Gapless mode in spin glasses with random anisotropic exchange

S. L. Ginzburg

B. P. Konstantinov Leningrad Institute of Nuclear Physics, Academy of Sciences, USSR
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It is shown that the order parameter in a spin glass with random anisotropic exchange is stable with respect to Gaussian fluctuations. At all temperatures below the phase-transition temperature, the correlator of these fluctuations is proportional, at small momenta k, to $1/k^2$; this means that a gapless critical mode is present in the system and indicates that the ground state of the system is degenerate.

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1. INTRODUCTION

It has been shown by the author[1] that in a magnetic material with random anisotropic and nonrandom isotropic exchange, a spin-glass phase exists. This result was derived in the approximation $p \to \infty$, where p is the number of components of the spin. This approximation is one of the numerous modifications of self-consistent field theory. On the other hand, in use of self-consistent field theory it is always necessary to clarify the question of the stability of the solution with respect to fluctuations of the self-consistent field. We remark that the question of fluctuations in the spin-glass problem is at present very urgent. The fact is that in all the spin-glass theories so far studied, it turns out that the order parameter is unstable with respect to fluctuations of the self-consistent field (see, for examples, Refs. 2–6). Therefore the construction of a spin-glass theory in which the order parameter is stable with respect to fluctuations is a very interesting problem.

The present paper is devoted to this question. In our problem, fluctuations occur in subsequent orders in $1/p$ (in Ref. 1, the zeroth order in $1/p$ was studied). Therefore for study of the problem of fluctuations, a systematic expansion in $1/p$ will be constructed. For this purpose, we shall use the method of replicas (see, for example, Ref. 4) and shall introduce an effective Hamiltonian. If one sets the first variation of this Hamiltonian with respect to the field variables equal to zero, one obtains equations for these variables. We shall show that these equations are completely equivalent to the equations of Ref. 1. We note that such a procedure corresponds to the self-consistent field approximation. We shall then construct the Hamiltonian for the fluctuations of the field variables. It turns out that the perturbation-theory series in this Hamiltonian gives the $1/p$ expansion.

We shall further study in detail Gaussian fluctuations of the fields. It is found that at all temperatures below the critical, the correlator describing these fluctuations contains a gapless mode. This means, first, stability of the system with respect to fluctuations (for instability there is a negative gap[2–6]), and second, the presence of a certain degeneracy (the same as in an ordinary ordered ferromagnet). We note that in a system with anisotropic exchange, spin is not conserved, and that there can be no spin waves in such a system. Therefore the indicated gapless mode is evidently connected with a degeneracy specifically characteristic of disordered systems with "frustration."[7] Previously, such degeneracy has been studied only numerically. In our case, it can be studied analytically.

The presence of a gapless mode in the system is the principal result of the present paper.

In concluding this section, we note that systems with random anisotropic exchange are encountered quite often. For occurrence of such exchange, two conditions must be satisfied: the presence of an interaction that does not conserve spin, and randomness. The simplest examples of this type are dipole forces in an amorphous ferromagnet and a random rotating anisotropy.[8,9]

2. DERIVATION OF AN EFFECTIVE HAMILTONIAN

We choose the Hamiltonian of a magnet with random anisotropic and nonrandom isotropic exchange in the following form[1]:

$$\frac{H}{T}=-\frac{1}{T}\sum_{\substack{ik\\ \alpha}} v_{ik} m_i^\alpha m_k^\alpha - \frac{1}{T}\sum_{\substack{ik\\ \alpha\beta}} J_{ik}^{\alpha\beta} m_i^\alpha m_k^\beta + \frac{1}{2a}\sum_{i\alpha}(m_i^\alpha)^2 + \frac{\lambda}{8p}\sum_{i\alpha\beta}(m_i^\alpha)^2(m_i^\beta)^2. \tag{1}$$

Here m_i^α are the spin variables (i enumerates the sites, α the components), $J_{ij}^{\alpha\beta}$ is the random exchange integral, v_{ik} is the nonrandom, T is the temperature, λ is an interaction constant, and a is a constant. The random matrix $J_{ik}^{\alpha\beta}$ has a Gaussian distribution: each $J_{ik}^{\alpha\beta}$ is a statistically independent quantity with a Gaussian distribution function,

$$P(J_{ik}^{\alpha\beta}) \sim \exp\left\{-\frac{p}{4I_{ik}}\sum_{\alpha\beta}(J_{ik}^{\alpha\beta})^2\right\},$$

$$\langle J_{ik}^{\alpha\beta} J_{lm}^{\gamma\delta}\rangle = \frac{1}{p} I_{ik}\{\delta_{il}\delta_{km}\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{im}\delta_{kl}\delta_{\alpha\delta}\delta_{\beta\gamma}\}. \tag{2}$$

