Orbital angular momentum and orbital dynamics: ^{3}He-A and the Bose liquid

G. E. Volovik and V. P. Mineev

L. D. Landau Institute of Theoretical Physics, Academy of Sciences of the USSR

General expressions for the superfluid current and the orbital angular momentum in a superfluid liquid with the structure of the A phase of ^{3}He at $T = 0$ are derived which go over in the limit $\Delta \gg \varepsilon_F$ into the corresponding expressions for a Bose liquid consisting of molecules with spin $\hbar 1$. It is shown that the local intrinsic angular momentum in ^{3}He-A, unlike the corresponding quantity in a Bose liquid, is defined only in variations, and its spontaneous part is small on account of the smallness of $(\Delta /\varepsilon_F)^2$. The Lagrangian formalism is used to construct the nonlinear orbital dynamics of ^{3}He-A at $T = 0$. It is shown that in the presence of textures in the l-vector field the dynamical equations are not Galilean invariant, which is another manifestation of the difference between ^{3}He-A and a Bose liquid. The noninvariance stems from the fact that, because of the absence of a gap in the excitation spectrum, the nonsingular textures in ^{3}He-A possess normal momenta, as a result of which the normal-component density is nonzero even at $T = 0$, and the velocity of the normal component does not drop out of the dynamical equations.

PACS numbers: 67.50.Fi

1. INTRODUCTION

The A phase of superfluid ^{3}He is formed through the Cooper pairing of the helium atoms in the state with orbital angular momentum $l = 1$ and a component of the momentum along the preferred direction equal to $l_z = 1$. The orbital part Ψ of the order parameter of ^{3}He-A is given by the spherical harmonic $Y_{11}(\mathbf{n})$ in the coordinate system with the z axis oriented along the direction of the angular momentum $\mathbf{l} = \mathbf{e}_1 \times \mathbf{e}_2$ of the Cooper pair:

$$\Psi = \Delta_0(T) Y_{11}(\mathbf{n}) = \sqrt{3/2}\,\Delta_0(T)\,(\mathbf{e}_1 + i\mathbf{e}_2,\, \mathbf{n}). \qquad (1)$$

Here $\Delta_0(T)$ is the energy gap, $\mathbf{n} = \mathbf{k}/k$ is the direction of the opposite momenta of the particles in the Cooper pair, and $\mathbf{e}_1$ and $\mathbf{e}_2$ are two orthonormal vectors. It is natural to expect the ^{3}He-A liquid as a whole to possess a spontaneous intrinsic orbital angular momentum even when it is in the completely homogeneous state. The computation of its magnitude was first carried out in Anderson and Morel's now classical paper[2] (see also Ref. 2 for a review), and has since then been the subject of numerous publications,[3-15] which have continued to appear up till now. The values obtained for the angular momentum density L lie in the range from $\hbar\rho/2$ to $\hbar\rho(\Delta_0/\varepsilon_F)^2$, i.e., they differ by up to six orders of magnitude. There are also discrepancies in the expressions for the superfluid current and in the structures of the hydrodynamic equations corresponding to the orbital degrees of freedom in ^{3}He-A.

The complexity of the indicated range of problems is due to the fact that the dimensions ξ of the Cooper pairs are significantly greater than the pair spacing a, and therefore the orbital angular momentum of the liquid is not simply the sum of the angular momenta of the Cooper pairs. Meanwhile, we should obtain upon the intensification of the pairing-inducing interaction, i.e., in the $\Delta_0 \gg \varepsilon_F$ or, equivalently, $\xi \ll a$ limit, in place of a system of Cooper pairs, a system of isolated identical molecules, each of which possesses a spin $\hbar 1$ (i.e., an A-phase structure Bose liquid), the orbital properties of which are much easier to investigate; for in the Bose case the intrinsic-angular-momentum density will be equal to the sum, $\frac{1}{2}\hbar\rho\mathbf{l}$, of the spins of all the molecules.

The idea of proceeding to the limiting Bose case belongs to Mermin and Muzikar,[16] who have obtained for the superfluid current an expression that is valid both in the weak coupling limit, which simulates the A phase, and in the very strong coupling limit, i.e., for the Bose liquid:

$$\mathbf{j} = \rho\mathbf{v}_s + \frac{1}{2}\,\mathrm{rot}\,\tfrac{1}{2}\hbar\rho\mathbf{l} - \frac{1}{2}\hbar C_0\mathbf{l}(\mathbf{l}\,\mathrm{rot}\,\mathbf{l}). \qquad (2)$$

Here $\mathbf{v}_s$ is the superfluid velocity and the coefficient C_0 varies continuously from zero in the case of the Bose liquid to the density ρ in the weak coupling limit.

The object of the present paper is to find the general features of, and the differences between, the orbital properties of the A phase and the Bose liquid, i.e., to consider from the standpoint of the indicated passage to the limit the problem of the intrinsic orbital angular momentum and the orbital dynamics. We shall show that it is precisely the deviation of C_0 from zero in the A phase of ^{3}He that gives rise to the important differences between the A phase and the Bose liquid, a fact which is reflected both in the unusual properties of the intrinsic orbital angular momentum of the liquid and in the existence in the inhomogeneous A phase of a normal component even at $T = 0$. We shall use the phenomenological approach. This approach describes the system in both limiting cases, and, what is more, our results go over in the weak coupling limit into the microscopic-theory results obtained by the Green-function or matrix-kinetic-equation method.

