Multielectron dimple on the surface of liquid helium

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(Submitted 24 March 1981)
Zh. Eksp. Teor. Fiz. **81**, 951–966 (September 1981)

In a sufficiently strong electric field that clamps the electrons to a helium surface, a situation is possible wherein the entire charge is gathered in a bounded region called a multielectron dimple. Such dimples are stable to thermal fluctuations if the number of electrons is $N \gg 10$, and they can be described by a linear equation if $N \ll 10^7$. Under the indicated restrictions on N, the dimple is calculated by a variational method in a macroscopic approximation. Field dependences are obtained for the dimple energy, the charged-region radius, the charge-distribution density, the hydrodynamic mass, and the mobility. Light refraction by the surface of the dimple is examined in detail. It is proposed to use optical observations to determine the dimple charge. To this end, the field dependences of the dimensionless focal lengths are calculated for the refraction of light by the center of the dimple (focusing) and by the boundary of the charged region (defocusing).

PACS numbers: 67.90. + z, 73.25. + i, 78.20. − e

The instability of a charged surface of liquid helium was predicted theoretically by Gor'kov and Chernikova.[1] They have shown that in a sufficiently strong clamping field the surface should acquire a macroscopic periodic structure, with energy considerations favoring a structure of hexagonal symmetry. Leiderer and Wanner[2] have recently succeeded in observing experimentally this phenomenon by means of light reflection from the restructured helium surface. The hexagonal symmetry of the produced structure was confirmed, and it was also shown that under certain conditions there can exist limited regions with hexagonal short-range order up to individual dimples into which all the electrons from the surface are gathered.

In our preceding paper[3] we discussed the possible mechanisms of nonlinearity of a charged surface. It was shown that the main nonlinearity is due to the onset of surface sections free of electric charge. As indicated in Ref. 3, the period of the produced structure should first vary with increasing field, so as to ensure a minimum energy. At a certain field value, the charge-free regions come in contact, so that the distribution takes the form of the periodically arranged isolated sections. The charge of each of them will hereafter be conserved, and since the average charge density is maintained constant, the lattice period should also remain constant. With further increase of the field, the sizes of the charged sections decrease, so that in the limit we obtain a lattice of weakly interacting dimples.

An investigation of the equilibrium shape of a charged surface encounters in the general case great difficulties. It is useful to solve first the problem of a single multielectron dimple, since such dimples are the simplest experimentally observable objects whose existence and properties are due to the same interactions as the instability of a uniformly charged surface. We present accordingly in Sec. 1 a qualitative description of the investigated phenomenon, obtain quantitative estimates of the quantities encountered in the problem, and derive equations but determine the equilibrium shape of the surface. These equations are analyzed in Sec. 2 as applied to the problem of single dimple, so that a numerical solution of the problem can be obtained using optimal variational functions. In Sec. 3 are summarized the results of a numerical calculation of the energy of the dimple, of the radius of the charged region, the charge distribution, and the hydrodynamic mass and mobility of the dimple. Section 4 deals with the refraction of light by the dimple, an experimental investigation of which can contribute, in particular, to the determination of the dimple charge.

1. QUALITATIVE DISCUSSION, DERIVATION OF EQUATIONS

The free surface of helium tends to assume a horizontal position under the influence of the surface tension and of the force of gravity. This interaction is quantitatively characterized by a surface-tension coefficient α and by the product ρg, where ρ is the density of the helium and g is the free-fall acceleration. The spatial scale of the deformation of the surface under the influence of a concentrated external force is determined by the capillary radius $(\alpha/\rho g)^{1/2} \approx 0.05$ cm, if it is assumed that $\alpha \approx 0.35$ dyn/cm and $\rho \approx 0.145$ g/cm^3 for He^4. In experiments with a charged surface there are two controllable parameters—the surface density of the charge $n(\mathbf{r})$ ($\mathbf{r}$ is the coordinate on the unperturbed surface) and the external electric field E. These quantities have the same dimensionality, and their characteristic scale is $(\alpha\rho g)^{1/4} \approx 800$ V/cm. Dividing this result by the electron charge, we obtain for the characteristic electron density a value 5×10^9 cm^{-2}. The scale of the charge in a square with a side equal to the capillary radius is $\alpha^{5/4}(\rho g)^{-3/4}$ and corresponds to the charge of 1.3×10^7 electrons. It follows from this that the discrete character of the electric charge can be neglected, inasmuch as the calculations involve only the surface density of the charge.

In the case of a charged surface it is necessary to take into account, besides the surface tension and the force of gravity, also the interaction of the charge with

 0038-5646/81/090505-08$02.40

the field E and the Coulomb interaction between the charges. If we measure the length in units of $(\alpha/\rho g)^{1/2}$, the field E and the charge density $n(\mathbf{r})$ in units of $(\alpha\rho g)^{1/4}$, and the energy $\mathscr{E}$ in units of $\alpha^2/\rho g$, then the total energy of the charge surface can be written in the form

$$\mathscr{E}=\int[1+(\nabla\xi)^2]^{1/2}d^2r+\frac{1}{2}\int\xi^2d^2r-E\int n\xi d^2r + \frac{1}{2}\int\frac{n(\mathbf{r})n(\mathbf{r}')}{|\mathbf{l}-\mathbf{l}'|}d^2rd^2r', \quad (1)$$

where $\xi(\mathbf{r})$ is the deviation of the liquid level from horizontal and $\mathbf{l}$ stands for the three-dimensional vector $(\mathbf{r}, \xi)$.