For what follows, we shall find it convenient to use the method of replicas. In this case, the averaging over $J_{ik}^{\alpha\beta}$ is carried out by a standard method (see, for example, Ref. 4). As a result, we obtain the Hamiltonian for n spin variables $m_{i\mu}^\alpha$ ($\mu = 1 \ldots n$). We must remember that we are interested in the limit $n \to 0$. After

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standard manipulations, we get the following Hamiltonian H_c:

$$\frac{H_c}{T}=-\frac{1}{T}\sum_{\substack{ik\mu\\ \alpha}} v_{ik}m_{i\mu}{}^{\alpha}m_{k\mu}{}^{\alpha}-\frac{1}{pT^2}\sum_{\substack{ik\alpha\beta\\ \mu\nu}} I_{ik}m_{i\mu}{}^{\alpha}m_{k\mu}{}^{\beta}m_{i\nu}{}^{\alpha}m_{k\nu}{}^{\beta}+\frac{1}{2a}\sum_{i\alpha\mu}(m_{i\mu}{}^{\alpha})^2+\frac{\lambda}{8p}\sum_{i\alpha\beta\mu}(m_{i\mu}{}^{\alpha})^2(m_{i\mu}{}^{\beta})^2. \quad (3)$$

This Hamiltonian is inconvenient in that the large parameter p is not explicitly separated in it. In order to separate it, we apply a method used in Ref. 10 for an ordered ferromagnet and in Ref. 11 for the problem of the motion of an electron in a random potential. We use the formula

$$\exp\left\{\sum_{pq=1}^{n}K_{pq}\eta_p\eta_q\right\}=[4|K|\pi^n]^{-1/2}\times\int_{-\infty}^{\infty}dz_1\ldots dz_n\exp\left\{\sum_{p=1}^{n}\eta_pz_p-\frac{1}{4}\sum_{pq=1}^{n}(1/K)_{pq}z_pz_q\right\}. \quad (4)$$

Here $(1/K)_{pq}$ is the matrix inverse to K_{pq}, and $|K|$ is the modulus of the determinant of the matrix K_{pq}. Applying formula (4) to the terms of fourth order in $m_{i\mu}^{\alpha}$ in (3), we get the following Hamiltonian:

$$\frac{H_c}{T}=-\frac{1}{T}\sum_{ik\mu\alpha}v_{ik}m_{i\mu}{}^{\alpha}m_{k\mu}{}^{\alpha}+\frac{1}{2a}\sum_{i\alpha\mu}(m_{i\mu}{}^{\alpha})^2-i\sum_{i\mu\alpha}\xi_{i\mu}(m_{i\mu}{}^{\alpha})^2+\frac{2p}{\lambda}\sum_{i\mu}\xi_{i\mu}{}^2-\sum_{i\alpha\mu\nu}Q_{i\mu\nu}m_{i\mu}{}^{\alpha}m_{i\nu}{}^{\alpha}+\frac{pT^2}{4}\sum_{ik\mu\nu}(1/I)_{ik}Q_{i\mu\nu}Q_{k\mu\nu}. \quad (5)$$

Here $(1/I)_{ik}$ is the matrix inverse to I_{ik}; $\xi_{i\mu}$ and $Q_{i\mu\nu}$ are new field variables, integration over which is carried out with weight $\exp\{-H_c/T\}$. Such an integration is implied everywhere hereafter. We omit the constant factors that arise when formula (4) is used, because they cancel out in the calculation of the mean values. We note that $Q_{i\mu\nu}$ is a symmetric tensor.

The Hamiltonian (5) is quadratic in the fields $m_{i\mu}^{\alpha}$. It is therefore possible to integrate over them. Then the large parameter p is separated out in explicit form. The Hamiltonian for the remaining variables—the tensor $\xi_{i\mu}$, diagonal in the indices μ and ν, and the symmetric tensor $Q_{i\mu\nu}$—is conveniently represented in the following form:

$$H_c/T=p\Phi(\hat{\xi},\hat{Q}),$$

$$\Phi(\hat{\xi},\hat{Q})=\frac{2}{\lambda}\sum_{i\mu}\xi_{i\mu}{}^2+\frac{T^2}{4}\sum_{ik\mu\nu}\left(\frac{1}{I}\right)_{ik}Q_{i\mu\nu}Q_{k\mu\nu}+\frac{1}{2}\operatorname{Sp}\ln\hat{A},$$

$$(\hat{A})_{ik\mu\nu}=\frac{1}{2a}\delta_{ik}\delta_{\mu\nu}-\frac{1}{T}v_{ik}\delta_{\mu\nu}-Q_{i\mu\nu}\delta_{ik}-i\xi_{i\mu}\delta_{\mu\nu}\delta_{ik}. \quad (6)$$

