Surface oscillations of a Fermi-liquid layer

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The influence of effects due to the finite character of a system on the character of surface vibrations of a normal Fermi liquid is investigated. In a free Fermi-liquid film, the spontaneous violation of translational symmetry leads to restoration of the classical capillary spectrum of the long-wave surface oscillations, which in this case play the role of gapless Goldstone excitations that restore the broken symmetry. The weak damping of these waves, however, is by far not of classical origin and is determined only by the finite thickness of the film. In the case of a Fermi liquid of finite depth, the classical capillary spectrum is not restored and the surface oscillations remain a purely damped mode, just as in a semi-infinite system.

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1. INTRODUCTION

Fomin[1] and the author[2] have proposed a theory of surface Fermi-liquid oscillations, based on the kinetic equation with a self-consistent field. On the other hand, to describe low-lying vibrational states of atomic nuclei, a self-consistent variant of the theory of finite Fermi systems was developed in Ref. 3, and the very same surface oscillations in nuclei are treated in fact within the framework of this variant. It is of interest to examine the relation between these two approaches. It turns out that the kinetic theory[1,2] is a quasiclassical asymptotic form of the self-consistent theory of finite Fermi systems,[3] and on the other hand, the self-consistent approach of the theory of finite Fermi systems is an adiabatic approximation, whereas the kinetic theory makes it possible to consider also high-frequency oscillations.

As found earlier,[2] surface waves in a semi-infinite Fermi liquid in the collisionless regime are a purely damped mode [we have in mind the low-lying branch of the surface oscillations (55) of Ref. 2]. These excitations disintegrate as a result of the Landau damping mechanism, i.e., by decay into individual particle-pole excitations. It is known at the same time that the vibrational states of atomic nuclei are well defined weakly damped collective excitations of the system. It is therefore of interest to examine the influence of the finite nature of a Fermi system on the character of surface oscillations and to investigate the mechanism of suppressing the Landau damping.

In this paper we consider surface oscillations in two systems: a free film of liquid ^{3}He, and liquid ^{3}He of finite depth. Our analysis provides an answer to the questions raised above, and permits a study of the possibility of propagation of undamped surface waves in these systems.

2. OSCILLATION OF A FREE FERMI-LIQUID FILM

We consider an infinite plane layer of a normal Fermi liquid of thickness L at zero temperature. The dispersion equation for the quantum capillary waves in this system is obtained in exactly the same manner as for a spherical drop of Fermi liquid (atomic nucleus),[3] and is of the form

$$\int \frac{dU}{dx}[A(x,x',\mathbf{k};0)-A(x,x',0;0)]\frac{dU}{dx'}dxdx' + \int\frac{d\rho}{dx}[\mathcal{F}(x,x',\mathbf{k})-\mathcal{F}(x,x',0)]\frac{d\rho}{dx'}dxdx' = \int\frac{dU}{dx}[A(x,x',\mathbf{k};\omega)-A(x,x',\mathbf{k};0)]\frac{dU}{dx'}dxdx', \quad (1)$$

where $\rho(x)$ is the density of the quasiparticles, $U(x)$ is the self-consistent potential, the x axis is directed across the considered layer, $\mathbf{k}$ is the two-dimensional wave vector of the wave directed along the layer; the $\mathbf{k}$-harmonics of the particle-hole propagator and of the effective interaction of the quasiparticles are defined respectively as

$$A(x,x',\mathbf{k};\omega)=\int d(\mathbf{r}_\perp-\mathbf{r}_\perp')\exp\{-i\mathbf{k}(\mathbf{r}_\perp-\mathbf{r}_\perp')\} \times\int G^q(\mathbf{r},\mathbf{r}';\varepsilon+\omega/2)G^q(\mathbf{r},\mathbf{r}';\varepsilon-\omega/2)\frac{d\varepsilon}{2\pi i}, \quad (2)$$

$$\mathcal{F}(x,x',\mathbf{k})=\int\mathcal{F}(\mathbf{r},\mathbf{r}')\exp\{-i\mathbf{k}(\mathbf{r}_\perp-\mathbf{r}_\perp')\}d(\mathbf{r}_\perp-\mathbf{r}_\perp'), \quad (3)$$

where G^q is the quasiparticle Green's function and $\mathbf{r}_\perp$ is the component of the vector-coordinate $\mathbf{r}$ along the surface of the system. Expression (1) was obtained under the assumption that the effective interaction does not contain velocity forces.

In the same manner as in a spherical drop of a Fermi liquid,[3] it can be shown that the rigidity of the considered system, which coincides with the left-hand side of Eq. (1), has the classical hydrodynamic form, and the dispersion equation (1) reduces to the form

$$2\sigma k^2=Q(\mathbf{k},\omega), \quad (4)$$

where σ is the surface-tension coefficient, and $Q(\mathbf{k},\omega)$ denotes the right-hand side of Eq. (1). The doubling of the rigidity in (4) compared with the case of a semi-infinite system is due to the fact that it now receives contributions from both the lower and the upper surfaces of the layer.