Not least among the causes of the astonishing discrepancies in the results of the computations of the orbital angular momentum in ^{3}He-A is the absence of simple physical ideas about the angular momentum of a liquid consisting of particles whose dimensions are much greater than their spacing, and which possess identical angular momenta $\hbar 1$. Therefore, we shall be-

gin the discussion of the orbital properties of ^{3}He-A with a discussion of a naive mechanical model.

2. A NAIVE MODEL OF THE ORBITAL ANGULAR MOMENTUM

Let us consider a Bose liquid consisting of identical molecules with density $\frac{1}{2}\rho$, identically directed orbital angular momenta $\hbar l$, and dimensions much smaller than their mean spacing. Clearly, the angular momentum of a unit volume isolated inside such a liquid will be $L = \hbar\rho l/2$ (below we set $\hbar = m = 1$, where m is the mass of the atoms of the liquid). The angular-momentum variation is equal to

$$\delta L = L_0 \delta l + \tfrac{1}{2}l\delta\rho. \tag{3}$$

Here it is reasonable to call the coefficient attached to δl (i.e., $L_0 = \frac{1}{2}\rho$) the spontaneous orbital angular momentum of the liquid; the coefficient attached to $\delta\rho$, the reaction of the angular momentum to a change in the density, or the induced angular momentum.

Let us now begin to increase the dimension ξ of the molecules in our Bose liquid, and examine whether the local intrinsic angular momentum will be defined when the dimension of the molecules becomes greater than their spacing a. For this purpose, let us isolate in the liquid some region of unit volume. This can be done in two ways, namely, the boundary of the isolated unit volume can be considered to be either fictitious, in which case it will cross some molecules, so that the volume will contain on the average $\frac{1}{2}\rho$ molecules, or real, in which case the volume will contain exactly $\frac{1}{2}\rho$ molecules. Clearly, in the second case the value of the orbital angular momentum of the isolated volume does not depend on the molecular dimensions. This fact can also be explained, using the picture of molecular currents. Let the angular momentum l of each of the molecules be produced by a molecular ring current. Then, if the volume contains exactly $\frac{1}{2}\rho$ molecules irrespective of their dimensions, even though the molecular currents inside the volume on the average cancel each other out, there exists along the surface a current that gives the angular momentum

$$\mathbf{L}_2 = \int dV[\mathbf{r} \times \mathbf{j}] = \tfrac{1}{2}\rho l.$$

In contrast, in the first case the currents cancel each other out at any point within the volume or on the surface, and $\mathbf{L}_1 = 0$. Moreover, an uncompensated part of the angular momentum arises in the first case only on account of the presence of particles whose dimension is smaller than their spacing. The fraction of such particles in the area S of the cross section of our volume in the plane perpendicular to the direction of l (the overlap of the molecules along the direction of l is unimportant) is

$$\frac{S}{\xi^2} \Big/ \frac{S}{a^2} = \frac{a^2}{\xi^2}.$$

Consequently, $\mathbf{L}_1 \approx a^2\rho l/\xi^2$. Recalling that the molecule (Cooper pair) dimension $\xi = v_F/\Delta_0$, where v_F is the velocity at the Fermi surface, we can represent the estimate for $\mathbf{L}_1$ in another form: $\mathbf{L}_1 \approx (\Delta_0/\varepsilon_F)^2\rho l$.

The responses of the local angular momentum of the liquid to changes in the density, $\delta\rho$, and direction, δl, correspond to the two cases defined above. Indeed, if we change the density of the liquid in a given volume isolated inside the liquid with the aid of a fictitious boundary by a certain amount $\delta\rho$, then the orbital angular momentum of this volume changes by $\delta L = L_2\delta\rho/\rho = \frac{1}{2}l\delta\rho$, since the number of molecules inside the given volume actually increases by $\frac{1}{2}\delta\rho$. If, on the other hand, leaving the density unchanged, we change the l of each molecule in the volume in question by a certain small amount δl, then the orbital angular momentum of this volume changes by $\delta L = |L_1|\delta l$.

Thus, the variation of the orbital-angular-momentum density for a liquid consisting of particles whose dimensions are greater than their mean spacing will have the form

$$\delta L = L_1 \delta l + L_2 \delta\rho/\rho, \tag{4}$$

so that

$$\frac{\partial L_1}{\partial\rho} \neq \frac{\partial(L_2/\rho)}{\partial l}.$$

This means that in such a liquid only the variation of the local angular momentum, and not the momentum itself, is defined.

Now we can explain the colossal spread, indicated in the Introduction, in the values obtained for L for the A phase in the various investigations. Specifically, in some papers[3,4,8] the authors computed the magnitude L_1 of the spontaneous angular momentum, which turned out to be equal (at $T = 0$ and in the absence of Fermi-liquid corrections) to

$$L_0 \equiv L_1 = \frac{1}{8}\rho\left(\frac{\Delta_0}{\varepsilon_F}\right)^2 \ln\frac{\varepsilon_F}{\Delta_0}. \tag{5}$$

In other papers either the authors calculated directly the response of the orbital angular momentum to a density change,[14] i.e., L_2, which turned out to be equal (at $T = 0$) to

$$L_2 = \tfrac{1}{2}\rho, \tag{6}$$

or they found the magnitude of the orbital angular momentum in a vessel with a fixed number of particles,[7,8,13] which is naturally also equal to $L_2 = \frac{1}{2}\rho$ at $T = 0$ [or $L_2 = \frac{1}{2}\rho_s(T)$ at temperatures different from zero].