In (6) we have used hats to designate matrices in $ik\mu\nu$ space. Hereafter we shall find it convenient to suppose that there in in the system a random magnetic field $h_{i\alpha}$ that has a Gaussian distribution and interacts with the m_i^{α} according to the formula $h_{i\alpha}m_{i\alpha}$. In this case, only $\hat{A}$ varies in (6). There results a correction to $\hat{A}$ of the following form:

$$(\Delta\hat{A})_{ik\mu\nu}=-\frac{h^2}{2T^2}\delta_{ik}\Lambda_{\mu\nu},\quad \langle h_{i\alpha}h_{k\beta}\rangle=\delta_{\alpha\beta}\delta_{ik}h^2. \quad (7)$$

In (7), $\Lambda_{\mu\nu}$ is a tensor whose elements are all unity. We note at once that

$$\hat{\Lambda}^2=n\hat{\Lambda}, \quad (8)$$

where n is the dimensionality of "replica" space; we are interested in the limit $n\to 0$. We note that the Hamiltonian in (6) is not invariant with respect to rotation in replica space, since (6) contains the diagonal tensor $\hat{\xi}$. This is a consequence of the fact that the last term in (3) has cubic symmetry, not spherical. Formula (6) gives the effective Hamiltonian that we need.

3. EQUATIONS OF THE SELF-CONSISTENT FIELD

As usual, the equations of the self-consistent field are obtained by varying the Hamiltonian with respect to the field variables $\hat{\xi}$ and $\hat{Q}$ and setting the derivative equal to zero. Then we get from (6) and (7)

$$Q_{i\mu\nu}=\frac{2}{T^2}\sum_k I_{ik}G_{kk\mu\nu},\quad G_{ik\mu\nu}\equiv(\hat{G})_{ik\mu\nu},$$

$$\xi_{i\mu}=i\frac{\lambda}{4}G_{ii\mu\mu}=i\frac{\lambda T^2}{8}\sum_k\left(\frac{1}{I}\right)_{ik}Q_{i\mu\mu},\quad (9)$$

$$(\hat{G}^{-1})_{ik\mu\nu}=\frac{1}{a}\delta_{ik}\delta_{\mu\nu}-\frac{2}{T}v_{ik}\delta_{\mu\nu}-2Q_{i\mu\nu}\delta_{ik}-2i\xi_{i\mu}\delta_{\mu\nu}\delta_{ik}-\frac{h^2}{T^2}\delta_{ik}\Lambda_{\mu\nu}.$$

Since $\hat{\xi}$ is expressed in terms of the diagonal part of $\hat{Q}$, (9) is an equation for the operator $\hat{Q}$. We note first of of all that for a stationary solution, $\hat{G}_{ii}$ and $\hat{Q}_i$ are independent of the site i. Therefore all the operators in (9) depend only on the indices μ and ν. We get for them the following operator equations:

$$Q_{\mu\nu}=\frac{2I_0}{T^2}G_{0\mu\nu},\quad \xi_\mu=i\frac{\lambda}{4}G_{0\mu\mu}=i\frac{\lambda T^2}{8I_0}Q_{\mu\mu},$$

$$\hat{G}_0=d^3\int\frac{d\mathbf{k}}{(2\pi)^3}\hat{G}(k),\quad I_0=I(k=0),\quad (10)$$

$$\hat{G}^{-1}(k)=\left[\frac{1}{a}-\frac{2v(k)}{T}\right]\hat{E}-2\hat{Q}-2i\hat{\xi}-\frac{h^2}{T^2}\hat{\Lambda},$$

where $\hat{G}(k)$, $v(k)$, and $I(k)$ are the corresponding functions in the momentum representation, d^3 is the volume of the elementary cell, and $\hat{E}$ is the unit matrix in replica space.