Expression (1) contains terms that are quadratic in ξ and n (the energy in the gravitational field and in the field E), as well as more complicated terms (the surface and Coulomb energies). To determine the shape of the surface $\xi(\mathbf{r})$ and the charge distribution $n(\mathbf{r})$ it is necessary to minimize Eq. (1) with respect to these functions, a practically impossible task in the general case. We therefore turn to a situation wherein the dimensionless charge density is small, $n(\mathbf{r})\ll 1$, so that in a certain range of fields the specific force nE acting on a unit surface is also small. In this case the surface displacement is $\xi\sim En\ll 1$, so that expression (1) can be simplified by retaining only the surface-energy contribution $\propto(\nabla\xi)^2$ and by putting $\mathbf{l}=\mathbf{r}$ in the denominator of the last term. As will be shown below, at $E\gtrsim 1$ the important values are $r\sim E^{-2}$. The inequality $\xi\ll r$ (or, equivalently, $|\nabla\xi|\ll 1$) is thus equivalent to the inequality $nE^3\ll 1$. This means in turn that at $n\ll 1$ there exists a field region $1\lesssim E\ll n^{-1/3}$ in which the energy of the charged surface is given by an energy quadratic in ξ and n, namely

$$\mathscr{E}=\frac{1}{2}\int[(\nabla\xi)^2+\xi^2]d^2r-E\int n\xi d^2r+\frac{1}{2}\int\frac{n(\mathbf{r})n(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^2rd^2r'. \quad (2)$$

We now assess the possibility of neglecting the temperature effects. This calls for the temperature to be less than all the energies encountered in the problem. We have taken the energy unit to be the quantity $\alpha^2/\rho g \approx 10^9$ eV, which exceeds with a tremendous margin all the conceivable temperatures. The energy $eE\xi$, which characterizes the interaction of an individual electron with an external field E, is much lower. This energy is measured in units of $e\alpha^{3/4}(\rho g)^{-1/4}\approx 40$ eV $\approx 4.6\times 10^5$ K. In the case of a dimple with dimensionless charge $Q\ll 1$ we have $\xi\sim EQ$, so that at $E\sim 1$ the energy $eE\xi$ is of the order of 1 K at $Q\approx 10^{-6}$. In terms of the number N of the electrons in the dimple this means that expression (2) is valid at $10\ll N\ll 10^7$, where the lower limit signifies neglect of the temperature effects, and the upper means going from expression (1) to the quadratic expression (2).

We wish to solve the problem of the equilibrium shape of the surface, i.e., of the minimum of the functional (2), at a given surface charge Q. This condition can be taken into account if we add to it, before we take the variation of (2), the term

$$-\lambda\int n(\mathbf{r})d^2r,$$

where the Lagrange multiplier λ must be obtained after solving the equations for $\xi(\mathbf{r})$ and $n(\mathbf{r})$ from the condition

$$\int n(\mathbf{r})d^2r=Q.$$

As a result of the variation we obtain the system of equations

$$-\Delta\xi(\mathbf{r})+\xi(\mathbf{r})=En(\mathbf{r}), \quad (3)$$

$$-E\xi(\mathbf{r})+\int\frac{n(\mathbf{r})}{|\mathbf{r}-\mathbf{r}'|}d^2r'=\lambda, \quad n(\mathbf{r})>0. \quad (4)$$

It is seen that the Lagrange multiplier λ coincides with the total electrostatic potential in the regions occupied by the electrons. Equation (4) was obtained by variation with respect to $n(\mathbf{r})$. This method of deriving the equations implies tacitly that the sign of the variation is arbitrary. If we recall that $n(\mathbf{r})\geq 0$, then we must confine the limit of applicability of Eq. (4) to those values of $\mathbf{r}$ where $n(\mathbf{r})>0$.

Equations (3) and (4) are suitable for the description of the most general case of charge distribution on a surface, provided separate Lagrange multipliers are introduced for each of the isolated charged regions, and that these multipliers are determined after solving the problem for specified charges in each region. In the simplest case there is only one such region, which we shall call a dimple.

We now present a qualitative description of the investigated phenomenon. To this end, it is convenient to exclude from the functional (2) the surface deformation $\xi(\mathbf{r})$ and to rewrite the functional only in terms of the charge density $n(\mathbf{r})$. This is a trivial task if it is recognized that the solution of the homogeneous equation (3) is given by a Bessel function $K_0(r)$, and leads to the result

$$\mathscr{E}=\frac{1}{2}\int n(\mathbf{r})V(\mathbf{r}-\mathbf{r}')n(\mathbf{r}')d^2rd^2r', \quad V(r)=\frac{1}{r}-\frac{E^2}{2\pi}K_0(r).$$

It is seen that because of the interaction with the clamping field E there is produced between the point charges on the helium surface, in addition to the usual Coulomb repulsion, also an attraction force determined by the potential $E^2K_0(r)/2\pi$. This potential decreases exponentially at $r\gg 1$ and increases like $\ln(1/r)$ as $r\to 0$. In both cases the attraction cannot compete with the Coulomb repulsion. It is clear therefore that with increasing field E the potential $V(r)$ must have a minimum at some finite r. Calculation shows that the minimum appears at $E_m\approx 3.159$ and corresponds to $r_m\approx 1.332$. We indicate for comparison that the instability of a uniformly charged surface sets in at $E_c=(4\pi)^{1/2}\approx 3.545$. At the instant of appearance of the minimum $V(r_m)\approx 0.3274$, so that the bound-state energy exceeds the energy of the charges removed to infinity. With increasing field, the minimum of $V(r)$ becomes deeper and goes through zero at $E\approx 3.670$ and $r_0\approx 0.5951$. In fields $E\gg 1$ the minimum of the potential is reached at $r=2\pi E^{-2}$ and is equal to $-E^2\ln(E/\pi)$.

Thus, in a sufficiently strong field E, two point charges on the surface of the helium are capable of forming a bound state with a radius of the order of the capillary constant. The main properties of these states,

namely their existence in a field region bounded from below, the positive energy in sufficiently weak fields, and the independence of the characteristic size on the value of the charge, will be shown below to remain in force also in the case of a dimple with a continuous charge distribution.

2. VARIATIONAL SOLUTION OF THE PROBLEM OF THE SINGLE DIMPLE

It is clear from general considerations that a single dimple should have circular symmetry, so that the charged region is a circle with a certain radius R. Outside the circle we have $n(\mathbf{r})=0$, and the solution of Eq. (3) is given by the Bessel function $K_0(r)$:

$$\xi(\mathbf{r})=AK_0(r), \quad r>R,$$

where the coefficient $A\sim QE$ is determined by the solution of the problem at $r<R$.