Although Brinkman and Cross's[10] critical comments directed at Ishikawa's[7] and Mermin's[8] investigations, in which operations are performed with a singular Cooper-pair wave function Φ, are valid, the origin of the answer $L = \frac{1}{2}\rho l$ obtained in these papers is, as has already been pointed out, not at all connected with the singularity of Φ, but simply with the consideration of a vessel with a fixed number of particles. The arguments adduced in Combescot's paper,[12] where an angular momentum $\frac{1}{2}\rho l$ is assigned to each particle and $-\frac{1}{2}\rho l$, to each hole, so that $L = 0$ in the particle-hole symmetry approximation, are apparently invalid. Although they yield the correct value for the magnitude of the spontaneous angular momentum $L_0 \ll \rho/2$, they contradict our intuitive idea that the total angular momen-

tum of a system of N particles does not depend on the dimension of the pairs, and is equal to $\hbar N/2$.

Let us also mention Fomin's paper,[11] which occupies a special position. In this paper Fomin proves on the basis of general symmetry arguments the equality of the spontaneous orbital angular momentum L_0 in the A phase and the spin angular momentum S_1 in the A_1 phase of ^{3}He. The latter is also given by the expression (5). There is no induced spin angular momentum in the A_1 phase (i.e., $S_2 = 0$), for the A_1 phase is a mixture of superfluid and normal components with practically the same [to within $(\Delta_0/\varepsilon_F)^2$] density and equal [also to within $(\Delta_0/\varepsilon_F)^2$] in magnitude but oppositely directed magnetizations. Therefore, simultaneous changes in the densities of the two components do not lead to a change in the spin angular momentum.

3. INVARIANT EXPRESSION FOR THE LOCAL INTRINSIC ANGULAR MOMENTUM

The expressions (4)–(6) can be derived by a simple phenomenological method that is valid for both limiting cases of a Fermi and a Bose liquid. The following expression has been proposed by us[6] and Mermin and Muzikar[16] for the superfluid current:

$$\mathbf{j} = -\frac{1}{2}\sum_{\mathbf{k}}\mathbf{k}\left(\frac{\partial\Phi_{\mathbf{k}}}{\partial\mathbf{k}}\nabla n_{\mathbf{k}} - \frac{\partial n_{\mathbf{k}}}{\partial\mathbf{k}}\nabla\Phi_{\mathbf{k}}\right), \quad \Phi_{\mathbf{k}} = \arctan\frac{\mathbf{e}_2\mathbf{k}}{\mathbf{e}_1\mathbf{k}}. \tag{7}$$

Here $\Phi_{\mathbf{k}}$ is the phase of the pair wave function (1) and $n_{\mathbf{k}}$ is the particle distribution function. The integration by parts of (7) leads to the following expression:

$$\mathbf{j} = \frac{1}{2}\sum_{\mathbf{k}}n_{\mathbf{k}}\nabla\Phi_{\mathbf{k}} + \frac{1}{2}\nabla_i\left(\sum\mathbf{k}n_{\mathbf{k}}\frac{\partial\Phi_{\mathbf{k}}}{\partial k_i}\right) - \frac{1}{2}\sum_{\mathbf{k}}\mathbf{k}n_{\mathbf{k}}\left(\nabla\frac{\partial}{\partial\mathbf{k}} - \frac{\partial}{\partial\mathbf{k}}\nabla\right)\Phi_{\mathbf{k}}. \tag{8}$$

Using the equalities

$$\nabla\Phi_{\mathbf{k}} = 2\mathbf{v}_s + \frac{(\mathbf{k}\times\mathbf{l})e_{ijl}k_i l_j\nabla l_l}{[\mathbf{k}\times\mathbf{l}]^2}, \quad \frac{\partial\Phi_{\mathbf{k}}}{\partial\mathbf{k}} = \frac{[\mathbf{l}\times\mathbf{k}]}{[\mathbf{l}\times\mathbf{k}]^2},$$

$$\nabla\frac{\partial\Phi_{\mathbf{k}}}{\partial\mathbf{k}} - \frac{\partial}{\partial\mathbf{k}}\nabla\Phi_{\mathbf{k}} = 2\pi(\mathbf{l}\operatorname{rot}\mathbf{l})(\mathbf{k}\mathbf{l})\delta(\mathbf{k}_{\perp}), \tag{9}$$

where $\delta(\mathbf{k}_{\perp})$ is a delta function and $\mathbf{k}_{\perp} = \mathbf{k} - \mathbf{l}(\mathbf{k}\cdot\mathbf{l})$, we obtain for the current the expression (2), first obtained by Mermin and Muzikar[16]:

$$\mathbf{j} = \rho\mathbf{v}_s + \frac{1}{2}\operatorname{rot}\frac{1}{2}\rho\mathbf{l} - \frac{1}{2}C_0\mathbf{l}(\mathbf{l}\operatorname{rot}\mathbf{l}),$$

$$C_0 = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}dk_z k_z^2 n(0,0,k_z). \tag{10}$$

The orders of magnitude of the terms in (2) correspond with the orders of magnitude of the terms in (8); $k_z = \mathbf{k}\cdot\mathbf{l}$. The parameter C_0 is equal in the superfluid-Fermi-gas limit to $k_F^3/3\pi^2$, i.e., is approximately equal to the density ρ. The deviation of C_0 from ρ is due to the pairing-induced change in the density. In other words, $C_0 = \rho(\Delta_0 = 0)$ (see Ref. 16). In the limiting case of a Bose gas, in which $n_{\mathbf{k}} = \rho\delta(\mathbf{k})$, the parameter C_0 is equal to zero.