To find the spectrum of the waves in the film, it is necessary to calculate the right-hand side of Eq. (1), i.e., the quantity $Q(\mathbf{k},\omega)$. We use for this purpose the λ representation of the particle-hole propagator[4]:

$$A(x,x',\mathbf{k};\omega)=\sum_{n,n'=0}^{\infty}\int\frac{d\mathbf{p}_\perp d\mathbf{p}_\perp'}{(2\pi)^2}\delta^{(2)}(\mathbf{p}_\perp-\mathbf{p}_\perp'-\mathbf{k}) \times\frac{n(\varepsilon_\lambda)-n(\varepsilon_{\lambda'})}{\varepsilon_\lambda-\varepsilon_{\lambda'}-\omega}\varphi_n^*(x)\varphi_n(x')\varphi_{n'}(x)\varphi_{n'}^*(x'), \quad (5)$$

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where $\lambda=(\mathbf{p}_\perp, n)$, $\varepsilon_\lambda = p^2/2m + \varepsilon_n$ is the quasiparticle energy, $\varphi_n(x)$ is the wave function of the transverse motion of the quasiparticle and satisfies the equation

$$-\frac{1}{2m}\frac{d^2}{dx^2}\varphi_n + U(x)\varphi_n = \varepsilon_n\varphi_n. \tag{6}$$

Using the representation (5), we obtain

$$Q(\mathbf{k},\omega) = -2\omega\sum_{n,n'=0}^{\infty}\int\frac{d\mathbf{p}_\perp d\mathbf{p}_\perp'}{(2\pi)^2}\delta^{(2)}(\mathbf{p}_\perp-\mathbf{p}_\perp'-\mathbf{k}) \times\frac{n(\varepsilon_\lambda)-n(\varepsilon_{\lambda'})}{\varepsilon_\lambda-\varepsilon_{\lambda'}}\frac{1}{\varepsilon_\lambda-\varepsilon_{\lambda'}-\omega}\left(\frac{dU}{dx}\right)^2_{nn'}. \tag{7}$$

The coefficient 2 in the right-hand side of (7) is the result of summation over the spin variables. The wave equation (6) allows us to express the matrix element of the derivative of the self-consistent field, which enters in (7), in terms of the matrix element of the coordinate:

$$\left(\frac{dU}{dx}\right)_{nn'} = m(\varepsilon_n-\varepsilon_{n'})^2(x)_{nn'}. \tag{8}$$

Assuming that the layer width is much larger than the diffuseness of the surface of the system, $L \gg 1/p$, we can use the quasiclassical approximation to solve the wave equation (6). In this case

$$\varepsilon_n = \frac{\pi^2}{2mL^2}(n+{}^1/_2)^2 - U_0, \tag{9}$$

$$(x)_{nn'} = L\pi^{-2}[(-1)^{n-n'}-1][(n-n')^{-2}-(n+n')^{-2}], \tag{10}$$

where U_0 is the depth of the self-consistent potential $U(x)$ in the central region of the layer.

As a result of the double energy denominator in (7), the main contribution to $Q(\mathbf{k},\omega)$ is made by the region of summation over λ and λ' near the Fermi surface. The contribution to Q from the summation over the remote regions, as shown by estimates, is of the order of $\omega^2 p_F\rho_0/\varepsilon_F$, and this, as will be shown ultimately, is an inessential correction. To carry out the summation near the Fermi surface we therefore make the following substitution:

$$\frac{n(\varepsilon_\lambda)-n(\varepsilon_{\lambda'})}{\varepsilon_\lambda-\varepsilon_{\lambda'}} \approx \frac{dn}{d\varepsilon}(\varepsilon), \tag{11}$$

$$\varepsilon = \frac{(\mathbf{p}_\perp+\mathbf{p}_\perp')^2}{8m} + \frac{\pi^2}{8mL^2}(n+n'+1)^2 - U_0. \tag{12}$$

We note that for the reason indicated above we shall also neglect the second term $\sim(n+n')^{-2}$ in the matrix element $(x)_{nn'}$ (10).

Taking all the foregoing into account, the expression for $Q(\mathbf{k},\omega)$ is transformed into

$$Q(\mathbf{k},\omega) = -2\omega\sum_{\varkappa=-\infty}^{\infty}[(-1)^\varkappa-1]^2\frac{L}{\pi}\int_0^\infty dP_\varkappa\int\frac{d\mathbf{P}_\perp}{(2\pi)^2} \times\frac{dn}{d\varepsilon}(\varepsilon)\frac{1}{\mathbf{Pq}/m-\omega}\left(\frac{P_x^2}{mL}\right)^2 \tag{13}$$

where the following notation was introduced

$$\mathbf{P}_\perp = {}^1/_2(\mathbf{p}_\perp+\mathbf{p}_\perp'),\quad P_x = \pi(n+n'+1)/2L,$$
$$\varkappa = n-n',\quad q_x = \pi\varkappa/L,\quad \mathbf{q} = \{q_x, \mathbf{k}\}.$$

On going from (7) to (13) the summation over $(n+n')$ was replaced by integration with respect to dP_x.