An analytic solution of Eqs. (3) and (4) cannot be obtained inside the circle. We have therefore carried out the variational calculation of $\xi(\mathbf{r})$ and $n(\mathbf{r})$ using the following trial functions:

$$\xi(\mathbf{r})=\sum_{s=0}^{m}\Xi_s\left(\frac{r}{R}\right)^{2s}, \quad r<R, \tag{5}$$

$$n(\mathbf{r})=\sum_{s=0}^{m}N_s n_s(r,R), \tag{6}$$

$$n_s(r,R)=\frac{2s+1}{2\pi R^2}\left(1-\frac{r^2}{R^2}\right)^{s-1/2}, \quad r<R, \quad n_s(r,R)=0, \quad r>R. \tag{7}$$

The expansion (6) was used because the electric potential for a charge density of the form $n_s(r,R)$ takes at $r<R$ the form of a polynomial of r^2/R^2 of degree s, so that Eq. (4) for any finite m can be solved exactly by expressing N_s in terms of Ξ_s. This connection between the charge density and the potential is given by the relation

$$\int\frac{n_s(r',R)}{|\mathbf{r}-\mathbf{r}'|}d^2r'=\frac{\Gamma(1/2)\Gamma(s+3/2)}{R\Gamma(s+1)}F\left(\frac{1}{2};-s;1;\frac{r^2}{R^2}\right)=\frac{1}{R}\sum_{k=0}^{s}C_{ks}\left(\frac{r}{R}\right)^{2k},$$

$$C_{ks}=(-1)^k\frac{\Gamma(k+1/2)\Gamma(s+3/2)}{\Gamma^2(k+1)\Gamma(s-k+1)}, \tag{8}$$

where F is a hypergeometric function and Γ is a gamma function. To derive (8) we must change over to a Fourier representation with allowance for the fact that

$$\int\frac{e^{i\mathbf{kr}}}{r}d^2r=\frac{2\pi}{k} \tag{9}$$

$$n_s(k,R)=\int e^{i\mathbf{kr}}n_s(r,R)\,d^2r=2^{s+1/2}\Gamma(s+3/2)J_{s+1/2}(kR)\,(kR)^{-s-1/2},$$

where $J_{s+1/2}(r)$ is a Bessel function, and use next the equation (6.574) of the handbook by Gradshteĭn and Ryzhik.[4] After substituting (5) and (6), using (8), and separating the powers of r^2, Eq. (4) takes the form

$$\frac{1}{R}\sum_{s=0}^{m}C_{ks}N_s=E\Xi_k+\lambda\delta_{k0}. \tag{10}$$

Inversion of the matrix $\hat{C}$ allows us now to express N_s in terms of Ξ_k, and obtain after substituting (6) in (3) an expression that contains only $\xi(r)$ and Ξ_k. It is more convenient, however to return to the initial functional (2) and rewrite it in terms of the variables Ξ_k. Integrating by parts in the interval (R,∞) the term with $(\nabla\xi)^2$ in Eq. (2) and substituting

$$\xi(R)=\sum_{s=0}^{m}\Xi_s, \quad \xi'(R)=\frac{K_0'(R)}{K_0(R)}\sum_{s=0}^{m}\Xi_s, \tag{11}$$

we obtain

$$\frac{1}{2}\int[(\nabla\xi)^2+\xi^2]d^2r=-\pi\xi(R)\xi'(R)+\pi\int_R^\infty\xi(\xi-\Delta\xi)rdr+\pi\int_0^R(\xi'^2+\xi^2)rdr$$
$$=\frac{1}{2}\sum_{sk}\Xi_sA_{sk}(R)\Xi_k. \tag{12}$$

We have taken account here of the fact that the integral that contains the factor $(\xi-\Delta\xi)$ yields, according to Eq. (3) a zero contribution at $r>R$. The remaining two terms are written using the matrix $\hat{A}(R)$, defined by the relations

$$A_{00}=\pi R^2+2\pi RK_1(R)/K_0(R),$$
$$A_{sk}=\pi[4sk/(s+k)+R^2/(s+k+1)+2RK_1(R)/K_0(R)].$$

For the second term in (2), using (5) and (6), we obtain

$$-E\int n(\mathbf{r})\xi(\mathbf{r})d^2r=-E\sum_{k,s=0}^{m}\Xi_kB_{ks}N_s,$$

$$B_{ks}=\frac{\Gamma(k+1)\Gamma(s+3/2)}{\Gamma(k+s+3/2)}.$$

Excluding now N_s and using (10), we obtain for the variational functional

$$\mathscr{E}(\lambda)=1/2\Xi\hat{\mathscr{A}}\Xi-\lambda\Xi\Xi^{(0)}, \quad \Xi\equiv\{\Xi_k\},$$
$$\hat{\mathscr{A}}=\hat{A}(R)-E^2R\hat{B}\hat{C}^{-1}, \quad \Xi_k^{(0)}=ER(\hat{B}\hat{C}^{-1})_{k0}.$$

It remains to write down the conditions that connect $\xi(r)$ at $r<R$ [expression (5)] with $\xi(r)$ at $r>R$. For this purpose it must be recognized that the function $n(r)$ in the right-hand side of (3) vanishes in proportion to its square root as $r\to R$, so that the coefficient N_0 in the expansion (6) is equal to zero at the value of R that minimizes the energy (this statement will be proved below). As a consequence, the solution of Eq. (3) for $\xi(R)$ must be continuous at the point $r=R$ together with its first two derivatives.

The continuity of the function and of the first derivative were already used in the derivation of (11) and (12). The conditions for matching the first and second derivatives

$$\Xi\Xi^{(1)}=0, \quad \Xi_k^{(1)}=2k+RK_1(R)/K_0(R),$$
$$\Xi\Xi^{(2)}=0, \quad \Xi_k^{(2)}=2k(2k-1)-R^2-RK_1(R)/K_0(R), \tag{13}$$

can be taken into account by multiplying the scalar products in (13) by the Lagrange multipliers λ_1 and λ_2 respectively and adding to $\mathscr{E}(\lambda)$. As a result we obtain the functional

$$\mathscr{E}(\lambda,\lambda_1,\lambda_2)=1/2\Xi\hat{\mathscr{A}}\Xi-\lambda\Xi\Xi^{(0)}-\lambda_1\Xi\Xi^{(1)}-\lambda_2\Xi\Xi^{(2)}, \tag{14}$$

which must be minimized with respect to Ξ, while the Lagrange multipliers λ, λ_1, and λ_2 are determined by the conditions (13) and by the condition

$$\int n(\mathbf{r})d^2r=\sum_{s=0}^{m}N_s=Q, \tag{15}$$

which specifies the charge of the dimple.