It is evident from (10) that if we solve (10) by perturbation theory with respect to h^2, then in consequence of (8) the operator $\hat{Q}$ will be expressed in terms of $\hat{E}$ and $\hat{\Lambda}$. Therefore we shall seek a solution in the form

$$\hat{G}_0=G_0\hat{E}+q\hat{\Lambda},\quad \hat{Q}=\frac{2I_0}{T^2}(G_0\hat{E}+q\hat{\Lambda}),\quad \hat{\xi}=i\frac{\lambda}{4}P_0\hat{E},\quad P_0=G_0+q. \quad (11)$$

As was shown in Ref. 1, q is an Edwards-Anderson parameter. Expressions for $\hat{Q}$ and $\hat{\xi}$ are obtained from (10) and the expression for $\hat{G}_0$ in (11). If we substitute these expressions in the equation for $\hat{G}(k)$ in (10), use (8), and take into account that in our case $n\to 0$, we get

$$\hat{G}(k)=G(k)\hat{E}+\left(\frac{4I_0}{T^2}q+\frac{h^2}{T^2}\right)G^2(k)\hat{\Lambda},$$

$$G^{-1}(k)=\frac{1}{a}+\frac{\lambda P_0}{2}-\frac{4I_0}{T^2}G_0-\frac{2v(k)}{T}. \quad (12)$$

Substituting (12) in the expression for $\hat{G}_0$ in (10) and equating coefficients of the operators $\hat{E}$ and $\hat{\Lambda}$, we get the following equations for the parameters G_0 and q:

$$G_0=d^3\int\frac{d\mathbf{k}}{(2\pi)^3}G(k),\quad q\left[1-\frac{4I_0}{T^2}K_1(0)\right]=\frac{h^2}{T^2}K_1(0),$$

$$K_1(k)=d^3\int\frac{d\mathbf{p}}{(2\pi)^3}G(p)G(\mathbf{p}+\mathbf{k}). \quad (13)$$

Equations (13) coincide with the corresponding Eqs. (4)

of Ref. 1 [for a random magnetic field, it is necessary in these equations to replace $G^2(k=0)$ by Π_0 in the expression for P_0]. These equations were studied in detail in Ref. 1. Here, therefore, we note only the following important fact. It follows from the equation for q in (13) that when $h=0$, there are two possibilities: either $q=0$, or

$$\frac{4I_0}{T^2}K_1(0)=1. \tag{14}$$

Therefore at all temperatures below the critical, the condition (14) is satisfied.

We note also the physical meaning of the operator $G(k)$. In disordered systems there are two types of two-particle correlators:

$$\langle\langle m_i^\alpha m_k^\beta\rangle_T\rangle_c, \quad \langle\langle m_i^\alpha\rangle_T\langle m_k^\beta\rangle_T\rangle_c, \tag{15}$$

where $\langle\ldots\rangle_T$ and $\langle\ldots\rangle_c$ denote thermodynamic and configurational averages. It is easy to show that the diagonal part of $\hat{G}$ determines the first correlator, the nondiagonal the second. Since the coefficient of $\hat{E}$ is equal to the difference of the diagonal and nondiagonal parts, it determines the irreducible correlator (the difference of the first and second). The coefficient of $\hat{\Lambda}$ determines the second correlator, which, as is easily shown, coincides with the function $K(r)$ in Ref. 1.

Thus we see that the correlation-function operator contains all the necessary information about two-particle averages.

4. THE HAMILTONIAN FOR THE FLUCTUATIONS

We denote the solution of Eqs. (10) and (11) for $\hat{Q}$ and $\hat{\xi}$ by $\hat{Q}_0$ and $\hat{\xi}_0$, and we set

$$\hat{Q}=\hat{Q}_0+\hat{R}, \quad \hat{\xi}=\hat{\xi}_0+\hat{\Psi}. \tag{16}$$

Then after uncomplicated calculations we obtain the following expression for Φ:

$$\begin{gathered}\Phi(\hat{\xi},\hat{Q})=\Phi_0+\Phi(\hat{\Psi},\hat{R}),\\ \Phi_0=\frac{2}{\lambda}\operatorname{Sp}\hat{\xi}_0^2+\frac{T^2}{4}\operatorname{Sp}\left(\hat{Q}_0\frac{1}{\hat{I}}\hat{Q}_0\right)+\frac{1}{2}\operatorname{Sp}\ln\hat{A}_0,\\ \Phi(\hat{\Psi},\hat{R})=\frac{2}{\lambda}\operatorname{Sp}\hat{\Psi}^2+\frac{T^2}{4}\operatorname{Sp}\left\{\hat{R}\frac{1}{\hat{I}}\hat{R}\right\}\\ -\operatorname{Sp}\{\hat{G}(\hat{R}+i\hat{\Psi})\}^2-\sum_{k=3}^{\infty}\frac{2^{k-1}}{k}\operatorname{Sp}\{\hat{G}(\hat{R}+i\hat{\Psi})\}^k,\\ (\hat{I})_{ik\mu\nu}=I_{ik}\delta_{\mu\nu}.\end{gathered} \tag{17}$$

In (17), the trace is taken both over the coordinate and over the operator indices. The operator $(\hat{G})_{ik\mu}$ is the Fourier transform of the function $\hat{G}(k)$, which is determined by the solution of Eq. (10). The quantity $\Phi(\hat{\Psi},\hat{R})$ determines the Hamiltonian for deviations of the field variables from their solution in the self-consistent field approximation, $\hat{\Psi}$ and $\hat{R}$. We note that $\Phi(\hat{\Psi},\hat{R})$ in (17) is correct both for $T>T_c$ and for $T<T_c$. The only difference is that when $T<T_c$, an Edwards-Anderson parameter q occurs and $\hat{G}$ is no longer proportional to the unit operator $\hat{E}$.