The integral in (13) can be easily calculated:

$$\int_0^\infty\frac{dP_x}{\pi}\int\frac{d\mathbf{P}_\perp}{(2\pi)^2}\frac{dn}{d\varepsilon}(\varepsilon)\frac{P_x^4}{\mathbf{Pq}/m-\omega}$$
$$= \frac{m^2p_F^4}{2\pi^2}\frac{1}{q}\left\{s\left(s^2w-\frac{1}{3}\right)+\left(\frac{k}{q}\right)^2\left[3sw-5s\left(s^2w-\frac{1}{3}\right)\right] + \left(\frac{k}{q}\right)^4\frac{1}{8}\left[\frac{3}{s}(w+1)-30sw+35s\left(s^2w-\frac{1}{3}\right)\right]\right\}, \tag{14}$$

$$q = (q_x^2+k^2)^{1/2},\quad s = \frac{\omega}{qV_F},\quad w = \frac{s}{2}\ln\frac{s+1}{s-1}-1. \tag{15}$$

As will be seen from the final result, the dispersion equation (4) has only a low-frequency solution, i.e., $|\omega/kV_F| \ll 1$ [the really investigated dispersion equation (1) is an adiabatic expansion, therefore a search for solutions of this equation is meaningful only in the low-frequency limit]. We therefore expand the expression in the right-hand side of (14) in the small parameter s, using the expansion of the w-function (15) at $|s| \ll 1$, $\mathrm{Im}\, s > 0$:

$$w \approx -1 - {}^1/_2 i\pi s + s^2 + \ldots \tag{16}$$

Next, summing in (13) over $\varkappa$, we obtain with the aid of the relation[5]

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2+z^2} = \frac{\pi}{4z}\,\mathrm{th}\frac{\pi z}{2} \tag{17}$$

the dispersion equation in the following form:

$$\sigma k^2 = \left\{\frac{kL}{\mathrm{sh}\,kL}\left[1+(kL)^2\,\mathrm{th}\frac{kL}{2}\right]\right\}\frac{m\rho_0}{k}\omega^2\,\mathrm{th}\frac{kL}{2} + \frac{9}{8}i\rho_0p_F\omega\left(\frac{kL}{\pi}\right)^4\sum_{n=-\infty}^{\infty}\left[(2n+1)^2+\left(\frac{kL}{\pi}\right)^2\right]^{-5/2}. \tag{18}$$

In the short-wave limit $kL \gg 1$ the term in the curly brackets in the right-hand side of (18) becomes exponentially small: $\sim e^{-kL}$. The sum over n in the second term can be calculated in this case by replacing the summation with integration. We thus find that at $kL \gg 1$

$$\sigma k^2 = {}^3/_4 i\rho_0p_F\omega. \tag{19}$$

This expression coincides exactly with the dispersion law of quantum capillary waves in a semi-infinite Fermi liquid [Eq. (55) of Ref. 2]. The term $\sim\omega^2$ in (55) of Ref. 2 is inessential.

In the case of long waves $kL \ll 1$, which is of practical interest, the term in the curly brackets of (18) tends to unity and we obtain the dispersion equation

$$2\sigma k^2 = m\rho_0L\omega^2 + {}^9/_2 iC\rho_0p_F(kL/\pi)^4\omega,\qquad C = \sum_{n=0}^{\infty}(2n+1)^{-5} \approx 1.00452. \tag{20}$$

The real part of the spectrum, determined by Eq. (20), coincides with the spectrum of classical hydrodynamic capillary waves in a film:

$$\sigma k^2 = \frac{m\rho_0}{k}\omega^2\,\mathrm{th}\frac{kL}{2}.$$

What is interesting, however, is that the damping of the obtained waves, in contrast to hydrodynamics, is in no way connected with the viscosity of the liquid, but is determined only by the finite thickness L of the film. The absorption coefficient of these waves, which is defined as the imaginary part of the wave vector k, is equal to

$$\gamma = \operatorname{Im} k = \frac{9}{4} C \frac{p_F}{m} \left(\frac{m\rho_0}{2\sigma}\right)^{5/2} \left(\frac{\omega}{\pi}\right)^4 L^{11/2}. \tag{21}$$

Thus, as seen from the analysis of the dispersion equation (1), the fact that the system is finite (in our case, in one dimension) leads to an abrupt decrease of the damping of the surface mode. The reason is that the finite thickness of the film brings about discrete quantum states of the transverse motion of the quasi-particles [see (9)], which in turn hinders the action of the Landau damping mechanism. The Landau damping, as is well known, is connected with the breakup of the collective mode into indivudual particle-hole excitations. The discreteness of the single-particle states narrows down the phase volume of the finite particle-hole states and by the same token hinders the decay of the collective mode.

