which has one of its pieces formed by (3.7) from G to $\mu\to\infty$ and corresponds to Minkowski space (for example, for $z\leqslant 0$). The other piece (for $z>0$) is formed by the integral curves which begin at G and first pass in the region $e>0$ to $\eta\to 0$, $\mu\to\infty$ (3.11) (singular state at $z=z_1$) and then in the region $e<0$ to the point $\eta\to\infty$, $\mu\to\infty$ (3.12) ($|z|\to\infty$). This solution can be interpreted as flow from the region $z>0$ into vacuum (Minkowski space), the velocity of the flow for $z=0$ ($e=0$) in the s system being determined in accordance with (2.5c) by the relation

$$v^2=c^2\mu_G=c^2(2\eta_G+3\gamma)^2/9\gamma^2$$

with the expression (3.8). For $6/5<\gamma<2$, one can construct a solution with one of the pieces formed by (3.7) from G to $A(z=z_1)$. The second piece for $|z|>|z_1|$ is formed by the integral curves which begin at A and pass in the region $e>0$ to $\eta\to 0$, $\mu\to\infty$ and then in the region $e<0$ either to $\mu\to\infty$, $\eta\to\infty$ or to $G(|z|\to\infty)$, or to the point A with finite z and with subsequent repetition.

Of particular interest are solutions for which one of the pieces is the Friedmann solution (3.1) for $|z|\geqslant|z_1|$ (or for $|z|\leqslant|z_1|$). The second piece for $|z|\leqslant|z_1|$ (or $|z|\geqslant|z_1|$) in the piecewise analytic solution can be the solution (3.9) or, for $1<\gamma<4/3$, the solution corresponding to the integral curve from B to D (Figs. 4 and 2). In this case, a weak discontinuity propagates through the Friedmann solution with the velocity of sound in accordance with the law $x^1=z_1x^0$. Taking (3.1) for $z<z_1$ and $z>z_2$ and the solution (3.9) for $z_1<z<z_2$, we obtain a nonlinear packet of finite width propagating through the Friedmann solution.

A shock wave on which the hydrodynamic variables have a discontinutiy is described by the singular hypersurface $z=z^*=\text{const}$ (with unit normal n^i), which in the s system (2.5b) is determined by the condition $y^1=y^{1*}$ (with $n^i=n^1\delta^i_1$).[8] The Rankine–Hugoniot conditions for the jump of the hydrodynamic quantities across the shock front (the states in front of and behind the shock are identified by the indices 1 and 2)[3]

$$(T^{ik}n_k)_1=(T^{ik}n_k)_2$$

can be written in the s system (2.5b) in the form

$$(eu^0u_1)_1=(eu^0u_1)_2,\quad [\gamma eu^1u_1-(\gamma-1)e]_1=[\gamma eu^1u_1-(\gamma-1)e]_2.\tag{4.1}$$

In the s system, the components of the metric tensor and its first derivatives are continuous across the shock front, while the components of the Ricci tensor have a discontinuity, except for the ones that remain continuous by virtue of (4.1).[9]

By means of Eqs. (2.2), (2.12), and (2.5c), (2.5a), and (2.5b), the relations (4.1) can be transformed to

$$\mu_1\mu_2=(\gamma-1)^2,\tag{4.2a}$$

$$\frac{\gamma-1+\mu_1}{(2\eta_1+3\gamma^2)\mu_1\eta_1}[(3\gamma+2\eta_1)^2-9\gamma^2\mu_1]=\frac{\gamma-1+\mu_2}{(2\eta_2+3\gamma^2)\mu_2\eta_2}[(3\gamma+2\eta_2)^2-9\gamma^2\mu_2].\tag{4.2b}$$

Equation (4.2a) is obtained by dividing the expressions (4.1) and is the analog of the Prandtl conditions,[3] and (4.2b) is the second of the relations (4.1). In the plane (μ, η), the discontinuity across the shock wave corresponds to the jump from the point μ_1, η_1 with $\gamma-1 \leqslant \mu_1 \leqslant 1$ (supersonic state) to the point μ_2, η_2 with $(\gamma-1)^2 \leqslant \mu_2 \leqslant \gamma-1$ (subsonic state, $e_2 > e_1$).

If the shock wave propagates through the Friedmann solution (3.1) $(2\eta_1 + 3\gamma = 0)$, then in accordance with (4.2) the points of the state 2 must lie on the straight line

$$2(2-\gamma)\eta = 3\gamma(\mu-1), \quad \mu_1\mu_2 = (\gamma-1)^2. \tag{4.3}$$

At the same time, they are situated on the interval of (4.3) from the point $B(\mu_1 = \mu_2 = \gamma - 1)$ to the point $C(\mu_1 = 1, \mu_2 = (\gamma-1)^2)$. If the Friedmann solution is realized behind a shock front (for example, behind the front of a converging shock wave), we have $2\eta_2 + 3\gamma = 0$, and the points of the state 1 lie on the straight line (4.3) between the point $B(\mu_1 = \mu_2 = \gamma - 1)$ and the point $A(\mu_1 = 1, \mu_2 = (\gamma-1)^2)$. The straight line (4.3) is shown in Figs. 2 and 4.

