A relativistic Lagrangian for a hot magnetized plasma and its adiabatic invariant

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We provide a relativistic Lagrangian for a hot magnetized plasma in arbitrary motion. This manifestly covariant Lagrangian can serve as a useful basis for relativistic geometrical optics. It gives rise to a conserved pseudophoton current, also called wave vector action, and allows for a covariant treatment of nonlinear wave–wave interactions.

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§1. THE RELATIVISTIC EIKONAL METHOD

Generally one cannot solve Maxwell's equations exactly for waves propagating through an inhomogeneous medium. The problem becomes even more complex if one wishes to do this for a medium in accelerated motion or in curved space-time, as is for example necessary if one wishes to describe wave propagation through a pulsar magnetosphere or through the accreting gas of a black hole. The standard treatments using plane wave solutions to describe wave propagation in a homogeneous plasma in a flat space time[1] are no longer applicable. In the geometrical optics approximation,[2] where the wavelengths λ are small in comparison with the "inhomogeneity scale" l, a refined version of Hamilton's theory of rays,[3] first described by Weinberg,[4] leads to radiative-transfer equations which determine the change in wave vector, polarization state, and intensity.

In this eikonal method one ignores internal reflection. A proof does not exist, but the general conjecture is that the amplitudes of the reflected waves go to zero as $\exp(-l/\lambda)$. Thus they cannot be obtained by a method which is essentially a power series expansion in λ/l. In the eikonal method it is assumed that the four-potential of the waves $\delta A^i(x)$ can be written as

$$\delta A^k(x)=f(x)a^k(x)e^{i\psi(x)}, \tag{1.1}$$

where the scalar f is the amplitude, the suitably normalized four-vector $a^k(x)$ is the polarization vector, and $\psi(x)$ is the eikonal. $f(x)$ and $a^k(x)$ are slowly varying functions of x, whereas the eikonal $\psi(x)$ describes the rapid oscillations of the waves, $\psi_{;i}:=k_i$, where k_i is the wave four-vector. The four components of k_i are not independent, but for a given medium they are related by a scalar relation which is characteristic for the medium and which is called the dispersion relation,

$$D=D(k_i, x_j)=0.$$

In order to solve the eikonal equation

$$D(\psi_{;i}, x_j)=0$$

and to construct $\psi(x)$ one used the method of ray tracing,[4] i.e. one solves for the ordinary differential equations

$$\begin{gathered}\dot{x}^i=dx^i/dl=\partial D/\partial k_i,\\ \dot{k}^i=k^i{}_{;j}\dot{x}^j=-\partial D/\partial x_i,\quad k_{i;j}=k_{j;i},\end{gathered} \tag{1.2}$$

and $\psi(x)$ is then given by

$$\psi(x)=\int k_i\dot{x}^i\,dl. \tag{1.3}$$

Locally the waves described by (1.1) are therefore plane waves, and the long-range effects of the medium and the geometry are taken into account by propagation laws. The most general way of describing wave propagation would certainly be that of phenomenological electrodynamics[3,5] where one uses the induction tensor δH^{ab} together with the field tensor

$$\delta F^{ab}=\delta A^{b;a}-\delta A^{a;b}.$$

The tensor δH^{ab} satisfies the field equation

$$\delta H^{ab}{}_{;b}=\frac{4\pi}{c}j^a_{ext} \tag{1.4}$$

and is related to δF^{ab} by the permeability four-tensor. To lowest order eikonal approximation this relation can be written[5]

$$\delta H^{ab}(x)=\varepsilon^{ab}{}_{cd}(x,k)\delta F^{cd}(x), \tag{1.5}$$

and, in the absence of external currents, leads to

$$\begin{gathered}\delta F^{ab}=if(k^a a^b-k^b a^a)e^{i\psi},\\ \delta H^{ab}k_b=if\varepsilon^{ab}{}_{cd}k_b(k^c a^d-a^c k^d)e^{i\psi}=0.\end{gathered} \tag{1.6}$$

The trivial symmetry relations for ε^{ab}_{cd} are

$$\varepsilon^{ab}{}_{cd}=-\varepsilon^{ba}{}_{cd}=-\varepsilon^{ab}{}_{dc},$$

and for a loss-free medium due to the Onsager relations

$$\varepsilon_{abcd}=\varepsilon^{*}_{cdab}, \tag{1.7}$$

where the asterisk denotes the complex conjugate quantity. Introducing $L^{ad}=\varepsilon^{abcd}k_b k_c$, Eq. (1.6) can be written

$$L^{ad}a_d=0,\quad L^{ad}=(L^{da})^*. \tag{1.8}$$

Due to the symmetry relations of ε, this equation has the trivial solution $a_d=k_d$. This implies that $\det\|L^{ad}\|=0$, i.e. $\det\|L^{ad}\|=0$ is not the dispersion relation. It can be shown[5] that a covariant dispersion relation is obtained if one chooses a specific gauge for a_d, for example $a_d k^d=0$, and adds this gauge condition

$$L^{ad}a_d \rightarrow (L^{ad}+k^a k^d)\, a_d$$

to $L^{ad}a_d$. $\mathrm{Det}\|L^{ad}+k^a k^d\|=0$ is then the dispersion relation (including the spurious mode $k_i k^i=0$ corresponding to $k^i a_i=0$).

Equation (1.8) determines further the polarization vector a^i (up to a gauge mode). In order to determine the amplitude f, however, dynamical quantities are needed (and one has to proceed to first order approximation to obtain these).

In geometric optics[3] Landau and Lifshitz use the physical argument that in a loss-free medium energy must be conserved on the average, leading to the requirement div S=0 from which the amplitude can be obtained by simple ray tracing since S points in the direction of the group velocity. In a medium in arbitrary motion energy is not conserved but the number of "photons," as represented by the numbers of rays, still is. Multiplying Eq. (1.8) by a_a^* we obtain

$$\bar{L}=a_a^* L^{ad} a_d=0.$$

It is easy to show that $\bar{L}$ is proportional to the dispersion relation,[5] and in geometrical optics the space part of

$$N^i=\frac{\partial}{\partial k_i}\bar{L}=a_a^* a_d \frac{\partial}{\partial k_i} L^{ad} \tag{1.9}$$

coincides with $\omega^{-1}\mathbf{S}$, i.e. it satisfies $N^{\alpha}{}_{;\alpha}=0$. The relativistic generalization of the conserved current therefore will be (1.9), if we can show that $N^i{}_{;i}=0$ in general. N^i is sometimes called wave-vector action, and its conservation gives rise, in the standard manner, to the adiabatic invariant $I=N^i d\Sigma_i=N^0 dV$, where dV is an aribtrary, infinitesimal volume element which is propagated along the rays of Eq. (1.2). In linear theory the average energy-momentum tensor $\bar{T}^{ik}$ is related to N^k as follows[5,6]

$$\bar{T}^{ik}=k^i N^k. \tag{1.10}$$

Now suppose the physical system under consideration (crystal, plasma, etc.) can be derived from a Lagrangian. In this case Eq. (1.10) is nothing but the averaged canonical energy momentum tensor,[2] and for the divergence of $\bar{T}^{ik}$ we would obtain

$$\bar{T}^{ik}{}_{;k}=-\bar{L}^{;i}. \tag{1.11}$$

Using $\bar{L}=0$ as the dispersion relation in (1.2) then immediately leads to ($\bar{L}^{;i}$ is the functional and not the total derivative with respect to x^i)

$$N^i{}_{;i}=0, \quad N^i=\frac{\partial}{\partial k_i}\bar{L}. \tag{1.12}$$

The method of an averaged Lagrangian, which was first considered by Whitham[7] and subsequently investigated by many authors, for example,[8,9] is therefore an elegant and powerful means to provide a manifestly covariant basis for relativistic geometrical optics. [Note that N^α was introduced for the first time by Sturrock[6] (1962), although his work did not have much influence on later development].