From the general point of view, a Fermi-liquid film is a system with spontaneous breaking of the translational symmetry, and this leads in fact to a restoration of the classical capillary spectrum of the long-wave oscillations. In the present case these oscillations assume the role of gapless Goldstone excitations which restore the broken symmetry.

It can be concluded on the basis of the foregoing that well-defined low-lying vibrational states exist in atomic nuclei only because the nucleus is finite so that the single-particle states are discrete. The same factor explains the relative success of the hydrodynamic approach to the description of surface vibrations in nuclei. The deviations of these vibrations from classical hydrodynamic behavior are connected primarily only with shell effects that arise as a result of the nonequilibrium behavior of the single-particle levels.

Waves of the considered type (20) can be realized in a free film of liquid ^{3}He. Even though we used in the foregoing calculations an effective interaction that contains only the zeroth harmonic in the momenta, the result expressed in the form (18) remains valid in the case of a nonzero first harmonic of the interaction. This can be demonstrated on the basis of a solution of the kinetic equation in a plane-parallel layer of a Fermi liquid. In addition, the kinetic equation makes it possible to investigate the presence of solutions not only in the low-frequency limit, as was done above, but also to consider the high-frequency case. For this reason, we describe briefly in the next section the solution of the kinetic equation.

3. KINETIC EQUATION IN A PLANE-PARALLEL FERMI-LIQUID LAYER

An analysis perfectly similar to that developed in the preceding paper[2] leads to the following formulation of the problem of finding the natural oscillations of a layer of a Fermi liquid. We consider the linearized kinetic equation

$$-i(\omega - \mathbf{k}\mathbf{V}_\perp) f + V_x \frac{\partial f}{\partial x} - \left(i\mathbf{k}\mathbf{V}_\perp + V_x \frac{d}{dx}\right)\left(\mathcal{F}_0 \rho' + \frac{\mathcal{F}_1}{p_F^2}\mathbf{p}\mathbf{j}\right) = -\nu\left(f + \frac{2\varepsilon_F}{3\rho_0}\rho' + \frac{1}{m^*\rho_0}\mathbf{p}\mathbf{j}\right) \tag{22}$$

in the region $-L/2 < x < L/2$, where the perturbation of the density ρ' and the flux of the number of particles j are defined by the relations

$$\rho'(x, \mathbf{k}) = \int \frac{dn}{d\varepsilon} f(\mathbf{p}, x, \mathbf{k}) 2 \frac{d\mathbf{p}}{(2\pi)^3}, \tag{23}$$

$$\mathbf{j}(x, \mathbf{k}) = \int \frac{dn}{d\varepsilon} \mathbf{p} f(\mathbf{p}, x, \mathbf{k}) 2 \frac{d\mathbf{p}}{(2\pi)^3}, \tag{24}$$

and ν is the frequency of the collisions. We seek for (22) a solution that satisfies the following boundary conditions: the conditions of specular reflection of the quasiparticles on the surfaces $x_s = -L/2, L/2$:

$$f(p_x) - f(-p_z) = 2p_x\left[-\mathcal{U}_0 + \frac{\mathcal{F}_1}{p_F^2} j_x(x_s, \mathbf{k})\right], \tag{25}$$

and the condition that the forces exerted by the liquid on the surfaces of the system and by the surfaces on the liquid be equal:

$$2\sigma \mathcal{U}_0 k^2 = i\omega[\Pi_{xx}(L/2, \mathbf{k}) - \Pi_{xx}(-L/2, \mathbf{k})], \tag{26}$$

where $\mathcal{U}_0$ is the amplitude of the surface oscillation velocity,

$$\Pi_{xx}(x, \mathbf{k}) = \frac{1}{m^*}\int \frac{dn}{d\varepsilon} p_x^2 [f(\mathbf{p}, x, \mathbf{k}) - \mathcal{F}_0 \rho'(x, \mathbf{k})] 2 \frac{d\mathbf{p}}{(2\pi)^3}, \tag{27}$$

is the xx-component of the momentum flux tensor, and $m^* = [1 + (1/3)F_1]m$ is the effective mass of the quasiparticles. The minus sign in the right-hand side of (26) is evidence that we are considering flexural oscillations of the layer, when both surfaces oscillate in phase.