In particular, the solution to the problem of a shock wave on the plane (μ, η) propagating in both directions through the Friedmann solution (3.1) is described for $|z| \geqslant |z^*|$ by the section of the straight line $2\eta_1 + 3\gamma = 0$ from $\mu = \infty$ to $\mu = \mu_1$ and for $|z| \leqslant |z^*|$ by an integral curve which starts on the straight line (4.3) (the state μ_2, η_2) and passes to the point D or to the point H (a wave of maximum intensity).

It is of particular interest to join through a shock wave the Friedmann solution and the Minkowski space-time (3.7) for waves of limiting intensity $(n^i n_i = 0)$ with the jump from A on the straight line (3.1) ($e_1 = 0$, Minkowski space in front of the wave).[10)] Analysis of the possibility of continuing the solution along the vacuum parabola (Figs. 2 and 4) shows that one can have a solution with a shock wave converging through Minkowski space [the section of (3.7) from G to A, $|z| \leqslant |z^*|$], the Friedmann space-time being behind the shock front (the state 2). For $1 < \gamma \leqslant 6/5$, one can also have a shock wave diverging through Minkowski space [the section of (3.7) from $\mu \to \infty$ to A, $|z| \geqslant |z^*|$], with the Friedmann solution realized behind the shock front.

For dust $(\gamma = 1)$ it follows in accordance with (4.1) that it is possible to have only a contact discontinuity with $\mu_1 = \mu_2 = 0$ and with arbitrary jump $e_2 - e_1$. Discontinuous solutions are formed by joining through the contact discontinuity at the singular point D (Fig. 1) either two $2\eta + 3 = 0$ solutions of the type (2.14) $(\gamma = 1)$ and (3.5) with matter $(e_{1,2} \neq 0)$ or a $2\eta + 3 = 0$ solution with matter $(e \neq 0)$ to one of the vacuum solutions $(e = 0)$ having at the pont D the asymptotic behavior

$$\mu \approx k_2(2\eta+3)^2, \quad k_2 = \text{const}, \quad z \approx z_0 \exp[-9k_2(2\eta+3)] \to z_0,$$
$$15\mu \approx 2\eta+3, \quad z \approx z_0(2\eta+3)^{-1/2} \to \infty,$$

in particular, to the solution (3.6) corresponding to Minkowski space.

For $e = p$ $(\gamma = 2)$ (Fig. 3), analytic solutions do not exist for $e > 0$. For $\gamma = 2$, piecewise analytic solutions are constructed by joining solutions at the singular points C and A. In accordance with (4.2), we have $\mu_1 = \mu_2 = 1$ across a shock wave for $\gamma = 2$ (Ref. 18); in this case, the 4-normal is isotropic. It is possible to join the Friedmann solution (3.1) to other solutions through a shock wave with jump from A to B and with jump from B to C (or vice versa) with fulfillment of the condition $e_2 > e_1$.

1)The only Friedmann solutions with self-similar structure[16] are the flat Friedmann model[2] with the metric (1.1), (1.4) with $f = 1$, $\delta = 2/3\gamma$ and (1.1), (1.5) with $f = \sin x^2$, as well as the open model in the vacuum case (Galilean space-time) with $f = 1$ (Refs. 1 and 4), which corresponds to (1.4) with $\delta = 0$ (Ref. 8) and is an asymptotic state for the open Friedmann model.

2)The Cauchy hypersurface $z = z_0$ is spacelike in the t region and timeline in the s region.

3)The solution for $\delta = 0$ given earlier in Ref. 8 corresponds to $d_2 - 1 \approx \delta/\zeta_0 \to 0$.

4)We note, as in Ref. 4, the distinguished values $\gamma = 6/5$ and $\gamma = 4/3$.

5)For the analogous situation in the anisotropic cosmological Bianchi model type V, however, the introduction of a discontinuity surface does not make it possible in the framework of this model to construct a solution in the X region for all values of the independent variable.[4]

6)Similar periodic solutions are also possible in the axisymmetric Bianchi model type V.[4]

7)The solutions for $e = 3p$ and $\gamma = 4/3$ are analyzed in Ref. 10.

8)The equation $|z| = z^*$ determines for $t < 0$ waves which converge to $x^1 = 0$ and for $t > 0$ waves which diverge from $x^1 = 0$.

9)In accordance with (2.5b), we have on the shock front (for $z = z^*$) $Y_1 = Y_2$, $T_1 > T_2$, $X_1 > X_2$ $(\mu_1 < 1)$.

10)For $\mu_1 = 1$, one cannot use the s system (2.5b) to consider the conditions across the discontinuity; instead, one must use a certain 1 system with metric of the type (2.1), in which the matter moves (and $v_1 = c$) and the components of the metric are finite, nonvanishing, and continuous across the discontinuity at $z = z_1$.

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