The variation $\delta\mathbf{L}$ of the local angular momentum can be found directly from the current by introducing the local changes $\delta\mathbf{l}$ and $\delta\rho$ and computing the response

$$\delta\mathbf{L} = \int d^3r[\mathbf{r}\delta\mathbf{j}]$$

under the condition that outside the volume in question $\delta\mathbf{l} = 0$ and $\delta\rho = 0$. We can use in the computation either the formula (2), or the expression (7) directly. In the latter case we have

$$\delta\mathbf{L} = \delta\left(\frac{1}{2}\rho\mathbf{l}\right) - \frac{1}{2}\sum_{\mathbf{k}}n_{\mathbf{k}}\left[\mathbf{k},\left(\delta\frac{\partial}{\partial\mathbf{k}} - \frac{\partial}{\partial\mathbf{k}}\delta\right)\Phi_{\mathbf{k}}\right] = \frac{1}{2}\mathbf{l}\delta\rho + \frac{1}{2}\delta\mathbf{l}(\rho - C_0). \tag{11}$$

Thus, we obtain for the spontaneous angular momentum L_0 the general formula

$$L_0 = \frac{1}{2}(\rho - C_0), \tag{12}$$

which yields the expected value $L_0 = \frac{1}{2}\rho$ in the limiting case of the Bose gas ($C_0 = 0$). In the limiting case of a Fermi gas with weakly interacting particles

$$L_0 = \frac{1}{2}(\rho(\Delta_0) - \rho(\Delta_0 = 0)) \sim \rho(\Delta_0/\varepsilon_F)^2. \tag{13}$$

When the Fermi-liquid effects are neglected, the coefficient attached to $(\Delta_0/\varepsilon_F)^2$ is given by the expression (5). The induced angular momentum of the liquid, i.e., the coefficient attached to $\delta\rho$ in (11) occurs in both limiting cases.

Thus, we have obtained the dependence of the local intrinsic angular momentum on the parameter C_0. In the case of the Bose liquid ($C_0 = 0$), the local angular momentum is well defined, and is equal to $\frac{1}{2}\rho\mathbf{l}$. In the case of the Fermi liquid ($C_0 \neq 0$), only the magnitude of the local-angular-momentum variation is defined. In the following section we shall confirm this conclusion by investigating the equations of the orbital dynamics. Let us point out that we can define the local angular momentum as

$$\mathbf{L} = \frac{1}{2}\mathbf{l}(\rho - C_0), \tag{14}$$

assuming that C_0 is a density-independent parameter, i.e., that C_0 is a dynamic invariant. In this case the variation of $\mathbf{L}$ with respect to ρ and $\mathbf{l}$ is given by (11).

4. THE INTRINSIC ANGULAR MOMENTUM GIVEN BY THE LINEAR DYNAMICAL EQUATIONS

It is usually thought that the subdivision of the angular momentum into a mechanical angular momentum of the current and an intrinsic angular momentum $\mathbf{L}$ has an element of arbitrariness about it. Indeed, the intrinsic angular momentum is usually eliminated from the hydrodynamic equations by redefining the current, $\mathbf{j} \rightarrow \mathbf{j} + \frac{1}{2}\operatorname{curl}\mathbf{L}$, and simultaneously symmetrizing the momentum flux tensor. The question of the intrinsic angular momentum is therefore said to have a semantic character. But in the case of systems with an order parameter with rotational degrees of freedom, there exists an equation of motion for these degrees of freedom that is invariant under current transformations, and this allows us to uniquely separate a local intrinsic angular momentum from the total angular-momentum density.

In ^{3}He-A the variables characterizing the rotational degrees of freedom are the vector $\mathbf{l}$ and the phase φ, which is at the same time the angle of rotation about the direction of $\mathbf{l}$. The hydrodynamic equations for $\mathbf{l}$ and φ, obtained in the linear approximation from the microscopic theory for $T = 0$ (see, for example, Ref.

17), have the following form:

$$\frac{\partial}{\partial t}\left(L_0\delta l-\frac{1}{4}1\left(\frac{\partial\mu}{\partial\rho}\right)^{-1}\delta\dot{\varphi}+\chi_{orb}[l\dot{\delta l}]\right)=-\left[1\frac{\delta F}{\delta 1}\right]-\frac{1}{2}1(\nabla j),$$

$$L_0=\frac{1}{8}\rho\left(\frac{\Delta_0}{\varepsilon_F}\right)^2\ln\frac{\varepsilon_F}{\Delta_0}, \qquad (15)$$

where F is the free energy. This equation is the equation of motion for the intrinsic angular momentum. Indeed, the right-hand side is equal to $-\delta F/\delta\theta$, where $\delta\theta$ is the angle of rotation of the order parameter, and is connected with φ and 1 by the relations $\delta 1 = \delta\theta \times 1$ and $\delta\varphi = -1\cdot\delta\theta$. Therefore, the left-hand side of (15) is the time derivative of the intrinsic angular momentum.