The minimization of (14) with respect to Ξ and the determination of the Lagrange multipliers reduces to elementary algebraic operations, the result of which is conveniently written using the third-rank matrix D, defined by the relation

$$D_{ij}=\Xi^{(i)}\hat{\mathscr{A}}^{-1}\Xi^{(j)}, \quad i, j=0; 1; 2. \tag{16}$$

By using the matrix $\hat{D}^{-1}$ which is the inverse of (16) we obtain for the vector Ξ, which minimizes the functional (14) and satisfies the conditions (13) and (15)

$$\Xi=Q\sum_{i=0}^{2}(\hat{D}^{-1})_{i0}\hat{\mathscr{A}}^{-1}\Xi^{(i)}. \tag{17}$$

The foregoing operations reduce the calculation of $\mathscr{E}$ at a given radius R to matrix-vector algebra. The next step is to find the minimum of $\mathscr{E}(R)$ with respect to R. This condition, as will be presently shown, coincides with the condition $N_0(R)=0$, where N_0 is a coefficient in the expansion (6).

Limitation of the region in which the dimple charge can become redistributed corresponds to introduction of an infinitely high potential well at $r=R$. By analogy with the problem of the charge distribution on a metallic disk,[5] we can conclude that in the general case $n(r)$ has at $r=R$ a singularity of the type $(1-r^2/R^2)^{-1/2}$. This circumstance is explicity taken into account by including the functions $n_0(r, R)$ in the expansion (6). The charge distribution $n(r)$ obtained when solving Eq. (4) ensures constancy of the potential at $r<R$. At $r>R$ the potential depends on r, and the character of this dependence determines the change of the energy upon redistribution of the dimple charge over a circle with a radius exceeding R. A charge distribution of the form $n_0(r, R)$ produces at $r>R$ a potential

$$\int\frac{n_0(r', R)}{|\mathbf{r}-\mathbf{r}'|}d^2r'=\frac{1}{rR}F\left(\frac{1}{2};\frac{1}{2};\frac{3}{2};\frac{R^2}{r^2}\right)\approx\frac{\pi}{2R}-\frac{2^{1/2}}{R}\left(\frac{r}{R}-1\right)^{1/2}.$$

Thus, if $N_0>0$, then at $r>R$ the potential has a square-root decrease from its constant value inside the region $r<R$. This means that it is energywise profitable to transfer into the region $r>R$ a certain part of the charge from the region $r<R$, thereby increasing the radius R of the dimple. It is clear that R should increase until N_0 vanishes. With further increase of the radius, N_0 becomes negative and as a result of which $n(r)$ reverses sign at $r<R$. Since $n(r)$ is positive, it is permissible to consider only N_0 such that $n(r)\geq 0$, therefore the condition $N_0(R)=0$ determines the minimum of the energy $\mathscr{E}(R)$. As follows from the statements made above, this minimum is reached at the boundary of the permissible region of variation of the parameters. It can be shown nevertheless that at this point we have not only $d\mathscr{E}/dR=0$, but also $d^2\mathscr{E}/dR^2=0$, so that the $\mathscr{E}(R)$ curve has an inflexion with a horizontal tangent (with further increase of the radius, the energy would continue to decrease, but the condition $n(r)\geq 0$ would be violated).

The condition $N_0(R)=0$, expressed in terms of the vector Ξ_k, takes the form

$$\sum_{k=0}^{m}(\hat{C}^{-1})_{0k}(E\Xi_k+Q(\hat{D}^{-1})_{00}\delta_{k0})=0. \tag{18}$$

Summarizing the content of this section, we indicate that the solution of the variational problem was reduced by us to a determination of the vector Ξ from relation (17) followed by a solution of Eq. (18) that defines the radius R of the charged region.

The actual calculations were performed using six terms ($m=5$) in the expansions (5) and (6), so that all the calculated quantities are accurate to better than 1%. The values of the energy and of the field E_D below which the dimple does not exist are accurate to four significant figures already at $m=3$. We note that Eq. (4) is a Fredholm integral equation of the first kind for $n(\mathbf{r})$, whose solution is unstable to small changes of $\xi(\mathbf{r})$. As a consequence, the matrix $\hat{C}$ is not quite exact, so that its inversion leads to a rapid loss of accuracy with increasing m. The choice of $m=4-5$ is a compromise between the tendency to increase the accuracy by increasing m and the need to limit m so as not to lose calculation accuracy because of the poor accuracy of the matrices that arise in the problem.

3. PRINCIPAL CHARACTERISTICS OF THE DIMPLE

The approach described above makes it possible to calculate numerically all the dimple characteristics of interest to us, namely, the energy $\mathscr{E}$ of the dimple, the radius R of the charged region, the charge distribution $n(r)$, the effective mass M, and the mobility μ. The quantities $\mathscr{E}$, M and μ^{-1} are quadratic in the dimple charge Q, and $n(r)\propto Q$. Accordingly, the plots show the relative quantities

$$\tilde{\mathscr{E}}=\mathscr{E}Q^{-2}, \quad \tilde{M}=MQ^{-2}, \quad \tilde{\mu}^{-1}=\mu^{-1}Q^{-2}, \quad \tilde{n}(r)=n(r)Q^{-1},$$

which depend only on the field E.

Figure 1 shows the dependence of the relative energy $\tilde{\mathscr{E}}$ on the field E. Calculation shows that the dimple exists only at fields $E>E_D=3.697$, with $\mathscr{E}(E_D)=Q^2\cdot 0.1185$. We indicate by way of comparison that the instability of a uniformly charged surface sets in a field $E_c=(4\pi)^{1/2}\approx 3.545$. With increasing field, the dimple energy goes through zero at $E_0\approx 4.062$, and at $E\gg 1$ it increases like

$$\mathscr{E}=-0.159Q^2E^2\ln(0.224E).$$

The same figure shows the dependence of the radius R of the charged region on E. For a dimple with a small charge Q, the radius depends only on the field and has

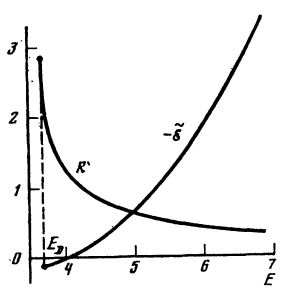

FIG. 1. Dependence of the relative dimple energy $\tilde{\mathscr{E}}$ and of the charged-region radius R on the field E.

at a field $E=E_D$ a maximum value $R_D\approx 2.83$. In strong fields we have

$$R(E)\approx 12.6E^{-2},\quad E\gg 1.$$

We discuss now the charge distribution $\tilde{n}(r)$. In a field E_D, the coefficient N_1 in the expansion (6) is small, and accordingly $n(r)$ decreases rapidly as $r\to R$, so that on the whole $n(r)$ turns out to be very similar to the function $\exp(-\alpha r^2)$. [The use of a Gaussian trial function for $n(r)$ was proposed earlier by Leiderer and Shikin.[6] We have used this approximation to calculate the field E_D and obtained $E_D\approx 3.711$. This differs by only 0.3% from our exact value 3.697.] At $E>E_D$ the function $n(r)$ has a square-root singularity as $r\to R$ and is qualitatively similar to the $\tilde{n}(r)$ curve presented earlier[3] for $E\to\infty$.