However, the price one pays is a loss of generality in that one does not always know how to construct a Lagrangian for a given physical system. Moreover, if one is interested in the phase of the amplitude one has to take recourse to the full system of differential equations.[4,5] On the other hand, if one has a Lagrangian at hand, many results follow with considerable ease from its invariance properties via E. Noether's theorems. In particular, Eq. (1.12) is a direct consequence of the eikonal ansatz (1.1) and holds rigorously and not only to lowest order eikonal approximation.

As mentioned above, to establish Eq. (1.12) ε^{ab}_{cd} must be known to first order eikonal approximation. This fact was overlooked by the authors of Ref. 5, and their result concerning the divergence of N^i is wrong.

A second gratifying aspect is that our treatment can immediately be extended to curved space-time.

The outline of the paper is as follows. In the §2 we review the previous work of Sturrock[10] and F. E. Low,[11] and we combine the procedure of both into a relativistic Lagrangian for a hot, magnetized plasma in arbitrary motion. We show that this Lagrangian leads to the correct "field" equations to all orders. Unfortunately, this Lagrangian is a hybrid in that its electromagnetic part is a density in the four-dimensional position space whereas the particle and interaction parts are densities in the seven-dimensional phase space. This leads to some modifications of the proof of Eq. (1.11) which are also given in §2. In §3 we explore the consequences of the eikonal ansatz (1.1) and go over to the averaged Lagrangian. In the following sections we show how useful physical results can be derived if the Lagrangian is expanded in powers of the small amplitudes of the fields. In §4 we drive the linearized Boltzmann-Vlasov equation from the requirement of gauge invariance. In §5 we derive the permeability tensor and prove the conservation of wave-vector action for a hot, magnetized plasma in arbitrary motion. In the final section we show how nonlinear phenomena can be treated and rederive the relativistic Manley-Rowe relations for nonlinear three-wave interaction.

§2. THE FIELD LAGRANGIAN FOR PARTICLE DISPLACEMENTS

It is well known how to describe particle motion in given exterior fields by means of a variational principle and so is the representation of field equations for the electromagnetic field by similar principles.[2] The synthesis of both into one single principle is complicated by the fact that particles and fields are described by different types of variables: "Lagrangian" for the particles and "Eulerian" for the fields.[8] Sturrock[10] and Low[11] have devised a procedure to overcome this difficulty at least in part. They introduce displacements ξ^k which describe the motion or particles under the action of a perturbation field such that the unperturbed trajectory $x^k = x^k(s)$ goes over into $\tilde{x}^k = x^k + \xi^k$. The displacement ξ^k will be a "field-like" variable if we require that $\tilde{x}^k=x^k+\xi^k(x)$. It is well known that such a labelling is not unique since the transformation

$$\tilde{\xi}^k(x)=\Delta x^k+\xi^k(x+\Delta x), \tag{2.1}$$
$$\tilde{\xi}^k=\xi^k+ds(u^k+\xi^k{}_{;l}u^l),$$

in which ds may be regarded as an arbitrary infinitesimal function, generates a set of representations of the same physical system.[10] This gives rise to a further gauge invariance of the theory (apart from the indeterminacy of the potentials of the electromagnetic field[2]) and leads to a "strong conservation law",[23,24] which, as we shall show, is just the conservation of particles for each species (whereas the gauge invariance of the electromagnetic potentials leads only to charge conservation of the total charge,[2] which is weaker, and only for a charge-separated plasma do the two coincide).

The result of the procedure, however, is formal, and only if the displacements are small so that the Lagrangian allows a power series expansion in the displacement amplitude does one obtain physically useful Lagrangians which can be treated in the standard manner.[2] However, the advantage of the more formal treatment is that it leads to manifestly covariant expressions, valid rigorously, and that it allows one to investigate easily the invariance properties of the Lagrangians to all orders of approximation.

Sturrock starts from the particle action

$$S=\sum_A S_{mA}+S_{mfA},$$

where the sum over A is over the different charged-particle species and where[2]

$$S_{mA}=-c\sum_{l=1}^{N_A} m_A\int ds_{Al}, \tag{2.2}$$

$$S_{mfA}=\frac{1}{c}\sum_{l=1}^{N_A} e_A\int A_k(x_{Al})\,dx_{Al}{}^k=\frac{1}{c}\int j_k(x)A^k(x)\,d^4x. \tag{2.3}$$

Henceforth we shall drop the index A and consider each kind of particles separately; in the final result one can easily restore the sum over all species. The equations of motion for the background plasma are (for each species)

$$b^i=u^i{}_{;k}u^k=\frac{e}{mc^2}F^i{}_k u^k. \tag{2.4}$$

The electromagnetic field equations follow from the total action $S_{tot}=S+S_f$

$$S_{tot}=S+S_f,$$

$$S_f=-\frac{1}{16\pi}\int F^{ik}F_{ik}\,d^4x,\quad F_{ik}=A_{k;i}-A_{i;k} \tag{2.5}$$

and read[2]

$$F^{ik}{}_{;k}=\frac{4\pi}{c}j^i. \tag{2.6}$$

If, under the action of a perturbation force, the particles at x go to $\tilde{x}$, the new equations will follow from the actions

$$\tilde{S}=-mc\sum_{l=1}^{N}\int d\tilde{s}_l+\frac{e}{c}\sum_{l=1}^{N}\int \tilde{A}_k\,d\tilde{x}_l{}^k \tag{2.7}$$

and

$$\tilde{S}=-\frac{1}{16\pi}\int \tilde{F}^{ik}\tilde{F}_{ik}\,d^4x+\frac{1}{c}\int \tilde{j}^i(x)\tilde{A}_i(x)\,d^4x \tag{2.8}$$

respectively, and give

$$\tilde{b}^i=\tilde{u}^i{}_{;k}\tilde{u}^k=\frac{e}{mc^2}\tilde{F}^i{}_k\tilde{u}^k,\quad \tilde{u}^i=\tilde{u}^i(s),\quad \tilde{u}^i\tilde{u}_i=-1, \tag{2.9}$$

$$\tilde{F}^{ik}{}_{;k}=\frac{4\pi}{c}\tilde{j}^i(x). \tag{2.10}$$

We derive now the above equations from the following "field" Lagrangian:

$$S=-\frac{1}{16\pi}\int \tilde{F}_{ik}\tilde{F}^{ik}d^4x+\frac{e}{c}\int N(x,u)\,(u^i+\dot{\xi}^i)\,\tilde{A}_i(x+\xi)\,d^4u\,d^4x$$
$$-mc\int N(x,u)\,[-(u^i+\dot{\xi}^i)(u_i+\dot{\xi}_i)]^{1/2}d^4u\,d^4x. \tag{2.11}$$