In the expression for $\Phi(\hat{\Psi},\hat{R})$, the expansion begins with terms quadratic in the fluctuations. Since, as is evident from (6), $H_c\sim p\Phi$, it is not difficult to show that if, in the calculation of the correlators, we take the quadratic part of Φ as the zeroth approximation and calculate the remaining terms by perturbation theory, then we obtain an expansion in the parameter $1/p$, which thus plays the role of an effective interaction constant. This can be seen also by another method: by making the substitution of variables $p\Psi^2\to\Psi^2$ and $pR^2\to R^2$. Then the parameter $1/p$ occurs in different degrees together with terms of higher order.

In the present paper, we shall restrict ourselves to study solely of Gaussian fluctuations determined by the quadratic part of (17). In concluding this section, we shall write this part of Φ in the explicit form

$$\begin{gathered}\Phi(\hat{\Psi},\hat{R})=\frac{2}{\lambda}\sum_{i\mu}\Psi_{i\mu}^2+\sum_{ik\mu\nu}G_{ik\mu\nu}^2\Psi_{i\mu}\Psi_{k\nu}+\frac{T^2}{4}\sum_{ik\mu\nu}(1/\hat{I})_{ik}R_{i\mu\nu}R_{k\mu\nu}\\ -\sum_{\substack{ik\mu\nu\\ \lambda\rho}}G_{ik\mu\nu}G_{ik\rho\lambda}R_{i\mu\rho}R_{k\nu\lambda}-2i\sum_{\substack{ik\mu\\ \nu\rho}}G_{ik\mu\nu}G_{ik\rho\nu}R_{i\mu\rho}\Psi_{k\nu}.\end{gathered} \tag{18}$$

5. CALCULATION OF THE CORRELATORS OF GAUSSIAN FLUCTUATIONS

We introduce the two correlators

$$D_{1\mu\nu}(\mathbf{r}_i-\mathbf{r}_k)=\langle\Psi_{i\mu}\Psi_{k\nu}\rangle, \quad D_{2\mu\nu\lambda\rho}(\mathbf{r}_i-\mathbf{r}_k)=\langle R_{i\mu\nu}R_{k\lambda\rho}\rangle, \tag{19}$$

where the averaging is carried out with weight $\exp\{-p\Phi(\hat{\Psi},\hat{R})\}$. On substituting (18) in (19), we get, after uncomplicated but unwieldy calculations, the following expressions for the Fourier transforms $D_{1\mu\nu}(k)$ and $D_{2\mu\nu\lambda\rho}(k)$:

$$\begin{gathered}D_{1\mu\nu}(k)=\frac{\lambda}{4p}\left\{\frac{1}{1+{}^1/_2\lambda\hat{\Pi}_0(k)}\right\}_{\mu\nu},\\ D_{2\mu\nu\lambda\rho}(k)=D_{0\mu\nu\lambda\rho}(k)+D_{0\mu\nu\rho\lambda}(k),\\ D_{0\mu\nu\lambda\rho}(k)=\frac{I(k)}{pT^2}\left\{\hat{E}+\frac{4I(k)}{T^2}\left[\hat{\Pi}(k)\right.\right.\\ \left.\left.-\frac{\lambda}{2}\hat{\Pi}(k)\hat{L}\frac{1}{\hat{E}+{}^1/_2\lambda\hat{L}\hat{\Pi}(k)\hat{L}}\hat{\Pi}(k)\right]\right\}_{\nu\rho}^{\mu\lambda},\\ (\hat{E})_{\gamma\delta}^{\alpha\beta}=\delta_{\alpha\beta}\delta_{\gamma\delta}, \quad (\hat{L})_{\gamma\delta}^{\alpha\beta}=\delta_{\alpha\gamma}\delta_{\alpha\beta}\delta_{\gamma\delta},\\ \hat{\Pi}(k)=\frac{\hat{K}(k)}{\hat{E}-4T^{-2}I(k)\hat{K}(k)},\\ \{K(k)\}_{\lambda\rho}^{\mu\nu}=d^3\int\frac{d\mathbf{p}}{(2\pi)^3}G_{\mu\nu}(p)G_{\lambda\rho}(\mathbf{p}+\mathbf{k}),\\ \{\hat{\Pi}_0\}_{\mu\nu}=\{\Pi(k)\}_{\mu\nu}^{\mu\nu}.\end{gathered} \tag{20}$$