We consider now the effective mass of the dimple M and its mobility μ. Both quantities are of hydrodynamic origin: the mass of the dimple is determined by the kinetic energy of the liquid that flows around a dimple that moves with constant velocity V_0, and the mobility of the dimple is determined by the energy dissipation of the liquid on account of the viscosity. To calculate the indicated quantities it suffices to obtain the energy of the liquid and the dissipation in an approximation quadratic in the velocity V_0. This means that in the calculation of the velocity field one can neglect the change of the shape of the dimple, i.e., assume that the deformation of the surface as the dimple moves takes the form $\xi(\mathbf{r}-\mathbf{V}_0t)$, where $\xi(\mathbf{r})$ is the shape of the dimple at rest.

A surface deformation in the form of a traveling plane wave $\xi=\xi_0\exp[i\mathbf{k}\cdot(\mathbf{r}-\mathbf{V}_0t)]$ of small amplitude ($\xi_0 k\ll 1$), the velocity component along the z axis perpendicular to the surface is

$$V_z=-i\xi_0(\mathbf{kV}_0)\exp[i\mathbf{k}(\mathbf{r}-\mathbf{V}_0t)-kz],$$

and for the component parallel to the surface we have

$$\mathbf{V}_\parallel=-\xi_0(\mathbf{kV}_0)(\mathbf{k}/k)\exp[i\mathbf{k}(\mathbf{r}-\mathbf{V}_0t)-kz].$$

When account is taken of these expressions, we can express the kinetic energy of the liquid in terms of the Fourier transform of the function $\xi(\mathbf{k})$:

$$\mathscr{E}_{\text{kin}}=\frac{\rho}{2}\int V^2d^3r=\frac{\rho}{16\pi^2}\int\frac{(\mathbf{kV}_0)^2}{k}\xi^2(\mathbf{k})\,d^2k.$$

Similarly, for the energy dissipation we obtain in first order in the viscosity η:

$$W=\frac{\eta}{2}\sum_{i,j}\int\left(\frac{\partial V_i}{\partial r_j}+\frac{\partial V_j}{\partial r_i}\right)^2d^3r=\frac{\eta}{2\pi^2}\int(\mathbf{V}_0\mathbf{k})^2k\xi^2(\mathbf{k})\,d^2k.$$

The quantity $2\mathscr{E}_{\text{kin}}/V_0^2$ can be naturally called the effective mass M of the dimple, while V_0^2/W can be called the effective mobility μ. We shall measure M in units of $\alpha^{3/2}\rho^{-1/2}g^{-3/2}\approx 1.8\times 10^{-5}$ g, and μ in units of $\alpha^{-1/2}(\rho g)^{1/2}\eta^{-1}\approx 6.7$ sec/g, if it is recognized that $\eta=30$ μP. We then obtain[1)]

$$M=\frac{1}{8\pi}\int_0^\infty\xi^2(k)k^2dk, \tag{19}$$

$$\mu^{-1}=\frac{1}{2\pi}\int_0^\infty\xi^2(k)k^4dk. \tag{20}$$

For the function $\xi(k)$, using (3) and (9), we obtain

$$\xi(k)=\frac{E}{1+k^2}n(k)=\frac{E}{1+k^2}\sum_{s=0}^{m}N_s\left(\frac{2}{kR}\right)^{s+1/2}\Gamma(s+3/2)J_{s+1/2}(kR). \tag{21}$$

Substitution of (21) in (19) and (20) gives rise to the integrals

$$I(m,n,p)=\int_0^\infty\frac{J_{m+1/2}(kR)J_{n+1/2}(kR)}{(1+k^2)^2(kR)^{m+n+1}}k^{2p}dk$$

$$=(-1)^{m+n+p}\frac{R^{3-2p}}{4}\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{(m+i)!(n+j)!}{i!(m-i)!j!(n-j)!2^{i+j}}$$

$$\times\left\{\frac{e^{-2R}}{R^{t+2}}\left(2+\frac{t+2}{R}\right)+(-1)^{t+1}\sum_{u=0}^{[t/2]}\frac{2^{i+t-2u}(1+u)}{(t-2u)!R^{3+2u}}\right.$$

$$\left.-\frac{1}{2}[(-1)^{m+i}+(-1)^{n+j}]\frac{t+2}{R^{t+3}}\right\},$$

where $t=m+n+i+j-2p+1$, and $[t/2]$ is the integer part of $t/2$. As a result we get for the mass and for the mobility of the dimple

$$M=\frac{E^2}{8\pi}\sum_{s,t=0}^{m}2^{s+t+1}\Gamma\left(s+\frac{3}{2}\right)\Gamma\left(t+\frac{3}{2}\right)I(s,t,1)N_sN_t;$$

$$\mu^{-1}=\frac{E^2}{2\pi}\sum_{s,t=0}^{m}2^{s+t+1}\Gamma\left(s+\frac{3}{2}\right)\Gamma\left(t+\frac{3}{2}\right)I(s,t,2)N_sN_t.$$

For a field E_D we have $M=0.0654Q^2$ and $\mu^{-1}=0.190Q^2$. The field dependence of the relative quantities $\tilde{M}$ and $\tilde{\mu}^{-1}$ is shown in Fig. 2. In the limit $E\gg 1$ we have[2)]

$$\tilde{M}=E^2/32-0.381,\quad \tilde{\mu}^{-1}=0.0253E^4-0.38E^2.$$

The characteristic dimple deceleration time $\tau=M\mu$ on account of viscosity does not depend on the dimple charge Q and is measured in units of $\alpha(g\eta)^{-1}\approx 12$ sec. At $E=E_D$ we have $\tau\approx 4.2$ sec, and at $E\gg 1$ the time is $\tau\approx 15\,E^{-2}$ sec. The dimple velocity at which its shape begins to be distorted is $(\alpha g/\rho)^{1/4}\approx 7$ cm/sec. To reach this velocity at $E=E_D$ it suffices to apply to the dimple a drawing field of the order of $0.1Q$ V/cm. A study of the oscillations of the dimple at a drawing-field frequency $\omega\gtrsim\tau^{-1}$ would make it possible to determine the mass M and the mobility μ, and also to investigate the dynamics of the dimple in the nonlinear regime.