Taking account of the fact that $-\frac{1}{2}\delta\dot{\varphi}(\partial\mu/\partial\rho)^{-1}$ is none other than a change, $\delta\rho$, in the density, we obtain for the intrinsic-angular-momentum variation the expression

$$\delta L=\frac{1}{2}(\rho-C_0)\delta 1+\frac{1}{2}1\delta\rho+\chi_{orb}[l\dot{\delta l}], \qquad (16)$$

which differs from the expression (11) in that (16) contains an additional induced moment that stems not from the change in the density, but from the rotation of 1 with angular velocity $[1\cdot\dot{1}]$. The last term in (16) can also be derived from the expression (7) (see Ref. 6) if we allow for the dependence of the distribution function n_k on $\dot{1}$. Thus, the dynamical equation for the order parameter confirms the correctness of the choice of the intrinsic angular momentum in the form (11), and the intrinsic angular momentum in the A phase, being a participant in the dynamics of the rotational degrees of freedom, has a clear physical meaning, just as in the case of an angular-momentum-carrying Bose liquid, for which $C_0 = 0$.

5. GENERALIZED ANGULAR MOMENTUM IN THE LAGRANGIAN FORMALISM

Let us now show how the hydrodynamic equations can be phenomenologically derived in a manner that will make them applicable to the two limiting cases under consideration. Let us use for this purpose the Lagrangian formalism. Equation (15) can be written in the form of the Euler equation

$$\frac{\partial}{\partial t}\frac{\delta\mathcal{L}}{\delta\dot{\theta}}-\frac{\delta\mathcal{L}}{\delta\theta}=0 \qquad (17)$$

where the Lagrangian $\mathcal{L}$ is given by

$$\mathcal{L}=F-\int d^3r L^g\dot{\theta}-\frac{1}{2}\int d^3r\chi_{orb}\dot{1}^2. \qquad (18)$$

Here L^g, the generalized angular momentum, differs from L only in that the term with the spontaneous angular momentum L_0 in it has a factor $\frac{1}{2}$ attached to it; specifically,

$$\delta L^g=\frac{1}{2}L_0\delta 1+\frac{1}{2}1\delta\rho. \qquad (19)$$

For simplicity, here the terms with χ_{orb} are not included in L^g. The factor $\frac{1}{2}$ stems from the fact that the term with L_0 in Eq. (17) is varied twice: once with respect to $\dot{\theta}$ and the second time with respect to θ, and both variations yield identical contributions unified in $L_0\dot{1}$.

We have proposed[6] for L^g a phenomenological expression,

$$\delta L^g=\sum_k l_k\delta n_k, \quad l_k=\frac{1}{2}\left[k\frac{\partial\Phi_k}{\partial k}\right], \qquad (20)$$

(where δn_k is the variation of the particle distribution function), that, as we shall now show, turns out to be applicable not only to the A phase, but also to an angular-momentum-carrying Bose liquid. The formula (20) can easily be justified with the use of only the form of the phase of the pair wave function Φ_k, more exactly, its dependence on the angle of rotation $\delta\theta$. If the system rotates with angular velocity $\dot{\theta}$, then the particle energy ε_k changes by the local value of the chemical potential $\frac{1}{2}\dot{\Phi}_k= l_k\cdot\dot{\theta}$. The generalized-angular-momentum variation δL^g is obtained from the free-energy variation

$$\delta F=\sum_k(\varepsilon_k+l_k\dot{\theta})\delta n_k$$

by differentiating it with respect to the angular velocity. As a result, we obtain (20).

Let us transform the formula (20) as follows:

$$\delta L^g=\sum_k l_k\delta n_k=\delta\left(\sum_k l_k n_k\right)-\frac{1}{2}\sum_k n_k\left[k\delta\frac{\partial\Phi_k}{\partial k}\right].$$

Taking account of the fact that

$$\sum_k l_k n_k=\frac{1}{2}\rho 1,$$

$$\delta\frac{\partial\Phi_k}{\partial k}=\pi(k\times 1)[1\times\delta 1]\delta(k_\perp)+\frac{[k\times\delta 1]}{[k\times 1]^2}+2\frac{[k\times 1](k\times 1)(k\times\delta 1)}{[k\times 1]^4}$$

we have

$$\delta L^g=\frac{1}{2}\delta 1(\rho-C_0)+\frac{1}{2}1\delta\rho. \qquad (21)$$

Thus, we have obtained a formula, (19), with the same value of $L_0=\frac{1}{2}(\rho-C_0)$ as in (11). This demonstrates the applicability of the definition (20) of L^g both in the case of ^3He-A and in the case of an angular-momentum-carrying Bose liquid. Using the formula (21) for the generalized angular momentum, we construct in the following section nonlinear orbital hydrodynamics for $T = 0$.

6. NONLINEAR HYDRODYNAMICS OF THE A PHASE AT $T=0$

The nonlinear hydrodynamics of the A phase at $T = 0$ should meet the following requirements. First, the equations should, on being linearized, go over into the equations obtained from the microscopic theory. Secondly, the correct equations should be obtained in the limit of an angular-momentum-carrying Bose liquid. As to the requirement that the terms containing the velocity v_n of the normal component vanish[18] at $T = 0$, for the A phase it is in the first place apparently unfulfillable and, in the second, not obligatory. As we shall see below, the terms with v_n do not vanish when the 1 field contains textures, which play the same role played by vortices in ^4He, producing a normal component even at $T = 0$.