To solve Eq. (22), we continue the distribution function $f(\mathbf{p}, x, \mathbf{k})$ into the intervals $(-L, -L/2)$ and $(L/2, L)$ in the following manner:

$$f(p_x, \mathbf{p}_\perp, x, \mathbf{k}) = f(-p_x, \mathbf{p}_\perp, L - x, \mathbf{k}) \tag{28}$$

at $L/2 < x < L$ and

$$f(p_x, \mathbf{p}_\perp, x, \mathbf{k}) = f(-p_x, \mathbf{p}_\perp, -L - x, \mathbf{k}) \tag{29}$$

at $-L < x < -L/2$. Using the definitions (23) and (24) we easily find that such a continuation of the distribution function leads to the following continuation equations for the macroscopic fluxes:

$$\rho'(x, \mathbf{k}) = \rho'(-L - x, \mathbf{k}), \quad \mathbf{j}_\perp(x, \mathbf{k}) = \mathbf{j}_\perp(-L - x, \mathbf{k}),$$
$$j_x(x, \mathbf{k}) = -j_x(-L - x, \mathbf{k})$$

at $-L < x < -L/2$, and to a similar relation with L replaced by $-L$ at $L/2 < x < L$.

It is easily seen that Eq. (22) is invariant to continuation of the distribution function (28), (29). To satisfy the boundary conditions (25) it is necessary to add to (22), at the junctions of the intervals $(-L, -L/2)$, $(-L/2, L/2)$, and $(L/2, L)$, the δ-function terms $2p_x V_x \mathcal{U}_0 \delta(-L/2 - x)$ and $-2p_x V_x \mathcal{U}_0 \delta(-L/2 - x)$.

We thus consider now the continued kinetic equation

$$-i(\omega-\mathbf{kV}_\perp)f+V_x\frac{\partial f}{\partial x}-\left(i\mathbf{kV}_\perp+V_x\frac{d}{\partial x}\right)\left(\mathcal{F}_0\rho'+\frac{\mathcal{F}_1}{p_F^2}\mathbf{pj}\right)$$

$$+2p_xV_x\mathcal{U}_0[\delta(-L/2-x)-\delta(L/2-x)]=-\nu\left(f+\frac{2\varepsilon_F}{3\rho_0}\rho'+\frac{1}{m^*\rho_0}\mathbf{pj}\right) \quad (30)$$

in the interval $-L<x<L$. Integrating (30) in a narrow vicinity of the point $x=L/2$ with respect to dx, we find that the condition of specular reflection on the surface $x_s=L/2$ (25) is automatically satisfied. The same holds true for the surface $x_s=-L/a$. We have thus found that the solution of the continued equation (30) coincides with the solution of the initial kinetic equation (22) in the interval $(-L/2, L/2)$, a solution that satisfies the correct boundary conditions of specular reflection (25) at $x_s=\pm L/2$.

We now continue the distribution function $f(\mathbf{p}, x, \mathbf{k})$ from the interval $(-L<x<L)$ in periodic fashion, over all of space. Obviously, to find a solution of (30) that is periodic over all of space it is convenient to use the discrete Fourier transformation

$$f(\mathbf{p},x,\mathbf{k})=\frac{1}{2L}\sum_{n=-\infty}^{\infty}f(\mathbf{p},\mathbf{q})\exp(iq_{x,n}x), \quad (31)$$

$$f(\mathbf{p},\mathbf{q})=\int_{-L}^{L}f(\mathbf{p},x,\mathbf{k})\exp(-iq_{x,n}x)\,dx, \quad (32)$$

where $q_{x,n}=\pi n/L$ and $\mathbf{q}=\{q_{x,n},\mathbf{k}\}$. Similar expansions hold also for the continued $\rho'(x,\mathbf{k})$ and $\mathbf{j}(x,\mathbf{k})$.

The Fourier transform of (30) is of the form

$$-i(\omega-\mathbf{qV})f-i\mathbf{qV}\left(\mathcal{F}_0\rho'+\frac{\mathcal{F}_1}{p_F^2}\mathbf{pj}\right)+4ip_xV_x\mathcal{U}_0\sin\frac{\pi n}{2}$$
$$=-\nu\left(f+\frac{2\varepsilon_F}{3\rho_0}\rho'+\frac{1}{m^*\rho_0}\mathbf{pj}\right). \quad (33)$$

This equation is solved in standard fashion, i.e., by reducing it to a system of linear algebraic equations with respect to $\rho'(\mathbf{q})$ and $\mathbf{j}(\mathbf{q})$.