4. LIGHT REFRACTION BY A MULTIELECTRON DIMPLE

Multielectron dimples were detected in experiment by the reflection of obliquely incident light from the deformed surface of helium. More information can be ob-

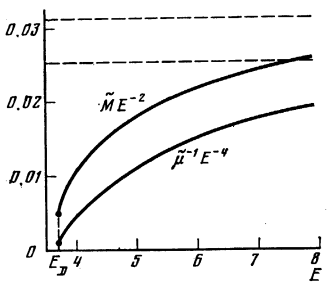

FIG. 2. Dependence of the relative dimple mass $\tilde{M}$ and of the relative mobility $\tilde{\mu}$ on the field E.

tained by investigating the pattern of the refraction of light incident perpendicular to this surface. In particular, it is possible in this way to estimate the order of magnitude of the charge of the dimple, and in the case of a dimple with small charge $Q \ll 1$ it is possible even to obtain its numerical value.

Let a parallel light beam be incident perpendicular to the unperturbed position of the surface and let it enter from a medium with refractive index n_1 into a medium with refractive index n_2. We investigate the distribution of the intensity of the transmitted light on a screen located at a distance d from the surface. The deformation of the surface is given by the function $\xi(\mathbf{r})$ (the deformation is defined as positive in the direction of propagation of the light), therefore the angle of reflection of the ray at the point r on the surface is

$$\alpha = |(n_2/n_1 - 1)\nabla\xi(\mathbf{r})|.$$

We note that the condition $Q \ll 1$ implies also the inequalities $|\nabla\xi| \ll 1$ and $\alpha \ll 1$. From the point r on the surface the ray is incident on the screen at the point

$$\mathbf{r}'(\mathbf{r}) = \mathbf{r} - d(n_2/n_1 - 1)\nabla\xi(\mathbf{r}),$$

which we shall call the image of the point r. It was shown above that for a single dimple

$$\xi(\mathbf{r}) = Q\tilde{\xi}(\mathbf{r}),$$

where the positive decreasing function $\tilde{\xi}(r)$ depends only on the value of r and on the field E as a parameter. The distribution of the light intensity on the screen is accordingly determined by the quantity

$$D = (n_2/n_1 - 1)Qd,$$

which is the effective distance from to the screen, and connects all the parameters of the problem except the field E. The sign of D can be either positive or negative. For example, $D > 0$ in the case of electrons clamped to the surface of helium, and $D < 0$ in the case of ions drawn from the helium. For a vacuum–helium interface ($n_1 = 1, n_2 = 1.057$) the value $D = 1$ corresponds to a depth $d \approx 0.88/Q$ cm.

We have thus found that the image of the dimple surface on a screen located at a distance D away from it is given by the function

$$\mathbf{r}'(\mathbf{r}) = \mathbf{r} - D\nabla\tilde{\xi}(\mathbf{r}). \tag{22}$$

The distribution of the light intensity on the screen $I(r')$ can be obtained from the condition that the light flux through an area on the surface and through the image of this area on the screen be equal. In the case of circular symmetry this leads to the expression

$$I(r') = \left|\frac{r(r')}{r'}\frac{dr(r')}{dr'}\right| = \left|\left(1 - \frac{D}{r}\frac{d\tilde{\xi}}{dr}\right)\left(1 - D\frac{d^2\tilde{\xi}}{dr^2}\right)\right|^{-1}, \tag{23}$$

where r (r') is a function inverse r' (r) (the point r is the original of the point r'). The light intensity far from the dimple (as $\tilde{\xi} \to 0$) is taken to be unity.

Expressions (22) and (23) are sufficient for the calculation of the intensity of the refracted light. We present first a qualitative analysis of the produced picture, using a particular example in which the field E has the value $E_0 \approx 4.06$ (in this field the dimple energy is $\mathscr{E} = 0$), after which we describe the method of determining the charge of the dimple from a light-refraction experiment.

In accordance with the already indicated properties of the function $\tilde{\xi}(r)$, its derivative $\tilde{\xi}'(r)$ is always negative, vanishes at $r = 0$ and $r = \infty$, and reaches an extremum at a certain $r \sim 1$. Since $\xi' < 0$, we have $r'(r) > r$ at $D > 0$ and $r'(r) < r$ at $D < 0$. This means that the dimple defocuses the light in the case $D > 0$ and focuses it at $D < 0$. The paths of the refracted rays at a field $E = 4.06$ is shown in Fig. 3. It follows from this figure that at sufficiently small D the rays do not intersect, i.e., there is a one-to-one correspondence between r and r'. After a certain value of r is reached, the rays cross, i.e., the function $r'(r)$ becomes nonmonotonic, and phenomena similar to focusing take place.

The monotonicity of $r'(r)$ is violated at those points where $dr'/dr = 0$, i.e., $D\tilde{\xi}''(r) = 1$. The function $\tilde{\xi}''(r)$ is shown in Fig. 4. It is seen that $\tilde{\xi}''$ reaches a mini-