It is convenient, in constructing the nonlinear hydro-

G. E. Volovik and V. P. Mineev

dynamics, to write the order parameter in the form

$$(e_1 + i e_2) e^{i\Phi},$$

thereby separating out the phase Φ of the condensate. Naturally, this phase will enter into real physical quantities only in combinations with the angles of rotation of the vectors e_1 and e_2 about l. Thus, the superfluid velocity v_s has the form

$$v_s = \tfrac{1}{2}\nabla\Phi + \tfrac{1}{2}e_{1i}\nabla e_{2i} = \tfrac{1}{2}\nabla\Phi - \tfrac{1}{2}l_i\nabla\theta_i, \qquad (22)$$

which is invariant with respect to a change in Φ by $\delta\Phi(r, t)$ with simultaneous rotation of e_1 and e_2 through the angle $\delta\theta = l\delta\Phi(r, t)$, which does not change the order parameter, and, consequently, should not change v_s as well.

The Lagrangian of the system has the following general form:

$$\mathscr{L} = \mathscr{L}_0\{l, \dot{l}, \rho, v_s\} + \int d^3 r \left(\tfrac{1}{2}\dot\Phi\rho - \dot\theta L^s \right). \qquad (23)$$

It should be varied with respect to all the variables l, ρ, v_s, Φ, L^s and θ under the conditions (21) and (22), as well as the condition $\delta l = \delta\theta \times l$. Notice that the constraint (21) has a nonholonomic character. For definiteness, we concretize $\mathscr{L}$ in the form

$$\mathscr{L}_0 = -\tfrac{1}{2}\chi_{orb}(\dot{l} + (v_s\nabla)l)^2 + F(\rho, v_s, l).$$

The standard variation of the Lagrangian under the conditions of the nonholonomic constraints leads, after the elimination of the Lagrange multipliers, to the following three Euler equations for l, ρ, and v_s:

$$L_0\dot{l} + \chi_{orb}\left([l, D^2 l] + (\nabla v_s)[l, Dl] - \tfrac{1}{2}\nabla_i l(\nabla_i l, Dl) \right)$$
$$+ \left[l \frac{\delta F}{\delta l} \right] + \tfrac{1}{2}\left(\frac{\partial F}{\partial v_s}\nabla \right) l, \qquad (24)$$

$$\dot\rho + \nabla\left(\frac{\partial F}{\partial v_s} - \chi_{orb}Dl_i\nabla l_i \right) = 0, \qquad (25)$$

$$\dot{v}_s = -\nabla\frac{\partial F}{\partial\rho} - \tfrac{1}{2}e_{ikl}l_i\nabla l_k\dot{l}_l, \qquad (26)$$

where $D = \partial/\partial t + v_s\nabla$ is the total-derivative operator. These equations in the linear approximation coincide with the equations obtained microscopically.

Let us now investigate how the Eqs. (24)–(26) for the Bose liquid and ^{3}He-A differ. In both cases the free energy has the form

$$F = \int d^3 r \left(\varepsilon(\rho, l) + \tfrac{1}{2}\rho v_s^2 + \tfrac{1}{2}v_s \operatorname{rot}\frac{\rho l}{2} - \tfrac{1}{2}C_0(lv_s)(l\operatorname{rot}l) \right), \qquad (27)$$

while the current

$$j = \rho v_s + \tfrac{1}{2}\operatorname{rot}\frac{\rho l}{2} - \tfrac{1}{2}C_0 l(l\operatorname{rot}l) - \chi_{orb}Dl_i\nabla l_i. \qquad (28)$$

As can be seen, in the nonlinear case the expression for the current contains, besides the terms entering into (2), a term connected with the dependence of the l field on the time. The same term characterizes the

case of the spin dynamics of the A and B phases,[18] where it includes the time derivatives of the angles of rotation (or the spin density). The expression for ε has the standard form:

$$\varepsilon(\rho, l) = \varepsilon_0(\rho) + \tfrac{1}{2}K_1(\nabla l)^2 + \tfrac{1}{2}K_2(l\operatorname{rot}l)^2 + \tfrac{1}{2}K_3[l\operatorname{rot}l]^2.$$

Naturally, the K coefficients in the l-dependent part $\varepsilon(\rho, l)$ of the energy have different values in the various limiting cases. This is, however, not the qualitative difference. The important difference stems from the last term in (27), which contains C_0, and, hence, vanishes in the case of the Bose liquid. Because of this term, Eq. (24) for ^{3}He-A turns out to be noninvariant under the Galilean transformation

$$r \rightarrow r - ut, \quad v_s \rightarrow v_s + u,$$

i.e., there appears on the left-hand side of (24) a correction equal to

$$\tfrac{3}{2}C_0(ul)(l\nabla)l. \qquad (29)$$

The Eqs. (25) and (26) are Galilean invariant if we consider C_0 to be an invariant of the motion.

Thus, the same coefficient C_0 that gives rise to the important difference between the orbital angular momenta of the A phase and the Bose liquid leads to the violation of Galilean invariance if

$$(l\nabla)l = -[l\operatorname{rot}l] \neq 0.$$

Furthermore, it can be shown that, for $C_0 \neq 0$, the Eqs. (24)–(26) do not satisfy the law of conservation of current. Indeed, we can, by differentiating the current j given by (28) with respect to the time and using Eqs. (24)–(26), easily verify that, for $C_0 \neq 0$, the time derivative of j cannot be represented in the form $\nabla_k\pi_{ik}$, i.e., that

$$\frac{\partial j_i}{\partial t} = \nabla_k\pi_{ik} + \frac{3}{2}C_0 l_i[l\dot{l}](l\nabla)l.$$

This paradox can be eliminated by introducing a normal component.