In (20), the operators $\hat{K}$, $\hat{\Pi}$, $\hat{L}$, and $\hat{E}$ act in a space that is the direct product of the two replica spaces, and $\hat{E}$ is the unit operator in such a space. In this space there is a subspace in which the replica indices coincide. The operator $\hat{L}$ is the unit operator in this subspace. It projects operators on this subspace. For example, the operator

$$\hat{\Pi}_0=\hat{L}\hat{\Pi}\hat{L} \tag{21}$$

is the projection of the operator $\hat{\Pi}$ on this subspace, as is evident from the last line of (20). It must be noted first of all that the correlators D_1 and D_2 are inversely proportional to p. This is quite natural, since for $p\to\infty$ the fluctuations are absent.

We shall now calculate the operator $\hat{\Pi}$. For this purpose it is convenient to introduce operators $\hat{E}_1$, $\hat{E}_2$, and $\hat{\Lambda}_1$, $\hat{\Lambda}_2$, which act in spaces whose direct product is our complete space. For example, $(\hat{E}_1\hat{\Lambda}_2)^{\mu}_{\lambda\rho} = \delta_{\mu\nu}\Lambda_{\lambda\rho}$. Then obviously $\hat{E} = \hat{E}_1\hat{E}_2$. We then get from (12), on taking (8) into account, the following expression for $\hat{\Pi}$:

$$\hat{\Pi} = \Pi_1\hat{E} + \Pi_2(\hat{E}_1\hat{\Lambda}_2 + \hat{E}_2\hat{\Lambda}_1) + \Pi_3\hat{\Lambda}_1\hat{\Lambda}_2,$$
$$\Pi_1(k) = K_1(k)\left\{1 - \frac{4I(k)}{T^2}K_1(k)\right\}^{-1},$$
$$\Pi_2(k) = K_2(k)\left\{1 - \frac{4I(k)}{T^2}K_1(k)\right\}^{-2},$$
$$\Pi_3(k) = K_3(k)\left\{1 - \frac{4I(k)}{T^2}K_1(k)\right\}^{-2} + \frac{8I(k)}{T^2}K_2^2(k)\left\{1 - \frac{4I(k)}{T^2}K_1(k)\right\}^{-3}, \quad (22)$$
$$K_2(k) = \left(\frac{4I_0}{T^2}q + \frac{h^2}{T^2}\right)d^2\int\frac{d\mathbf{p}}{(2\pi)^3}G^2(p)G(\mathbf{p}+\mathbf{k}),$$
$$K_3(k) = \left(\frac{4I_0}{T^2}q + \frac{h^2}{T^2}\right)^2 d^2\int\frac{d\mathbf{p}}{(2\pi)^3}G^2(p)G^2(\mathbf{p}+\mathbf{k}),$$
$$(\hat{\Pi}_0)_{\mu\nu} = (\Pi_1 + 2\Pi_2)\delta_{\mu\nu} + \Pi_3\Lambda_{\mu\nu}.$$

The function $K_1(k)$ was defined in (13). We note at once the following fact. As is evident from (14), for $h=0$ and $T < T_c$ the operator $\hat{\Pi}$ is very singular at small k. As is evident from (22), $\Pi_1(k) \sim k^{-2}$, $\Pi_2(k) \sim k^{-4}$, $\Pi_3(k) \sim k^{-6}$. This fact also leads, as we shall see, to the occurrence of a gapless mode in $\hat{D}_2$.

It is easy to show that the field Ψ is not critical, since in the critical region $\hat{D}_1 \sim 1/\hat{\Pi}_0 \to 0$. Therefore we shall restrict ourselves to study solely of the correlator D_2. Taking into account that

$$\left\{\hat{L}\frac{1}{1+{}^1/_2\lambda\hat{L}\hat{\Pi}\hat{L}}\right\}^{\mu\nu}_{\lambda\rho} = \left\{\frac{1}{1+{}^1/_2\lambda\hat{\Pi}_0}\right\}_{\mu\nu}\delta_{\mu\lambda}\delta_{\nu\rho}, \quad (23)$$

we get, after uncomplicated calculations, the following expresssion for D_2:

$$D_{2\mu\nu\lambda\rho} = (\hat{D}_2)^{\mu\lambda}_{\nu\rho},$$
$$\hat{D}_2 = \frac{I(k)}{pT^2}\left\{\hat{H} + \frac{4I(k)}{T^2}\frac{\Pi_1}{(1+{}^1/_2\lambda\Pi_1)^2}(\hat{C}\hat{H}\hat{C} + \lambda\Pi_1\hat{L}) + \frac{4I(k)}{T^2}\frac{\Pi_2}{(1+{}^1/_2\lambda\Pi_1)[1+{}^1/_2\lambda(\Pi_1+2\Pi_2)]}\hat{C}\hat{B}\hat{C} + \frac{8I(k)}{T^2}\frac{\Pi_3}{[1+{}^1/_2\lambda(\Pi_1+2\Pi_2)]^2}\hat{C}\hat{\Lambda}_1\hat{\Lambda}_2\hat{C}\right\},$$
$$\hat{C} = \hat{E} + {}^1/_2\lambda(\hat{E}-\hat{L})\Pi_1, \quad (\hat{H})_{\nu\rho}{}^{\mu\lambda} = \delta_{\mu\lambda}\delta_{\nu\rho} + \delta_{\mu\rho}\delta_{\nu\lambda},$$
$$(\hat{B})_{\nu\rho}{}^{\mu\lambda} = \delta_{\mu\lambda} + \delta_{\nu\rho} + \delta_{\mu\rho} + \delta_{\nu\lambda}. \quad (24)$$

The general formula (24) is quite unwieldy. To make it easier to understand, we separate $\hat{D}_2$ into four parts:

$$\hat{D}_2 = (\hat{E}-\hat{L})\hat{D}_2(\hat{E}-\hat{L}) + (\hat{E}-\hat{L})\hat{D}_2\hat{L} + \hat{L}\hat{D}_2(\hat{E}-\hat{L}) + \hat{L}\hat{D}_2\hat{L}. \quad (25)$$

The meaning of this decomposition is very transparent. From the definition of $\hat{L}$ in (20) it is clear that $\hat{L}\hat{D}_2\hat{L}$ is the projection of the operator $\hat{D}_2$ on the subspace $\mu=\nu, \lambda=\rho$. This means that $\hat{L}\hat{D}_2\hat{L}$ describes the correlators of the diagonal components of $\hat{R}$, while $(\hat{E}-\hat{L})\hat{D}_2(\hat{E}-\hat{L})$ describes the nondiagonal. The other two operators describe the correlation of the diagonal and nondiagonal components. Since $\hat{L}(\hat{E}-\hat{L}) = (\hat{E}-\hat{L})\hat{L} = 0$, while $\hat{L}^2 = \hat{L}$ and $(\hat{E}-\hat{L})^2 = (\hat{E}-\hat{L})$, we easily get from (24)

$$(\hat{E}-\hat{L})\hat{D}_2(\hat{E}-\hat{L}) = \frac{I(k)}{pT^2}(\hat{E}-\hat{L})\left\{\left(1+\frac{4I(k)}{T^2}\Pi_1\right)\hat{H} + \frac{4I(k)}{T^2}\frac{\Pi_2(1+{}^1/_2\lambda\Pi_1)}{1+{}^1/_2\lambda(\Pi_1+2\Pi_2)}\hat{B} + \frac{8I(k)}{T^2}\frac{\Pi_3(1+{}^1/_2\lambda\Pi_1)^2}{[1+{}^1/_2\lambda(\Pi_1+2\Pi_2)]^2}\hat{\Lambda}_1\hat{\Lambda}_2\right\}(\hat{E}-\hat{L}),$$

$$(\hat{E}-\hat{L})\hat{D}_2\hat{L} = \frac{4I^2(k)}{pT^4}\frac{1}{1+{}^1/_2\lambda(\Pi_1+2\Pi_2)} \times (\hat{E}-\hat{L})\left\{\hat{\Pi}_2\hat{B} + \frac{2\Pi_3(1+{}^1/_2\lambda\Pi_1)}{1+{}^1/_2\lambda(\Pi_1+2\Pi_2)}\hat{\Lambda}_1\hat{\Lambda}_2\right\}\hat{L},$$
$$\hat{L}\hat{D}_2\hat{L} = \frac{2I(k)}{pT^2}\hat{L}\left\{1+\frac{4I(k)}{T^2} \times \frac{\Pi_1+2\Pi_2}{1+{}^1/_2\lambda(\Pi_1+2\Pi_2)} + \frac{4I(k)}{T^2}\hat{L}\hat{\Lambda}_1\hat{\Lambda}_2\hat{L}\frac{\Pi_3}{[1+{}^1/_2\lambda(\Pi_1+2\Pi_2)]^2}\right\}. \quad (26)$$