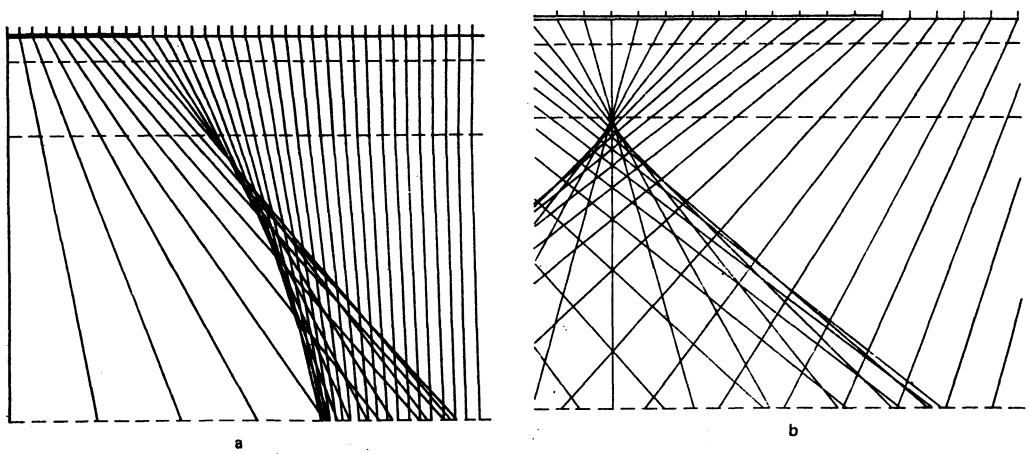

FIG. 3. Refraction of light by the dimple surface in a field $E = 4.06$. a) In the case $D > 0$ (defocusing), b) in the case $D < 0$ (focusing). The top solid line represents the helium surface, the double line shows the region occupied by the electrons, the dashed lines correspond to $d = F/4$, F, and $4F$. a) $F = F^+$, b) $F = F^-$.

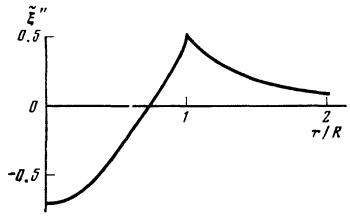

FIG. 4. Radial dependence of the relative curvature of the dimple surface for a field $E = 4.06$.

mum at $r = 0$. We denote $\tilde{\xi}''(0)$ by $1/F^-$, where $F^- < 0$ is the focal length for rays traveling along the dimple axis. With increasing r, the function $\tilde{\xi}''(r)$ first increases and reaches a maximum at $r = R$, after which it decreases monotonically to zero. (In fields close to E_D, there is one more maximum of $\tilde{\xi}''(r)$ at $r < R$, but its region of existence is quite narrow.) We designate $\tilde{\xi}''(R)$ by $1/F^+$, where $F^+ > 0$ corresponds to the focal length of the circular section of the dimple near $r = R$. At a field $E = 4.06$ we have $F^+ = 1.97$ and $F^- = -1.42$.

It is clear now that the function $r'(r)$ is monotonic so long as $F^- < D < F^+$. When D goes outside this interval, $r'(r)$ becomes nonmonotonic, and the inverse function $r(r')$ becomes multiply valued, namely, to each r' there correspond three values of $r(r')$. Figure 5 shows the plots of $r'(r)$ for the depth D indicated in Fig. 3. It follows from Fig. 5 that at $D > 0$ the nonmonotonicity of $r'(r)$ is reached first at $D = F^+$ and corresponds to the point $r = R$. With further increase of D, $r'(r)$ has a maximum and a minimum which are located respectively above and below $r'(R)$. At $D < 0$, nonmonotonicity arises first at $D = F^-$ at the point $r = 0$, and at $D < F^-$ the function $r'(r)$ has a maximum at the point $r = 0$ and a minimum at a certain $r > 0$.

Near the extrema, the function $r'(r)$ varies slowly with changing r. This means that in the sections of the screen there will be gathered near the extrema rays from a wide region of values of r, which leads to the onset of points or of rings with infinite brightness. In accordance with the number and positions of the extrema of $r'(r)$, at $D = F^+$ the intensity $I(r')$ has a singularity at the point $r' = r'(R)$, which is transformed with increasing D into two singluarities at $r_1' < r'(R)$ and at $r_2' > R'(R)$. If $D = F^-$, a singularity of $I(r')$ takes place at $r' = 0$. At $D < F^-$ this point singularity is preserved, but an additional ring singularity appears, whose radius increases with increasing $|D|$. The foregoing properties of $I(r')$ are illustrated in Fig. 6. It is seen that in addition to the infinite singularities $I(r')$ has also a kink singularity at $r' = r'(R)$, and this singularity is preserved at all D.

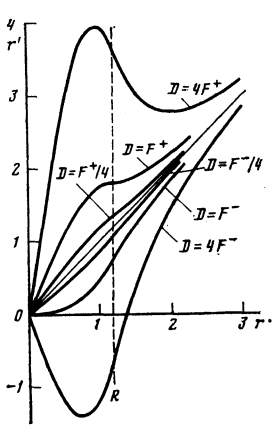

FIG. 5. Image $r'(r)$ of the dimple surface r on a screen r' at a field $E = 4.06$ and at various distances to the screen D.

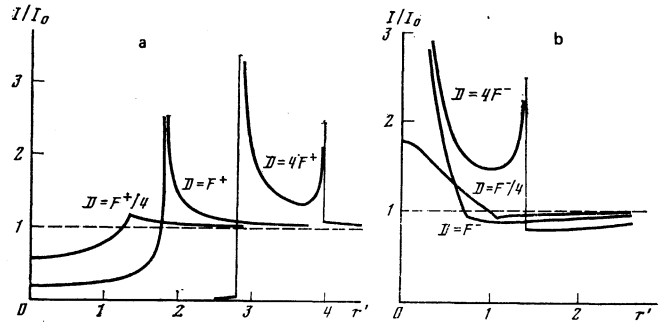

FIG. 6. Distribution of the light intensity at a field $E = 4.06$ and at various distances to the screen: a) $D > 0$, b) $D < 0$.

We now find the behavior of $I(r')$ near all the indicated singularities. At $r \sim R$, with allowance for Eq. (3) and for the previously discussed behavior of the charge density $n(r)$ as $r \to R$, we have

$$\begin{aligned} \tilde{\xi}'' &\approx 1/F^+ - \alpha(R-r)^{1/2}, \quad r<R, \\ \tilde{\xi}'' &= 1/F^+ - \beta(r-R), \quad r>R, \end{aligned} \tag{24}$$

where the coefficients $\alpha, \beta > 0$. Substitution of (24) in (23) followed by a transition to the variable r' in accordance with (22) yields

$$\begin{aligned} &I \propto (r'(R)-r')^{-1/2}, \quad r'<r'(R), \quad I \propto (r'-r'(R))^{-1/2}, \quad r'>r'(R), \quad D=F^+; \\ &I \propto (r'-r_1')^{-1/2}, \quad r>r_1', \quad I \propto (r_2'-r')^{-1/2}, \quad r'<r_2', \quad D>F^+, \end{aligned} \tag{25}$$

where r_1' and r_2' are the positions of the right-hand and left-hand singularities of $I(r')$. It follows therefore that at $D > F^+$ the intensity $I(r')$ has one-sided square-root singularities on the internal edges of the region of the nonmonotonicity of $r'(r)$, when three points $r(r')$ contribute to $I(r')$. On the outer edges of this region, however, when only the light refracted by the nonsingular point $r(r')$ contributes to $I(r')$, the value of $I(r')$ falls practically to zero on the left of r_1' and to unity on the right of r_2', as seen from Fig. 6a.