Here $\xi^i=\xi^i(x,u)$ has been extended to cover the hot plasma case where particles pass with different four-velocity u^i, $\xi^i=\xi^i(x,u)$ through each point in coordinate space. Consequently, the total derivation of ξ^i is now

$$\dot{\xi}^i=\frac{D}{ds}\xi^i=\xi^i{}_{;k}u^k+\xi^i{}_{,uk}b^k,\quad \xi^i{}_{,uk}=\frac{\partial}{\partial u^k}\xi^i. \tag{2.12}$$

$N(x,u)$ is the distribution function of the background plasma. We shall show below that due to the gauge invariance under transformation (2.1) N must obey the Boltzmann-Vlasov equation

$$\dot{N}=\frac{D}{ds}N=N_{;i}u^i+N_{,ui}b^i=0, \tag{2.13}$$

where b^i is given by (2.4) so that $b^i{}_{,ui}=0$. Note that instead of working in the physical seven-dimensional, curved phase space we prefer to work in a fictitious[12] eight-dimensional flat phase space by taking into account the identity $\tilde{u}^i\tilde{u}_i=-1$ by means of a δ function and the fact that $\tilde{u}^0\geq 1$ by means of a θ-function. Our function N is related to the usual f_0 by

$$N(x,u)=2\theta(u^0)\delta(-1-u^iu_i)f_0(x,u), \tag{2.14}$$

and the four-current j^i is defined

$$j^i(x)=\int u^iN(x,u)\,d^4u. \tag{2.15}$$

Integrating (2.12) over du^0, using $b^0{}_{,u0}=0$ and that N vanishes fast enough in velocity space, i.e. $u^0N(x,u)\to 0$ as $u^0\to\infty$, we obtain the well-known form of the Boltzmann-Vlasov equation for $f_0(x,u)=f(x,v/c)$, with $\gamma=u^0$ the Lorentz factor

$$\frac{\partial}{c\partial t}f+\frac{\mathbf{v}}{cv}\nabla f+(\nabla_v f)\frac{e}{mc^2}\left(\mathbf{E}+\left[\frac{\mathbf{v}}{cv}\times\mathbf{B}\right]\right)=0, \tag{2.16}$$

and (2.14) becomes

$$\rho=\int f(x,v)\,d^3v,\quad j^\alpha=\int v^\alpha f(x,v)\,d^3v$$

as it must.

As we are considering only electromagnetic forces, Eq. (2.4) or (2.9) guarantees that the "current" $j^A=(Nu^a,Nb^a)$ in phase space is conserved

$$j^A{}_{;A}=(Nu^a)_{;a}+(Nb^a)_{,ua}=0,$$

which in turn guarantees particle-number conservation

$$dN=Nu^0d^3x\,d^4u=\tilde{N}\tilde{u}^0d^3\tilde{x}\,d^4\tilde{u} \tag{2.17}$$

under the action of a perturbation force. $\tilde{N}=\tilde{N}(\tilde{x},\tilde{u})$ is the new distribution function and obeys

$$\frac{D}{d\tilde{s}}\tilde{N}(\tilde{x},\tilde{u})=0, \tag{2.18}$$

where $D/d\tilde{s}$ is to be taken along the perturbed orbit. It does not seem to be possible to derive Eq. (2.18) from the defining equation (2.17)

$$\tilde{N}(\tilde{x},\tilde{u})=N(x,u)\frac{d\tilde{s}}{ds}\frac{\partial^4 x}{\partial^4\tilde{x}}\frac{\partial^4 u}{\partial^4\tilde{u}}, \qquad (2.19)$$

but we shall show later that in the linearized case this is in fact possible, i.e. the linearized Vlasov equation follows from gauge invariance under transformations (2.1) and the equations of motion (2.9). Let us postulate then (2.17) for the time being. It is easy to show then that (2.11) leads back to (2.7) and (2.8), respectively, which guarantees Lorentz invariance of our "Lagrangian" and also its gauge invariance under transformations (2.1).

Let us next show that a variation of (2.11) with respect to the "field" $\xi^i(x,u)$ leads to Eq. (2.9). The variation is to be performed in phase space and leads to

$$\frac{\delta\hat{L}}{\delta\xi^i}=\left(\frac{\partial\hat{L}}{\partial\dot{\xi}^i}\right)^{\cdot}-\frac{\partial\hat{L}}{\partial\xi^i}=\left(\frac{\partial\hat{L}}{\partial\xi^i_{;a}}\right)_{;a}+\left(\frac{\partial\hat{L}}{\partial\xi^i_{,ua}}\right)_{;ua}-\frac{\partial\hat{L}}{\partial\xi^i}=0, \qquad (2.20)$$

where

$$\hat{L}=-mcN[-(u^i+\dot{\xi}^i)(u_i+\dot{\xi}_i)]^{1/2}+\frac{e}{c}N(u^i+\dot{\xi}^i)\tilde{A}_i(x+\xi)=N\check{L}, \qquad (2.21)$$

and the definition for $\dot{\xi}^i$ [Eq. (2.12)] should be remembered. We find

$$\frac{\delta\hat{L}}{\delta\xi^i}=N\frac{d\tilde{s}}{ds}\left(mc\frac{D}{d\tilde{s}}\tilde{u}^i-\frac{e}{c}\tilde{F}^{ik}\tilde{u}_k\right)+\dot{N}G^i=0,$$
$$G^i=mc\tilde{u}^i+\frac{e}{c}\tilde{A}^i(\tilde{x}). \qquad (2.22)$$

Invariance under the gauge transformation (2.1) leads to (2.22) contracted by $\tilde{u}_i$ as a strong conservation law, (i.e. for arbitrary $\tilde{u}$ and $\tilde{A}$), which implies

$$\dot{N}=0,\quad \delta\check{L}/\delta\xi=0, \qquad (2.23)$$

as we mentioned earlier. Equation (2.22) agrees with that of Sturrock[10] derived for the cold plasma, and in the nonrelativistic limit our Lagrangian (2.11) goes over into Low's Lagrangian.[11] Note that variation with respect to $\tilde{A}(x)$ is trivial if one uses the equivalent form (2.8) which goes over into (2.11) by means of (2.19) and a relabelling of the coordinates

$$\tilde{F}(x)^{ik}{}_{;k}=\frac{4\pi}{c}\tilde{j}^i(x)=\frac{4\pi e}{c}\int\tilde{N}(x,u)u^i d^4u. \qquad (2.24)$$