We note that the method of solving the kinetic equation (22) in a half-space, which was used earlier,[2] is fully equivalent to the method developed here, the only exception being that one period of the continued distribution function spans the entire x axis in the case of the half-space problem. The last circumstance has made it necessary to use an integral rather than a discrete Fourier transformation. Therefore, disregarding the discrete character of $q_{x,n}$, the solution obtained in Ref. 2 for the kinetic equation in the half-space $x<0$ in the $\mathbf{q}$ representation, is a solution of Eq. (33), the only difference being that the quantities q_x and $\mathcal{U}_0$ in the semi-infinite solution [Eqs. (42)-(47) of Ref. 2] must replaced by

$$q_{x,n}=\frac{\pi n}{L}\text{ and }-2i\mathcal{U}_0\sin\frac{\pi n}{2}.$$

To find the spectrum of the oscillations of the film it is necessary to use the remaining boundary condition (26), i.e., to calculate the quantity

$$\Pi_{xx}\left(\frac{L}{2},\mathbf{k}\right)-\Pi_{xx}\left(-\frac{L}{2},\mathbf{k}\right)=\frac{1}{2L}\sum_{n=-\infty}^{\infty}\Pi_{xx}(\mathbf{q})\left(2i\sin\frac{\pi n}{2}\right). \quad (34)$$

Since $\mathcal{U}_0$ enters as a common factor preceding $\Pi_{xx}(\mathbf{q})$ [see Eqs. (42)-(47) of Ref. 2], the additional factor in front of $\mathcal{U}_0$, equal to $-2i\sin(\pi n/2)$, is simply multiplied by a weighting factor in the sum (34), as a result of which we have $4\sin^2(\pi n/2)$.

We see thus that ultimately the summation in (34) is only over odd n, and the dispersion equation takes the form

$$\sigma\mathcal{U}_0k^2=i\omega\frac{1}{L}\sum_{n=-\infty}^{\infty}\Pi_{xx}^{(s-inf)}(q_x=\pi(2n+1)/L,\mathbf{k}), \quad (35)$$

where $\Pi_{xx}(\mathbf{q})$ is already replaced by the solution of the infinite problem without any redefinition of $\mathcal{U}_0$, and this solution takes in the collisionless limit $(\nu=0)$ the form[2]

$$\Pi_{xx}^{(s-inf)}(\mathbf{q})=i\mathcal{U}_0m^*\rho_0\frac{V_F^2}{\omega}\Bigg(\frac{1}{\Delta_0(s)}\Big\{-{}^2/_3F_1F_0s^2w$$
$$+{}^3/_2F_0(1+{}^1/_3F_1)\left[\frac{k^2}{q^2}(w+1)+\left(2-3\frac{k^2}{q^2}\right)s^2w\right]^2$$
$$+{}^3/_2F_1s^2\left[\frac{k^2}{q^2}w+\left(2-3\frac{k^2}{q^2}\right)(s^2w-{}^1/_3)\right]^2\Big\}$$
$$-\frac{6F_1}{\Delta_1(s)}\left(1-\frac{k^2}{q^2}\right)\frac{k^2}{q^2}s^2(w-s^2w+{}^1/_3)^2+6\Big\{s^2(s^2w-{}^1/_3)$$
$$+\frac{k^2}{q^2}s^2[3w-5(s^2w-{}^1/_3)]+{}^1/_8\frac{k^4}{q^4}[3(w+1)-30s^2w+35s^2(s^2w-{}^1/_3)]\Big\}\Bigg) \quad (36)$$

$$\Delta_0(s)=(1+{}^1/_3F_1)-[F_0(1+{}^1/_3F_1)+s^2F_1]w, \quad (37)$$

$$\Delta_1(s)={}^1/_2F_1(1+w-s^2w)-(1+{}^1/_3F_1). \quad (38)$$

The equations $\Delta_0(c_0/V_F)=0$ and $\Delta_1(c_1/V_F)=0$ determine the propagation velocities of the longitudinal (c_0) and transverse (c_1) zero sounds.[6]

We shall consider below only the collisionless regime $(\nu=0)$, inasmuch as in the hydrodynamic limit ($\nu/\omega\gg1$) the investigated equations yield nothing new compared with hydrodynamics,[7] as expected. In the low-frequency limit $|\omega/kV_F|\ll1$ we can expand (36) in the small parameter s, using Eq. (16). Calculation by means of Eq. (35), using (17), leads to the dispersion equation (18) of the capillary waves, as noted above.

In the high-frequency limit $|\omega/kV_F|\gtrsim1$, it is convenient to investigate the wave spectrum by changing from summation in (35) to integration with respect to dq_x, using the summation formula (17):

$$\frac{1}{L}\sum_{n=-\infty}^{\infty}\varphi\left(\frac{\pi}{L}(2n+1)\right)=\int_{C_q}\text{th}(iLq_x/2)\varphi(q_x)\frac{dq_x}{2\pi}, \quad (39)$$

where C_q is a closed contour that circles around the real q_x axis in the positive direction. When the contour integration (39) is used in place of summation, the calculation of the spectrum in the high-frequency limit hardly differs from the corresponding calculation in the semi-infinite problem as carried out by Fomin.[1]

The result if particularly simple when the velocities of the longitudinal and transverse zero sounds greatly exceed the Fermi velocity ($c_0, c_1\gg V_F$):

$$(2-\xi_1^2)^2\,\text{th}\left[-\frac{kL}{2}(1-\xi_0^2)^{1/2}\right]$$
$$=4(1-\xi_0^2)^{1/2}(1-\xi_1^2)^{1/2}\text{th}\left[-\frac{kL}{2}(1-\xi_1^2)^{1/2}\right]+\frac{5\sigma}{m\rho_0V_F^2}k\xi_1^2(1-\xi_0^2)^{1/2}, \quad (40)$$

where $\xi_0=\omega/kc_0$ and $\xi_1=\omega/kc_1$. We have left out from (40) the inessential terms

$$\sim i(V_F/c_1)^3\xi_1^{-1}\,\text{th}(-kL/2),$$

which define the weak damping of the mode on account of the disintegration into particle-hole pairs.