7. DENSITY OF THE NORMAL COMPONENT AT $T = 0$

The presence of a normal component in the A phase at $T = 0$ is not connected with the shortcomings of our method; it is a property of the A phase, stemming directly from the absence of a gap in the excitation spectrum with $k_\perp = 0$. Indeed, the fact that the system is not Galilean invariant in the presence of a texture with $(l \cdot \nabla)l \neq 0$ indicates the existence of a preferred coordinate system in which the texture is at rest. The nonconservation of current indicates that there is associated with the moving texture an additional nonsuperfluid current that has not been considered by us. Thus, the textures play the role of impurities, producing a normal component even at $T = 0$. The normal component in ^{3}He-A consists of quasiparticles that, owing to the absence of a gap in the $k_\perp = 0$ spectrum even at $T = 0$, collect in the potential wells produced by the textures. The Bose liquid does not contain a normal component, is Galilean invariant, and conserves the superfluid

current. The density of the normal component of the A phase at $T = 0$ can easily be estimated.

Let us use the expression for the excitation spectrum in the A phase:

$$E_k = (\xi_k^2 + \Delta_0^2 [kl]^2/k^2)^{1/2} + \frac{1}{2}(kv)\Phi_k, \qquad (30)$$

$$(kv)\Phi_k = 2kv_s + (kl)[kl](kv)l/[kl]^2. \qquad (31)$$

The quasiparticle distribution function ν_k at $T = 0$ is a θ function of $k \cdot v_n - E_k$, where v_n is the velocity of the normal component, and is different from zero at arbitrarily small gradients of l and arbitrarily small $v_n - v_s$ because of the absence of a gap at $k_\perp = 0$. As a result, there always exists in the presence of a texture an excitation current

$$P = \sum_k k\theta(kv_n - E_k). \qquad (32)$$

Varying P with respect to $v_n - v_s$, we obtain in the linear (in $v_n - v_s$) approximation the magnitude of the normal density:

$$P_i = \rho_{ik}^n (v_k^n - v_k^s), \quad \rho_{ik}^n = \sum_k k_i k_k \delta(E_k), \qquad (33)$$

where E_k no longer depends on v_s. The evaluation of the sum in (33) yields

$$\rho_{ik}^n = \rho \frac{v_F}{\Delta_0} l_i l_k |(l\nabla)l|. \qquad (34)$$

Thus, the presence of a texture with $(l \cdot \nabla)l \neq 0$ indeed leads to the appearance of a normal component in the A phase even at $T = 0$. The nonanalytic form of ρ^n as a function of the gradients should be noted. The existence of a $\rho^n(T = 0)$ and the difference of C_0 from zero are closely connected facts. They are each a consequence of the singularity of the pair wave function at $k_\perp = 0$: ρ^n arises on account of the vanishing of the gap at $k_\perp = 0$ and the related nonanalyticity, and C_0 is nonzero because of the δ-function singularity at $k_\perp = 0$. Let us also note that the coefficient K_3 attached to the term $[1 \times \text{curl } 1]^2 = ((l \cdot \nabla)l)^2$ in the energy $\varepsilon(\rho, 1)$ has a logarithmic singularity caused by the vanishing of the gap at $k_\perp = 0$ (Refs. 3 and 17). It can be seen directly from the expression (34) that $\rho^n \to 0$ in the Bose liquid (i.e., for $\Delta_0 \to \infty$).

The existence of the new variable v_n at $T = 0$ requires another equation for this variable. Such an equation is the momentum conservation law, which is no longer a consequence of the Eqs. (24)–(26), as it was in the case of the Bose liquid, but an independent equation. The derivation of this equation, as well as the extension of the results obtained and the generalization of the equations to the case of nonzero temperatures require further investigations. Here we only note that Eq. (24) is valid in the coordinate system in which $v_n = 0$, and should in the general case be supplemented by the term $\frac{3}{2}C_0(v_n \cdot 1)(1 \cdot \nabla)1$ in order to secure Galilean invariance. But under the conditions of a slowed-down normal component, i.e., for $v_n = 0$ (fourth-sound or pinned-texture regime), the system of equations (24)–(26) constitutes a complete system.

8. CONCLUSION

We have elucidated the differences between the properties of a superfluid Fermi liquid and a superfluid Bose liquid with identical order parameters with the structure of the A phase of ^{3}He. Although the possession of the same symmetry by the order parameters leads, as is well known,[19] to identical global superfluid properties (identical topologically stable defects and textures, identical properties of the persistent superflows, identical mechanisms of the nonstationary Josephson effect, identical vortex structures in a rotating vessel), the orbital dynamics of these liquids are significantly different. In the limit of a slightly nonideal Fermi gas with the structure of the A phase of ^{3}He, the singularities occurring in the excitation spectrum and the density of states give rise to new physical phenomena:

1) the expression for the current contains an additional term in comparison with the expression for the current in the Bose liquid;

2) the local orbital angular momentum of the liquid turns out to be defined only in variations, and its spontaneous part is small on account of the smallness of $(\nabla_0/\varepsilon_F)^2$;

3) the nonsingular textures possess normal momenta, as a result of which the density of the normal component does not vanish even at $T = 0$, and the velocity v_n does not drop out of the dynamical equations.