Invariance of the total action under coordinate transformations

$$x^i\to x^i+\varepsilon^i,$$

with constant ε^i leads to the pseudo-energy-momentum tensor

$$T^{ik}=A_l{}^{;i}\frac{\partial}{\partial A_{l;k}}\left(L_f+\int\hat{L}\,d^4u\right)-g^{ik}\left(L_f+\int\hat{L}\,d^4u\right)+\int\xi_l{}^{;i}\frac{\partial\hat{L}}{\partial\xi_{l;k}}d^4u, \qquad (2.25)$$

which obeys

$$T^{ik}{}_{;k}=-L^{\cdot i}-\int\hat{L}^{\cdot ui}d^4u. \qquad (2.26)$$

Here $\hat{L}^{\cdot ui}$ and $L^{\cdot i}$ are again the functional (not the total) derivatives with respect to u^i and x^i, i.e. only the explicit dependence of $\hat{L}$ and L on x and u is to be differentiated.

The complicated form of our pseudo-energy-momentum tensor is a consequence of the hybrid nature of our Lagrangians; L_f is a density in the four-dimensional position space whereas $\hat{L}_{fm}$ and $\hat{L}_m$ are densities in the eight-dimensional phase space. Using the field equations and performing some simple manipulations,[2] one easily arrives at (2.22). In order to reduce (2.26) to the standard result [Eq. (1.11) without averaging] we have to show that the integral vanishes. To this end we note [see (2.21)] that $\hat{L}$ can be written $\hat{L}=N\check{L}$ and that N does not depend on the field ξ, so that the functional derivative of N is just the total derivative

$$\hat{L}^{\cdot ui}=\check{L}\frac{d}{du_i}N+N\check{L}^{\cdot ui}.$$

Partial integration of the first term then gives [with the usual requirement that $N(u)$ vanishes fast enough at the boundary of velocity space]

$$\hat{L}^{\cdot ui}=N\left(\check{L}^{\cdot ui}-\frac{d}{du_i}\check{L}\right)=N\frac{\delta\check{L}}{\delta\xi}\delta\xi^{\cdot ui}, \qquad (2.27)$$

which in fact vanishes due to Eq. (2.23).

So far we have not made any approximations, which guarantees that our later results will inherit the invariance properties to all orders of approximation.

§3. THE AVERAGED LAGRANGIAN

In order to proceed we make the further assumption that the solutions of (2.9) and (2.10) are strictly periodic. It is then possible to introduce ψ as a new, independent variable and to formulate a modified variational principle. To this end we define new fields

$$B^i(x,\psi)=A^i(x)\eta^i(x,u,\psi)=\xi^i(x,u). \qquad (3.1)$$

For the partial derivatives we then have

$$A^i{}_{;k}=B^i{}_{;k}+B^i{}_{;\psi}\psi_{;k},$$
$$\xi^i{}_{;k}=\eta^i{}_{;k}+\eta^i{}_{;\psi}\psi_{;k},\quad \xi^i{}_{,uk}=\eta^i{}_{,uk}, \qquad (3.2)$$

and the new Lagrangian will be

$$L=L(B,\eta)=L(B,\eta,B^i{}_{;k}+B^i{}_{;\psi}\psi_{;k},\quad \eta^i{}_{;k}+\eta^i{}_{;\psi}\psi_{;k}). \qquad (3.3)$$

Now suppose we have a solution of Eqs. (2.9) and (2.10). $\psi_{;a}=k_a$ is then a known function of x, $k_a=k_a(x)$. We will insert this function into (3.2) and (3.3), so that[22] L is a function of ψ only:

$$L=L(x,u,B,\eta,B^i{}_{;l}+B^i{}_{;\psi}k_l,\quad \eta^i{}_{;l}+\eta^i{}_{;\psi}k_l,\quad \eta^i{}_{,ul}). \qquad (3.4)$$

Since B and η are periodic, so is L.

We now consider the following action

$$S=\int_0^{2\pi}d\psi\int d^4x\left(\int\hat{L}\,d^4u+L_f\right) \qquad (3.5)$$

and vary S first with respect to B^i. We obtain

$$\delta S=\int_0^{2\pi}d\psi\int d^4x\left[-\left(\frac{\partial L}{\partial B^i{}_{;l}}\right)_{;l}-\left(\frac{\partial L}{\partial B^i{}_{;\psi}}\right)_{;\psi}+\frac{\partial L}{\partial B^i}\right]\delta B^i$$
$$+\int d^4x\left[\delta B^i\frac{\partial L}{\partial B^i{}_{;\psi}}\right]_0^{2\pi} \qquad (3.6)$$

and an analogous equation for η^i. For periodic δB^i the "surface term" in (3.6) vanishes since L is periodic, and we obtain

$$\left(\frac{\partial L}{\partial B^i{}_{;l}}\right)_{;l}+\left(\frac{\partial L}{\partial B^i{}_{;\psi}}\right)_{;\psi}-\frac{\partial L}{\partial B^i}=0, \qquad (3.7)$$

which is nothing but the original

$$\left(\frac{\partial L}{\partial A^i{}_{;l}}\right)_{;l}-\frac{\partial L}{\partial A^i}=0. \tag{3.8}$$

Therefore we can introduce the averaged Lagrangian

$$\bar{L}=\int_0^{2\pi} L\, d\psi. \tag{3.9}$$

This Lagrangian is invariant under the gauge transformation $\psi\to\psi+\alpha$ for constant α, and therefore leads to a conserved current (E. Noether, first part one of her theorem[23])

$$N^i{}_{;i}=0,\quad N^i=\frac{\overline{\partial L}}{\partial k_i}. \tag{3.10}$$

For a direct proof one integrates the identity

$$\int_0^{2\pi}\frac{dL}{d\psi}d\psi=0$$

and uses the modified field equations (3.7). One then obtains

$$\int_0^{2\pi}\left\{\frac{\partial}{\partial x^i}\left(B^l{}_{;\psi}\frac{\partial L}{\partial B^l{}_{;i}}\right)+\int d^4u\left[\frac{\partial}{\partial x^i}\eta^l{}_{;\psi}\frac{\partial \hat{L}}{\partial \eta^l{}_{;i}}+\frac{\partial}{\partial u^i}\left(\eta_{;\psi}\frac{\partial \hat{L}}{\partial \eta_{,ui}}\right)\right]\right\}d\psi=0, \tag{3.11}$$

and inspection of (3.4) shows that in fact

$$\frac{\partial L}{\partial B^i{}_{;l}}B^i{}_{;\psi}+\frac{\partial L}{\partial \eta^i{}_{;l}}\eta^i{}_{;\psi}=\frac{\partial L}{\partial k_l}. \tag{3.12}$$

We have therefore shown that even for such a complicated nonlinear Lagrangian there still exists a conserved current if the Lagrangian allows for periodic solutions. This is certainly true for those small amplitude oscillations where the dispersion relation allows for real solutions so that the eikonal ansatz (1.1) is justified.