In the short-wave limit $kL \gg 1$, when the opposite surfaces hardly influence each other in practice, the dispersion equation (40) coincides with the dispersion equation for Rayleigh waves in a solid[7]:

$$(2-\xi_1^2)^2=4(1-\xi_0^2)^{1/2}(1-\xi_1^2)^{1/2}. \qquad (41)$$

The result (41) was obtained for a semi-infinite Fermi liquid by Fomin.[1] In the long-wave limit:

$$\max[(p_F L)^{-1/2},\ V_F/c_1] \ll kL \ll 1$$

Eq. (40) has the solution

$$\omega=3^{-1/2}(1-c_1^2/c_0^2)^{1/2}c_1 Lk^2, \qquad (42)$$

which coincides with the spectrum of the transverse waves in a thin elastic plate.[7]

We see thus that in the collisionless regime in the limit of high zero-sound velocities ($c_0, c_1 \gg V_F$) a thin Fermi-liquid film ($kL \ll 1$) exhibits simultaneously features of a classical liquid film and of an elastic plate. The role of the relaxation time of this liquid is played here by the parameter $\tau_0=(kV_F)^{-1}$: at $\omega \ll 1/\tau_0$ the system behaves like a classical liquid film and has a classical capillary oscillation spectrum. At $\omega \gg 1/\tau_0$ the system acts as an elastic plate that has a characteristic spectrum of transverse flexural waves. The small damping of the considered collective motions (both of the elastic and the capillary waves), however, is determined by the utterly nonclassical Landau mechanism.

Application of the developed theory to liquid ^{3}He shows that no high-frequency oscillations propagate in a liquid ^{3}He film because no transverse sound develops, as is manifest by the proximity of the velocity of the transverse sound to V_F. The failure of transverse zero sound to develop manifests itself in the fact that the stresses produced by it in the liquid cannot compensate for the corresponding stresses due to the propagation of the longitudinal zero sound ($c_0 \approx 3.5V_F$), a fact that destroys the mechanism of production of elastic waves.

4. SURFACE OSCILLATIONS OF A FERMI LIQUID OF FINITE DEPTH

With the aid of the method used in the preceding section we can analyze surface oscillations of a liquid having a finite depth L. To this end we need consider, as before, the linearized kinetic equation (22) in the region $-L/2 < x < L/2$. In the boundary conditions for this equation it must be taken into account that the bottom of the vessel ($x_b=-L/2$), in which the liquid is located, remains immobile. Thus, in the considered case we have the following boundary conditions:

$$f(p_x)-f(-p_x)=2p_x[-\mathscr{U}_0+\mathscr{F}_1 j_x(L/2,\mathbf{k})/p_F^2] \qquad (43)$$

at $x=L/2$

$$f(p_x)-f(-p_x)=0 \qquad (44)$$

at $x=-L/2$, and

$$\sigma\mathscr{U}_0 k^2=i\omega\Pi_{xx}(L/2,\mathbf{k}). \qquad (45)$$

The kinetic equation with these boundary conditions is solved by the same method as in the preceding section, i.e., by continuing the distribution function into the interval $-L<x<L$ [Eqs. (28), (29)] and using the Fourier transformations (31) and (32). The only difference is that because the boundary condition (44) at $x=-L/2$ differs from (25), it is necessary to retain in the continued kinetic equation (30) only one δ function at $x=L/2$. Proceeding in this manner, we usually find that the continued kinetic equation now takes the following form in the Fourier representation:

$$-i(\omega-\mathbf{qV})f-i\mathbf{qV}(\mathscr{F}_0\rho'+\mathscr{F}_1\mathbf{pj}/p_F^2)-2p_xV_x\mathscr{U}_0e^{-i\pi n/L}=0. \qquad (46)$$

We are considering here directly the case of the collisionless regime ($\nu=0$).