In the derivation of (26) we have taken into account that $\hat{H}$ commutes with $\hat{L}$ and that $\hat{L}\hat{H}\hat{L} = 2\hat{L}$, while $\hat{L}\hat{B}\hat{L} = 4\hat{L}$. If we substitute (22) in (26), we get explicit expressions for all the correlators. But the expression obtained is very unwieldy and hard to examine. Therefore we shall consider only special cases. We shall not be interested in random magnetic fields; therefore we shall hereafter set $h=0$. Then for $T > T_c$, $\Pi_2 = \Pi_3 = 0$, and we get from (22) and (26)

$$(\hat{E}-\hat{L})\hat{D}_2(\hat{E}-\hat{L}) = \frac{I(k)}{pT^2}(\hat{E}-\hat{L})\hat{H}(\hat{E}-\hat{L})\left[1-\frac{4I(k)}{T^2}K_1(k)\right]^{-1}, \quad (27)$$
$$(\hat{E}-\hat{L})\hat{D}_2\hat{L}_2 = 0,$$
$$\hat{L}\hat{D}_2\hat{L} = \frac{2I(k)}{pT^2}\hat{L}\frac{1+{}^1/_2\lambda K_1(k)}{1+[{}^1/_2\lambda - 4I(k)/T^2]K_1(k)}.$$

Since (14) is satisfied when $T = T_c$, therefore near T_c and at small k, as is evident from (27), $\hat{L}\hat{D}_2\hat{L}$ is finite but $(\hat{E}-\hat{L})\hat{E}_2(\hat{E}-\hat{L})$ is large. This means that the nondiagonal matrix elements $R_{\mu\nu}$ are a critical mode. Since (14) is satisfied everywhere when $T < T_c$, therefore at $k=0$, as follows from (22) and (26), $(\hat{E}-\hat{L})\hat{D}_2\hat{L}$ and $\hat{L}\hat{D}_2\hat{L}$ are finite, but $(\hat{E}-\hat{L})\hat{D}_2(\hat{E}-\hat{L})$ becomes infinite. At small k this correlator is

$$(\hat{E}-\hat{L})\hat{D}_2(\hat{E}-\hat{L}) = \frac{I_0}{pT^2}\left[1-\frac{4I(k)}{T^2}K_1(k)\right]^{-1} \times (\hat{E}-\hat{L})(\hat{H} + {}^1/_2\hat{B} + \hat{\Lambda}_1\hat{\Lambda}_2)(\hat{E}-\hat{L}). \quad (28)$$

It is evident from (28) that the critical correlator is proportional to $1/k^2$ at all $T < T_c$; this signifies the presence of a gapless mode.

Physically, this means the following. In analogy to Ref. 12, one can show that for $\mu \neq \nu, \lambda \neq \rho$

$$D_{Rij} = D_{2\mu\nu\mu\nu ij} - 2D_{2\mu\nu\mu\rho ij} + D_{2\mu\nu\lambda\rho ij} \sim \langle(\langle m_i m_j\rangle_T - \langle m_i\rangle_T\langle m_j\rangle)^2\rangle_c \sim \langle\chi_{ij}^2\rangle_c. \quad (29)$$

where χ_{ij} is the local susceptibility. In (29) it is understood that different indices are not equal to each other; for example, in $D_{2\mu\nu\mu\rho}$, $\rho \neq \mu$ and $\rho \neq \nu$. From (28) and (29), by taking into account the explicit form of H and B, we get the following expression for the Fourier transform $D_R(k)$:

$$D_R(k) = \frac{I_0}{pT^2}\left[1-\frac{4I(k)}{T^2}K_1(k)\right]^{-1} \sim \frac{1}{k^2}.$$

Thus at all $T < T_c$, the generalized susceptibility $D_R(k) \sim k^{-2}$. This is completely analogous to the behavior of the ordinary suceptibility in a ferromagnet, and it is an indication that there is a degenerate ground state in the system.

We note that the gapless mode appears only in the absence of a random magnetic field, which leads to the appearance of a gap and thus removes the degeneracy of the ground state.

In conclusion, we should like to mention the following important fact. As we have already said, the correlator D_1 is not critical. Therefore to calculate the non-Gaussian fluctuations it is necessary to take into account only the correlator D_2. On the other hand, as is evident from (20), the correlator D_2 is proportional to $I(k)$. Therefore if one considers a situation in which the ferromagnetic volume v is small but the random anisotropic exchange has a large radius R_0, then in the calculation of the non-Gaussian corrections the expansion will go not according to the parameter $1/p$, but according to the parameter $1/pR_0^2$. Thus p enters only as a multiplier. Therefore such an expansion is equivalent to the corresponding expansion in the usual Ising spin glass. This fact was mentioned in Ref. 13.

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