§4. SMALL AMPLITUDE WAVES: THE LINEARIZED VLASOV EQUATION

We now expand our Lagrangians (2.11) into a power series of the amplitude. We write

$$\tilde{A}^i=A^i+\delta A^i,$$

where A^i is the four-potential of the background plasma. Note that our definition is slightly inconsistent with (1.1) and that we are considering now only real quantities. L_f leads to only two terms.

$$L_f{}^1=-a_{i;k}F^{ik}/4\pi,\quad L_f{}^2=-f_{ik}f^{ik}/16\pi,\quad f_{ik}=a_{k;i}-a_{i;k}.$$

All higher order terms vanish. In Low's nonrelativistic treatment the same would be true for the matter Lagrangian, but the relativistic Lagrangian gives contributions to all orders.

We define the projection operator into the local rest frame $h^a_b=\delta^a_b+u^a u_b$ and write

$$[-(u^i+\dot{\xi}^i)(u_i+\dot{\xi}_i)]^{1/2}=(1-u^a\dot{\xi}_a)[1-h_{ab}\dot{\xi}^a\dot{\xi}^b/(1-u^a\dot{\xi}_a)^2]^{1/2}. \tag{4.1}$$

The power series expansion of (2.11) is then straightforward.

Adding to the first order Lagrangian the divergences (which do not change the field equations) $(1/4\pi)(a_iF^{ik})_{;k}$

$$C_1=N\left(mcu^i-\frac{e}{c}A_i\right)\xi_i$$

and C_1 with the first order Lagrangian is seen to vanish identically because of the field equations of the background plasma. To second order approximation, after adding C_2, where

$$C_2=-\frac{e}{c}N\left(a^i+\frac{1}{2}A^i{}_{;k}\xi^k\right)\xi_i,$$

we obtain

$$\hat{L}_m{}^2=-{}^1/{}_2 mcNh_{ab}\dot{\xi}^a\dot{\xi}^b+(eN/2c)(F_{ab}\xi^a\dot{\xi}^b+u_cF^c{}_{a;b}\xi^a\xi^b), \tag{4.2}$$

$$L_{fm}{}^2=-(eN/c)f^{ab}\xi_a u_b,\quad L_f{}^2=(16\pi)^{-1}f^{ik}f_{ik}.$$

Variation with respect to a^i gives Maxwell's equation

$$f^{ik}{}_{;k}=\frac{4\pi}{c}e\int d^4u[N(\xi^iu^k-u^i\xi^k)]_{;k}=\frac{4\pi}{c}\delta j^i, \tag{4.3}$$

and with respect to ξ gives the equation of motion

$$L^a=(h^a{}_b\dot{\xi}^b)^{\cdot}-\frac{e}{mc^2}(F^a{}_b\dot{\xi}^b+F^a{}_{b;c}u^b\xi^c)=\frac{e}{mc^2}f^a{}_bu^b. \tag{4.4}$$

We shall now show that δj^i in Eq. (4.3) is given by the usual definition for the current Eq. (2.15)

$$\delta j^i=\int \delta N u^i d^4u, \tag{4.5}$$

and that δN obeys the linearized Vlasov equation. To this end we define the perturbed distribution function by (2.19). To linear order we have

$$\begin{gathered}d\tilde{s}/ds=1-u^a\dot{\xi}_a,\\ \tilde{u}^a=u^a+h^a{}_b\dot{\xi}^b,\quad \partial\tilde{u}^a/\partial u^b=\delta_b{}^a+(h^a{}_c\dot{\xi}^c)_{,ub},\\ \tilde{x}^a=x^a+\xi^a,\quad \partial\tilde{x}^a/\partial x^b=\delta_b{}^a+\xi^a{}_{;b},\\ \Delta\equiv\frac{d\tilde{s}}{ds}\frac{\partial^4x}{\partial^4\tilde{x}}\frac{\partial^4u}{\partial^4\tilde{u}}=1-u^a\dot{\xi}_a-\xi^a{}_{;a}-(h^a{}_b\dot{\xi}^b)_{,ua},\end{gathered} \tag{4.6}$$

and from (2.19) together with (4.6) we obtain

$$\delta N=-(N\xi^a)_{;a}-(Nh^a{}_b\dot{\xi}^b)_{,ua}-(u\dot{\xi})N. \tag{4.7}$$

Inserting (4.7) into (4.5), we obtain after integrating by parts

$$\delta j^i=e\int[u^i-(N\xi^k)_{;k}-(Nh^a{}_b\dot{\xi}^b)_{,ua}-(u\dot{\xi})N]d^4u=\int[(u^i\xi^kN)_{;k}+\dot{\xi}^iN]d^4u.$$

Using $\dot{N}=0$, the last term can be written $(\xi^iN)^{\cdot}$, and a second integration by parts using $b^i{}_{,ui}=0$ yields (4.3). It remains to be shown that N of Eq. (4.7) obeys the linearized Vlasov equation:

$$\frac{D}{ds}\delta N=\delta\dot{N}=-\frac{e}{mc^2}N_{,ui}f^{ik}u_k. \tag{4.8}$$

Equations (4.6) and (4.7) are invariant only under the restricted gauge transformation $\xi^a\to\xi^a+\sigma u^a$ with constant σ, whereas (4.3) and (4.4) are invariant under the gauge transformation with arbitrary σ. Therefore we can choose the gauge $u^a\dot{\xi}_a=0$ without a loss of generality and still have the freedom of the restricted gauge transformation.

Using the identities

$$\frac{D}{ds}A^i{}_{;i}=\left(\frac{D}{ds}A^i\right)_{;i}-A^i{}_{,uj}b^j{}_{;i},$$

$$\frac{D}{ds}A^i{}_{,ui}=\left(\frac{D}{ds}A_i\right)_{,ui}-A^i{}_{;i}-A^i{}_{,uj}b^j{}_{,ui},$$

we obtain at $\dot{N}=0$

$$\delta\dot{N}=-(NL^a)_{,ua}, \tag{4.9}$$

where L^a was defined in (4.4) and is now

$$L^a = \ddot{\xi}^a - \frac{e}{mc^2}(F^a{}_b\dot{\xi}^b + F^a{}_{b;c}u^b\xi^c) = \frac{e}{mc^2} f^a{}_b u^b, \tag{4.10}$$

according to which $L^a_{\nabla_a} = 0$, so that the linearized Vlasov equation follows in fact. Equation (4.9) was derived (nonrelativistically) in Low's paper. Low noted that this does not yet finish the proof since the class of solutions of the equation of motion (4.4) can actually be larger than the class for which (4.9) holds.