Now, as in the preceding section, we can use directly the solution of the semi-infinite problem in the Fourier representation,[2] which, by using the obvious redefinition $\mathscr{U}_0e^{-i\pi n/L}$ for $\mathscr{U}_0$, will be the solution of Eq. (46). To determine the spectrum of the oscillations it is necessary to calculate the momentum flux tensor $\Pi_{xx}(L/2,\mathbf{k})$ [see Eq. (45)]. Using again the fact that a factor $e^{-i\pi n/L}$ that arises in the redefinition of $\mathscr{U}_0$ is contained simply as a common factor in the solution of Eq. (46) [this is the consequence of the linearity of Eq. (46)], we can write down a dispersion equation, which is none other than the last boundary conditions (45), in the form

$$\sigma\mathscr{U}_0k^2=i\omega\frac{1}{2L}\sum_{n=-\infty}^{\infty}\Pi_{xx}^{(s-inf)}\left(q_x=\frac{\pi n}{L},\mathbf{k}\right), \qquad (47)$$

where $\Pi_{xx}(\mathbf{q})$ is taken to be the solution of the semi-infinite problem (36) without a redefinition of $\mathscr{U}_0$.

To investigate the spectrum of the oscillations in the low-frequency limit $|\omega/kV_F| \ll 1$, we must use the expansion of the momentum flux tensor $\Pi_{xx}^{(s-inf)}(\mathbf{q})$ in the small parameter s. We thus obtain the dispersion equation

$$\sigma k^2=i\frac{9}{16}\rho_0p_F\omega\left(\frac{kL}{\pi}\right)^4\sum_{n=-\infty}^{\infty}\left[n^2+\left(\frac{kL}{\pi}\right)^2\right]^{-5/2}. \qquad (48)$$

The addition $\sim\omega^2$ to the right-hand side of (48) turns out to be negligibly small at arbitrary kL, in contrast to (18), and has therefore been left out of (48).

In the short-wave limit $kL \gg 1$, the spectrum (48) coincides with the corresponding spectrum of a semi-infinite system (19). In the long-wave limit $kL \ll 1$ however, this spectrum also remains purely damped:

$$\omega=-i16\sigma Lk^3/9\pi\rho_0p_F \qquad (49)$$

in contrast to Eq. (20) for the free film. The reason is that the reflection of the quasiparticles from the bottom of the vessel leads to a dephasing of the surface wave. In other words, the presence of the vessel bottom, which bounds the liquid, leads already to a real rather than spontaneous violation of the translational symmetry, which does not lead to the appearance of an undamped Goldstone mode.

The use of the condition of self-consistency in an external field[8] makes it possible to consider capillary-gravitational waves in the system, i.e., surface oscillations of a Fermi liquid in the presence of a external gravitational field. It turns out that a weak gravitational field alters in standard hydrodynamic fashion the rigidity of the system [σk^2 in the left-hand side of (48) is replaced by $\sigma k^2 + m\rho_0 g$, where g is the acceleration due to gravity] and has no effect whatever on the dynamics of the oscillations, i.e., on the right-hand side of (48).

To investigate the high-frequency spectrum $|\omega/kV_F| \gtrsim 1$ it is convenient to change from summation over n in (47) to integration with respect to dq_x, in analogy with the procedure used in the preceding section [see (39)]. Using the summation formula[5]

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+z^2} = \frac{\pi}{z} \operatorname{cth} \pi z, \tag{50}$$

we easily find that

$$\frac{1}{2L} \sum_{n=-\infty}^{\infty} \varphi\left(\frac{\pi n}{L}\right) = \int_{C_q} \operatorname{cth}(iLq_x)\varphi(q_x)\frac{dq_x}{2\pi}, \tag{51}$$

where C_q is the same contour as in (39).

A particularly lucid result is obtained in the limit of high zero-sound velocities ($c_0, c_1 \gg V_F$):

$$(2-\xi_1^2)^2 \operatorname{cth}[-kL(1-\xi_0^2)^{1/2}] = 4(1-\xi_0^2)^{1/2}(1-\xi_1^2)^{1/2} \operatorname{cth}[-kL(1-\xi_1^2)^{1/2}], \tag{52}$$

where ξ_0 and ξ_1 are defined as in (40). In the short-wave limit $kL \gg 1$, Eq. (52) coincides with the dispersion law for the Rayleigh waves (41), and in the long-wave limit it has the solution

$$\omega = 2(1-c_1^2/c_0^2)^{1/2} c_1 k, \tag{53}$$

which can be easily seen to coincide with the dispersion law of longitudinal waves in an elastic plate.[7]

In liquid ^{3}He, no such high-frequency waves propagate for the reason indicated in the preceding section.

The oscillations of the considered type (just as in a Fermi liquid of finite depth) can be realized in a free Fermi-liquid film. In the case of such oscillations, the opposite surface of the film move in antiphase, and for their analysis it suffices to reverse the sign of $\Pi_{xx}(-L/2, \mathbf{k})$ in the boundary condition (26). These oscillations can be called a surface "compression" mode, although no real compression of the Fermi liquid takes place here, only flow from the troughs to the crests. The surface compression mode was purposely analyzed with a Fermi-liquid of finite depth as the example, inasmuch as observation of this mode in liquid ^{3}He is easiest to realize precisely in this case.

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