In the case $D = F^-$ both factors of the (23) vanish at the point $r' = 0$, while at $D < F^-$ only one of them vanishes. Accordingly,

$$I(r') \propto (r')^{-1/3}, \quad D=F^-, \quad I(r') \propto 1/r', \quad D<F^-.$$

The ring singularity at $D < F^-$ is similar to the indicated singularity of $I(r')$ at r_2' [see (25) and Fig. 6b].

The foregoing results shows that the distribution of the intensity of light refracted by a dimple has a strongly pronounced structure, the study of which makes it possible to determine certain parameters of the dimple.

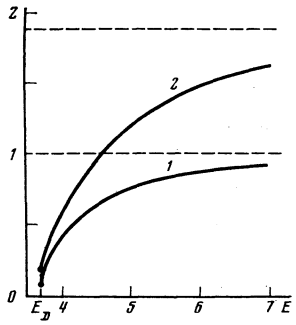

FIG. 7. Dependences of the dimensionless focal lengths F^+ and F^- on the field E. Curve 1 corresponds to $10^3[E^5F^+(E)]^{-1}$, and curve 2 corresponds to $10^3(E^5)\cdot|F^-(E)|^{-1}$.

We point out, e.g., that measurement of the radius $r'(R)$ [at which the kink of $I(r')$ is located] for two values of helium depth is sufficient to find the radius of the charged region from pure geometrical considerations. As shown above, in the limit of a small dimple charge the radius R depends only on the field (see Fig. 1). A comparison of the experimental results for R with Fig. 1 makes it possible to estimate qualitatively the value of Q, while a comparison of the radii of the different dimples may perhaps lead to certain conclusions concerning the dependence of the dimple charge on the conditions under which it was produced.

At the same time there exists also a direct method of determining the absolute charge of the dimple. This method is suitable in the limit of a small charge Q, i.e., for dimples with less than 10^7 electrons. The point is that at $Q\ll1$ the dimensionless focal lengths F^+ and F^-, which depend only on the field E, constitute a parameter combination of the form $(n_2/n_1-1)Qd^*$, where d^* are the real focal distances expressed in capillary radii. If it is assumed that the refractive indices and the capillary radius are known, then the experimental measurement of one of the focal distances at a certain field E is sufficient to determine the dimensionless charge Q of the dimple from the known values of $F^+(E)$ or $F^-(E)$. Plots of these quantities are given in Fig. 7. The numerical asymptotic values were obtained for F^* are

$$F^+=996E^{-5},\quad F^-=-530E^{-5},\quad E\gg1.$$

The number N of the electrons in the dimple is given by the expression

$$N=\alpha^{7/4}(\rho g)^{-5/4}F^{\pm}(E)/e(n_2-n_1)d^{\pm}, \tag{26}$$

where all the quantities except, of course, $F^*(E)$ should be expressed in dimensional units. It is assumed here that $|n_{1,2}-1|\ll1$, as in the case of helium. We emphasize that for practical utilization of (26) in conjunction with Fig. 7 it is necessary to be able to measure only the field E and the focal distances d^*, since α, ρ, g, n_1, and n_2 are well known.

Our results take practically complete care of the problem of a single dimple with a small charge. In concluding the article, we point out phenomena, quite difficult for numerical calculation but may be accessible to experimental observation.

We have found the field E_D that determines the lower limit of the existence of a dimple, assuming the latter to have circular symmetry. It is natural to assume, however, that if the field is somewhat stronger than E_D, the dimple becomes unstable to elliptic deformation. An indirect indication of such an instability may be the disintegration of the dimples when a low-amplitude capillary wave is excited.

With increasing field E, the nonlinear interactions become substantial even in the case of a dimple with a small charge $Q\ll1$. As follows from the qualitative discussion given at the beginning of the article, at a certain field $E\sim Q^{-1/3}$ the helium surface closes up over the dimple, which becomes thereby a multielectron bubble with radius $\sim Q^{2/3}$. Generalizing this result, we can state that in those cases when disruption of the periodic structure of the surface with departure of multielectron dimples towards the lower electrode takes place at large values of the field, the radius of the bubbles should depend on the field like E^{-2}.

We are grateful to L. P. Gor'kov and É. I. Rashba for a number of helpful discussions.

[1)]In our preceding paper[2] the value of μ^{-1} differed from (20) by an extra factor 4, and a corrected plot of μ^{-1} is shown in Fig. 2.

[2)]In the limit $E\gg1$, our results for $\mathscr{E}$, R, M, and μ differ from those previously obtained[6] by numerical factors.

[1]L. P. Gor'kov and D. M. Chernikova, Dokl. Akad. Nauk SSSR 228, 829 (1976) [Sov. Phys. Dokl. 21, 328 (1976)].
[2]P. Leiderer and M. Wanner, Phys. Rev. Lett. 42, 315 (1979).
[3]V. I. Mel'nikov and S. V. Meshkov, Pis'ma Zh. Eksp. Teor. Fiz. 33, 222 (1981) [JETP Lett. 33, 211 (1981)].
[4]I. S. Gradshtein and I. M. Ryzhik, Tablitsy integralov, summ, ryadov i proizvedenii (Tables of Integrals, Sums, Series, and Products), Fizmatgiz, 1971. [Academic, 1965].
[5]L. D. Landau and E. M. Lifshitz, Elektrodinamika sploshnykh sred (Electrodynamics of Continuous Media), Fizmatgiz, 1959 [Pergamon, 1960].
[6]V. P. Shikin and P. Leiderer, Pis'ma Zh. Eksp. Teor. Fiz. 32, 439 (1980) [JETP Lett. 32, 416 (1980)].

Translated by J. G. Adashko