We shall show that we can use the remaining gauge freedom to establish a one-to-one correspondence between the two classes of solutions [without changing the current (4.3) or (4.5), respectively]. Multiplying Eq. (4.10) by u_a, we obtain by means of (4.4) $L^a u_a = 0$. This may be written as

$$u_a\ddot{\xi}^a = (u^a\dot{\xi}_a)^{\cdot} - (\dot{u}^a\dot{\xi}_a) = \frac{e}{mc^2}(u_aF^a{}_b\dot{\xi}^b + u_aF^a{}_{b;c}u^b\xi^c) = -(\dot{u}_a\dot{\xi}^a), \tag{4.11}$$

i.e.,

$$(u^a\dot{\xi}_a)^{\cdot} = 0,$$

so that $u^a\dot{\xi}_a = C$ for all solutions of (4.10). If this constant C happens to be different from zero, we can gauge it to zero with a restricted gauge transformation $\xi^a \to \xi^a + C\cdot u^a$ (without affecting the physical components ξ^α, as is easily seen in the local rest frame).

The proof of (4.9) is somewhat tedious but can be simplified if one proceeds as follows. One starts from $\delta N = N\Delta - N(\tilde{x}, \tilde{u})$ and uses $\dot{N}(x,u) = 0$. This gives

$$-\dot{N}(\tilde{x}, \tilde{u}) = -N_{,ui}(\dot{b}^i - b^i),$$

which is just $-N_{,ui}L^i$ according to (4.4), (2.4) and (2.9)

$$\dot{\Delta} = L^a{}_{,ua} \tag{4.12}$$

and

$$-\dot{N}_{;a}\xi^a - \dot{N}_{,ua}h^a{}_b\xi^b = \dot{N}(\tilde{x}, \tilde{u}) = 0. \tag{4.13}$$

The remaining terms are then just $-N_{,a}L^a$, which proves (4.9). We have therefore shown that although the Vlasov equation does not follow as a variational equation from the Lagrangian, it is contained implicitly in the field equations and may replace Eq. (4.4).

§5. THE EQUIVALENT PERMEABILITY TENSOR

Having established that our basic equations (2.9) and (2.10) follow from the Lagrangians (4.1) and (4.2), we can now apply the result of §3, Eq. (3.10), if we make the eikonal ansatz (1.1). As the next step we use the virial theorem to eliminate ξ^i from the total averaged Lagrangian. Since L^2_m is quadratic in ξ and L_{mf} is linear, we obtain

$$2\bar{L}_m = -\bar{L}_{mf},$$

so that

$$\bar{L} = \bar{L}_m + \bar{L}_{mf} + \bar{L}_f = \bar{L}_f + {}^1/_2\bar{L}_{mf}.$$

Defining

$$\delta H^{ab} = -\frac{4\pi e}{c}\int N(\xi^a u^b - u^a\xi^b)\,d^4u + f^{ab}, \tag{5.1}$$

we obtain finally

$$L = -(16\pi)^{-1}\overline{f_{ab}\delta H^{ab}}, \tag{5.2}$$

which agrees with Ref. 5. Applying (3.10), we see that the wave action $N^i = \partial\bar{L}/\partial k_i$ is conserved.

We only have to determine the linear response tensor which relates ξ^i to f^{ik}. For a cold plasma this is straightforward. With the eikonal ansatz (1.1) we have to invert (4.10), which now reads to lowest order eikonal approximation

$$-(ku)^2\xi^a + i(ku)\Omega^a{}_b\xi^b = (e/mc^2)f^a{}_b u^b, \tag{5.3}$$

in order to obtain

$$\xi^a = \sigma^{ab}\omega_{bc}u^c, \tag{5.4}$$

where

$$\omega_{bc} = (e/mc^2)f_{bc}, \quad \Omega_{bc} = (e/mc^2)F_{bc},$$

and ${}^*\Omega^a_b = e^a_{bcd}\Omega^{cd}$ is the dual to Ω^a_b. The result is

$$\sigma^a = \frac{1}{(ku)^2}\,\frac{(ku)^2\delta^a{}_b + i(ku){}^*\Omega^a{}_b - {}^*\Omega^a{}_c{}^*\Omega^c{}_b}{-(ku)^2 + {}^1/_2(\Omega_{ij}\Omega^{ij})}. \tag{5.5}$$

The final result (5.5) can be easily derived from $\Omega^a_b u^b = 0$, which holds to lowest-order eikonal approximation, so that as a consequence we have ${}^*\Omega^a_c\Omega^c_d = 0$. Using further the identity

$$\Omega^a{}_b\Omega^b{}_c = {}^*\Omega^a{}_b{}^*\Omega^b{}_c - {}^1/_2(\Omega_{ab}\Omega^{ab})\delta^a{}_c,$$

one easily shows that (5.5) is the inverse of $(ku)^2\delta^a_b - i(ku)\Omega^a_b$. In the local rest frame of the plasma the space part of Eq. (5.5) reduces to the well-known conductivity tensor σ^α_β if we put

$$(ku)^2 = \omega^2, \quad {}^1/_2(\Omega_{ij}\Omega^{ij}) = \Omega_L{}^2 = \Omega^2,$$

where Ω_L is the Larmor frequency in the particle's rest frame. From (5.4) and (5.5) we read off the permeability tensor, using (5.1) and (1.5), (1.6)

$$\varepsilon^{ab}{}_{cd} = \delta_{[c}{}^{[a}\delta_{d]}{}^{b]} + (\omega_p{}^2/c^2)\sigma_{[c}{}^{[a}u^{b]}u_{d]}, \tag{5.6}$$

where

$$A_{[a}B_{b]} = {}^1/_2(A_aB_b - B_aA_b).$$

Note that in the case of a cold plasma it would still be possible to derive the first order approximation to σ^a_b. To the required accuracy one may replace $\Omega^a_{b;c}u^b$ by $b^a{}_{;c}$ and replace $i(ku)\Omega^{ab}$ in (5.3) by

$$i(ku)\Omega^{ab} + {}^1/_2\dot{\Omega}^{ab} - \Omega^{c(a}u_c{}^{;b)}, \quad \dot{\Omega}^{ab} = \Omega^{ab}{}_{;c}u^c.$$

Repeating the analysis of Ref. 5 shows that the important term is $\dot{\Omega}^{ab}$ which makes ε^{ab}_{cd} nonhermitian and guarantees that now $N^i{}_{;i} = 0$ to first order, as it must.