New problems can arise at $T \neq 0$. Let us briefly touch upon one of them. The dynamics of the A_1 phase has recently been intensively studied. It has been shown[20,21] that, although the angle of rotation φ of the spin vectors of the order parameter about the magnetic field and the phase Φ of the wave function enter into the order parameter in an invariant fashion as $\varphi + \Phi$, the dynamics of the variables φ and Φ are different. The φ dynamics is connected with the spin oscillations; the Φ dynamics, with the density oscillations. The responses of the spin S_z and the density ρ to $\dot{\varphi}$ and $\dot{\Phi}$ have the form:

$$\delta S_z = a_{11}\delta\dot{\varphi} + a_{12}\delta\dot{\Phi}, \quad \delta\rho = a_{21}\delta\dot{\varphi} + a_{22}\delta\dot{\Phi},$$

where the cross terms a_{12} and a_{21} are equal to zero in the A_1 phase, $a_{11} = \chi_\parallel$ is the spin susceptibility, and $a_{22} = -\frac{1}{2}\partial\rho/\partial\mu$.

A similar situation for the orbital angular momentum may occur in the A phase. Although the angle φ of rotation about 1 and the phase Φ enter into the order parameter in an invariant fashion, their dynamics may differ. The following equations obtain in the general case:

$$\delta L_z = a_{11}\delta\dot{\varphi} + a_{12}\delta\dot{\Phi}, \quad \frac{1}{2}\delta\rho = a_{21}\delta\dot{\varphi} + a_{22}\delta\dot{\Phi}.$$

At $T = 0$ we have

$$a_{11} = a_{21}, \quad a_{12} = a_{22}$$

[see (15) and (16)], so that $\delta L_z = \frac{1}{2}\delta\rho$. In the $T \neq 0$ case it may turn out that there exists a temperature range in which the two $\delta\rho$ and δL_z motions are independent. The oscillations of the z component of the angular momen-

tum at constant density will in this case be connected with temperature oscillations (second sound), like the longitudinal-magnetization oscillations in the A_1 phase that were predicted by Gurgenishvili and Kharadze[20] and Liu,[21] and experimentally observed by Corruccini and Osheroff.[22] That δL_z and $\frac{1}{2}\delta\rho$ become unbalanced at $T \neq 0$ can be discerned from Nagai's[23] results, which show, in particular, that $a_{12}/a_{22} = 1 - Y(T)$, where $Y(T)$ is the Yosida function, equal to unity at $T = T_c$ and zero at $T = 0$.

[1]P. W. Anderson and P. Morel, Phys. Rev. **123**, 1911 (1961).

[2]A. J. Leggett, Rev. Mod. Phys. **47**, 331 (1975).

[3]M. C. Cross, J. Low Temp. Phys. **21**, 525 (1975).

[4]G. E. Volovik, Pis'ma Zh. Eksp. Teor. Fiz. **22**, 234 (1975) [JETP Lett. **22**, 108 (1975)].

[5]G. E. Volovik, Pis'ma Zh. Eksp. Teor. Fiz. **22**, 412 (1975) [JETP Lett. **22**, 198 (1975)].

[6]G. E. Volovik and V. P. Mineev, Zh. Eksp. Teor. Fiz. **71**, 1129 (1976) [Sov. Phys. JETP **44**, 591 (1976)].

[7]M. Ishikawa, Prog. Theor. Phys. **55**, 2014 (1976); **57**, 1836 (1977).

[8]N. D. Mermin, Physica (Utrecht) **90B+C**, 1 (1977).

[9]M. C. Cross, J. Low Temp. Phys. **26**, 165 (1977).

[10]W. F. Brinkman and M. C. Cross, in: Progress in Low Temperature Physics, Vol. VII A (ed. D. F. Brewer), North Holland Publ. Company, 1978.

[11]A. I. Fomin, Phys. Lett. A **66**, 47 (1978).

[12]R. Combescot, Phys. Rev. B **18**, 6071 (1978).

[13]M. G. McClure and S. Takagi, Phys. Rev. Lett. **43**, 596 (1979).

[14]M. Ishikawa, Prog. Theor. Phys. **63**, 338 (1980).

[15]K. Miyake and T. Usui, Prog. Theor. Phys. **63**, 711 (1980).

[16]N. D. Mermin and P. Muzikar, Phys. Rev. B **21**, 980 (1980).

[17]M. Ishikawa, Prog. Theor. Phys. **57**, 1 (1977).

[18]R. Combescot and T. Dombre, Phys. Lett. A **76**, 293 (1980).

[19]G. E. Volovik and V. P. Mineev, Fizika i topologiya, Ser. Fiz. (Physics and Topology, Physics Series), Vol. 6, Znanie, Moscow, 1980.

[20]G. E. Gurgenishvili and G. A. Kharadze, Pis'ma Zh. Eksp. Teor. Fiz. **31**, 593 (1980) [JETP Lett. **31**, 557 (1980)].

[21]M. Liu, Phys. Rev. Lett. **43**, 1740 (1979).

[22]L. R. Corruccini and D. D. Osheroff, Phys. Rev. Lett. **45**, 2029 (1980).

[23]K. Nagai, J. Low Temp. Phys. **38**, 677 (1980).

Translated by A. K. Agyei