Instead of solving Eq. (4.10) for the hot plasma case (where u is an independent variable), we use the linearized Vlasov equation (4.8), which, as we have shown, is equivalent. To this end we employ a technique due to Sagdeev and Shafranov[13] (as described by Stix[14] and Clemnow and Dougherty[15]). To put the method in covariant form we borrow the formalism from Buneman.[16] One first solves Eq. (4.8) in Lagrangian coordinates for the perturbation δN so that it is sufficient to insert the zero-order orbits into the right-hand side of Eq. (4.8) since f^{ik} is already a quantity of first order. These orbits are parametrized so that at proper time $s' = s$ they pass through x^i and u^i, respectively. For a magnetized plasma the orbits are to lowest order

$$\bar{x}^i = x^i + \tau^i_k u^k + u^i(s'-s), \quad \tau^i_k = a\Omega^i_k + b\Omega^i_e \Omega^e_k + (s'-s)\delta^i_k,$$
$$a = c[\cos\Omega(s'-s) - 1]/\Omega^2, \quad b = c[\Omega(s'-s) - \sin\Omega(s'-s)]/\Omega^3, \quad (5.7)$$
$$\bar{u}^i = \dot{\tau}^i_k u^k, \quad \dot{\tau} = d\tau/ds.$$

Equations (5.7) are the appropriate generalization of the Lorentz rotators given by Buneman[16] [the notation is that used in Eq. (5.5) and below].

If one wishes to take into account particle drifts, higher order harmonics, and the change in the amplitude of gyration via an adiabatically conserved magnetic moment, covariant formulations are also available.[17] Replacing the Lagrangian by Eularian coordinates, one obtains[14]

$$\delta N(x,u) = -\frac{e}{mc^2}\int_{s_0}^{s} \bar{u}_l(s') f^{kl}(\bar{x}(s')) N_0(\bar{x}(s'), \bar{u}(s'))_{,\bar{u}k}\, ds' + \delta N(x_0, u_0). \quad (5.8)$$

To make the integral independent of the initial data we imagine the perturbation to be switched on adiabatically in the infinite past and introduce the integration variable $t = s' - s$ to obtain

$$\delta N(x,u) = \frac{e}{mc^2}\int_{-\infty}^{0} \bar{u}_l(t) f^{kl}(\bar{x}(t)) N_0(\bar{x}(t), \bar{u}(t))_{,\bar{u}k}\, dt. \quad (5.9)$$

By this trick the integral has become independent of the proper times. With our eikonal ansatz (1.1) we obtain finally

$$\delta N(x,u) = -(e/mc^2) f^{kl}(x) s_{kl}(u),$$
$$s_{kl} = \int_{-\infty}^{0} \bar{u}_l N_{,\bar{u}k} \exp[i(k_a \tau^a_b u^b + k_a u^a t)]\, dt. \quad (5.10)$$

Instead of continuing the discussion in general form, we shall illustrate the procedure by a simple example: we consider a one-dimensional relativistic gas. (For a treatment of a magnetized plasma with an isotropic relativistic Boltzmann distribution function cf. Buneman.[16]) In the absence of a magnetic field the particle trajectories are simply $\bar{u}^i = u^i$ and $\bar{x}^i + u^i(s'-s)$ and s_{kl} of Eq. (5.10) is given by

$$iu_l N_{,uk}(ku)^{-1}. \quad (5.11)$$

For the current we obtain

$$\delta j^a = -i\frac{e^2}{mc^2}\int u^a \frac{u_l N_{,uk}}{k_b u^b} f^{kl}\, d^4u. \quad (5.12)$$

In evaluating this integral, the Landau rule must be used. Putting $f^{ab} = i(k^a \delta A^b - \delta A^a k^b)$ and using Maxwell's equation (4.3) we obtain

$$(k^a \delta A^b - k^b \delta A^a) k_b = -\frac{4\pi e^2}{mc^2}\int \frac{u^a N_{,ub} u_c}{k_r u^r}(k^b \delta A^c - \delta A^b k^c)\, d^4u \quad (5.13)$$

or

$$(k^a k_b - k^2 \delta^a_b)\delta A^b = \sigma^a_b \delta A^b, \quad (5.14)$$
$$\sigma^a_b = \frac{4\pi e^2}{mc^2}\int \frac{d^4u}{k_r u^r}[u^a((N_{,uc}k^c)u_b - (u^e k_e) N_{,ub})]$$
$$= -\frac{4\pi e^2}{mc^2}\int \frac{N d^4u}{(k_r u^r)^2}[(k^a u_b + k_b u^a)(k_r u^r) - (k_r k^r) u^a u_b - (k^r u_r)^2 \delta^a_b].$$

The last equation of (5.14) was obtained by integtating by parts. The symmetry of σ^a_b and satisfaction of $\sigma^a_b k^b = 0$ then become obvious. Note that (5.14) coincides with (5.5) inserted into (5.1) at zero magnetic field. However, in general this is not true, as can be easily proved since (5.5) has only one resonance at the fundamental Larmor frequency Ω_L. We assume now that

$$N = 2n_0(x) f(u_1)\delta(u_2)\delta(u_3)\delta(-1 - u^r u_r)\theta(u^0),$$

where we have chosen x^1 for the direction of the anisotropy. Such a plasma may exist for example around a pulsar. We have

$$\sigma^2_2 = \sigma^3_3 = \omega_p^2/c^2 = 4\pi e^2 n_0/mc^2,$$

and (5.15)

$$\sigma^1_1 = -\frac{\omega_p^2}{c^2}\int \frac{\omega u^1(N_{;0}u_1 + N_{;1}u^0)}{u^0 - cku^1} d^4u = -\frac{\omega_p^2}{c^2}\int \frac{uf'(u)\,du}{(1-u^2)^{1/2} - nu} \qquad n = \frac{ck^1}{\omega},$$
$$\sigma^0_0 = -c\omega^{-1}\sigma^0_a k^a.$$

For longitudinal oscillations we obtain from $\delta A^\alpha = 0$ ($\alpha = 1, 2, 3$) the dispersion relation

$$k^\alpha k_\alpha = -\omega^2 \sigma^{\alpha\beta} k_\alpha k_\beta, \quad (5.16)$$

whereas for transverse waves in the Landau gauge $\delta A^0 = 0$, we get with the help of (5.14)

$$[k^\alpha k_\beta + (\omega^2/c^2 - k^\gamma k_\gamma)\delta^\alpha_\beta]\delta A^\beta = \sigma^\alpha_\beta \delta A^\beta, \quad (5.17)$$

and for the principal modes (parallel or perpendicular to x^1), for which $k_\alpha \delta A^\alpha = 0$ is possible, we have the dispersion relation

$$1 - n^2 = c^2\sigma/\omega^2, \quad (5.18)$$

where

$$\sigma = \sigma^1_1 = \sigma_\parallel \quad \text{or} \quad \sigma = \sigma^2_2 = \sigma^3_3 = \sigma_\perp = \omega_p^2/c^2.$$

To estimate the influence of a large velocity dispersion on the propagation of the mode parallel to the x^1 we approximate $f(u)$ by

$$f(u) = {}^1/_2 \theta e^{-\theta|u|},$$

where $2\theta^{-1} = kT = \langle\gamma\rangle$ is an effective temperature and measures the velocity dispersion. We obtain

$$\sigma_\parallel = \theta\frac{\omega_p^2}{c^2}\int_0^\infty \frac{ue^{-\theta u}\,du}{(1+u^2)^{1/2} - nu}. \quad (5.19)$$

In the limit as $\theta^{-1} \to 0$ we obtain $\sigma_\parallel = \omega_p^2/c^2$, as we must, and as $\theta \to 0$ we get

$$\sigma_\parallel \to \theta^2 \frac{\omega_p^2}{c^2}\frac{1+n}{1-n^2}$$

from which it follows that $1 - n^2 \to \omega_p^2/\omega^2\langle\gamma\rangle$, i.e., the plasma frequency is decreased by a factor $\langle\gamma\rangle^{-1/2}$, a well-known result.

In conclusion, we give the result for an isotropic distribution function. We choose x^3 for our reference axis and obtain[15]

$$\sigma^3_3 = \sigma_\parallel = \frac{\omega_p^2}{c^2}\int_0^{2\pi} d\psi \int_0^\pi \sin\theta\, d\theta \int_0^\infty \frac{u^2\,du}{u^0}\left[-\frac{\cos^2\theta f'}{1 - z\cos\theta}\right], \quad (5.20)$$

$$\sigma^1_1 = \sigma^2_2 = \sigma_\perp = \frac{\omega_p^2}{c^2}\int_0^\pi d\psi \int_0^\pi \sin\theta\, d\theta \int_0^\infty \frac{u^3 du}{u^0}\left[-f'\frac{\sin^2\theta\cos^2\psi}{1 - z\cos\theta}\right], \quad (5.21)$$

where the following convention is used

$$u^\alpha = (u\sin\theta\cos\psi, \quad u\sin\theta\sin\psi, \quad u\cos\psi),$$
$$\delta A^i = (0, \delta A, 0, 0), \quad k^i = (\omega/c, 0, 0, k).$$

The integral over the angles is elementary, and it is convenient to consider first $\sigma_\parallel + 2\sigma_\perp$, which shows that the integrals behave like $1/\langle\gamma\rangle \ln\langle\gamma\rangle$, again a known result.

§6. NONLINEAR WAVE-WAVE INTERACTION

A nonrelativistic treatment of the interaction of three waves at resonance was given for example by Vedenov,[18] so we confine ourselves here to pointing out the differences which arise in a relativistic treatment. As already mentioned at the beginning of §4, in the relativistic treatment we obtain contributions to the matter Lagrangian to all orders, whereas the Low Lagrangian stops with second order. We find using Eq. (4.1)

$$L_m{}^3 = {}^1/_2 mcN(u^a\dot{\xi}_a)(h_{ab}\dot{\xi}^a\dot{\xi}^b) + ({}^1/_3 F_{ab;\,c}\dot{\xi}^a\xi^b\xi^c + {}^1/_6 F_{ab;c;d}u^a\xi^b\xi^c\xi^d)(eN/c), \tag{6.1}$$

$$L_{mf}{}^3 = ({}^1/_2 f_{ab;c}u^a\xi^b\xi^c + {}^1/_2 f_{ab}\dot{\xi}^a\xi^b)(eN/c), \tag{6.2}$$

$$L_f{}^3 = 0. \tag{6.3}$$

To check the gauge invariance of the above equations would be extremely tedious, as both the equations of the background medium and the first-order equations (4.3) and (4.4) must be used. Thus it now pays off that we have proved the gauge invariance of the exact equations (2.9) and (2.10). In order to derive Eqs. (6.1) and (6.2) we have added the term $-\frac{1}{2}\dot{C}_3$ where

$$C_3 = (a_{b;c}\xi^b\xi^c + {}^1/_3 A_{a;b;c}\xi^a\xi^b\xi^c)(eN/c). \tag{6.4}$$

The relativistic version of the resonance condition is

$$k_1{}^a + k_2{}^a = k_3{}^a. \tag{6.5}$$

Applying our exact relation

$$T^{ab}{}_{;b} = -L^{;a} \tag{6.6}$$

to the Lagrangian for three waves

$$L = L(a + a_2 + a_3, \quad \xi + \xi_2 + \xi_3), \tag{6.7}$$

where only one pair of variables (a, ξ) is assumed to be unknown[18] and the other two assumed to be determined from the second-order approximation, we find for each of the three waves in turn

$$k_1{}^a N_1{}^b{}_{;b} = -i(k_2{}^a - k_3{}^a)\bar{L}_3 = ik_1{}^a\bar{L}_3, \tag{6.8}$$

$$k_2{}^a N_2{}^b{}_{;b} = -i(k_1{}^a - k_3{}^a)\bar{L}_3 = ik_2{}^a\bar{L}_3, \tag{6.9}$$

$$k_3{}^a N_3{}^b{}_{;b} = -i(k_1{}^a + k_2{}^a)\bar{L}_3 = -ik_3{}^a\bar{L}_3, \tag{6.10}$$

from which we obtain the so-called[8] Manley-Rowe relations in their relativistic form

$$N_1{}^a{}_{;a} = N_2{}^a{}_{;a} = -N_3{}^a{}_{;a}. \tag{6.11}$$

To prove Eqs. (6.8)–(6.10) one writes $L = L^2 + L^3$, where L^2 is the quadratic and L^3 the cubic Lagrangian and uses the fact that the divergence of ${}^2N_a = \partial L^2/\partial k_a$ tends to zero because of the linearity of the equations, and furthermore that $\bar{L}_2 = 0$ because of the dispersion relation, and

$$k^a{}_{;b}N_2{}^b = -L^{2;a} \tag{6.12}$$

in accord with the ray tracing. Thus upon averaging one obtains the exact relation

$$(k^a N_3{}^b)_{;b} = \bar{L}_3{}^{;a} + \bar{L}_3{}^{,a}. \tag{6.13}$$

Here $\bar{L}_3{}^{,a}$ denotes $d/dx_a \times \bar{L}_3$, which is small compared to $\bar{L}_3{}^{;a}$ which is proportional to ik^a. For a homogeneous medium $k^a{}_{;b} = 0$, so that

$$k^a{}_{;b}N^b + k^a N^b{}_{;b} \approx k^a N^b{}_{;b},$$

at least as long as the inhomogeneities of the medium are not too large. In a strongly inhomogeneous medium, however, the resonance may not occur. For the actual calculation of the interaction forces the reader can consult the article by Vedenov,[18] as well as the paper by Galloway and Kim.[25]

It is obvious that the above methods can be applied to curved space-time, i.e., to plasma in gravitational fields; see also the work of Breuer and Ehlers.[19] What remains to be done is to extend the formalism in the way used by Dewar[20] for magnetohydrodynamics. So as to include the reaction of the waves on the background by the same formalism. Note further that the formalism cannot treat internal reflection or parametric resonance, which are of particular interest for pulsar magneto-spheres. The reader who is interested in applications of the developed theorm is referred to our earlier work.[21]

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