Static properties of distributed inhomogeneous Josephson junctions

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The process involving the entry of a vortex structure into a long Josephson junction whose parameters vary greatly but smoothly along the length of the structure is theoretically analyzed. A sine-Gordon-type equation that describes the rapid oscillations of the Josephson current along the junction is reduced with the aid of the asymptotic technique to an "abridged" equation that describes the slow variations of the parameters of such a vortex structure. The abridged equation admits of an analytic solution, and this allows us to find in its explicit form the critical junction current as a function of the magnetic field. The obtained results are in good agreement with the data obtained in recently performed experiments.

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1. INTRODUCTION

Recently, there has been an upsurge in interest in the investigation of the static and dynamic properties of distributed Josephson junctions, i.e., those junction whose length L is significantly longer than the Josephson penetration depth λ_J (see, for example, Refs. 1–4). This interest is explained by the fact that there can be realized in long junctions vortex (many-soliton) structures, which attract attention from the point of view of both general and applied physics.[5,6]

It is well known (see, for example, Ref. 5) that the behavior of the vortex structure in a junction is described by the distribution of the quantity φ, the order-parameter phase difference, over the plane of the junction. In the static situation (i.e., for $\partial\varphi/\partial t=0$), the function $\varphi(x, y)$ satisfies a nonlinear differential equation that, in the one-dimensional case of interest to us, has the form

$$\frac{d}{dx}\left(\mu^2(x)\frac{d\varphi}{dx}\right)=\rho^2(x)\sin\varphi-j_e(x). \tag{1}$$

The functions $\mu^2(x)$, $\rho^2(x)$, and $j_e(x)$ describe the distribution of the specific parameters of an inhomogeneous Josephson junction along the length of the junction.

In the general case Eq. (1) admits of only a numerical solution, and, what is more, the computational difficulties increase rapidly as the ratio L/λ_J increases[7] because of the rapid oscillations of the quantity φ (the period is shorter than, or of the order of, λ_J). At the same time asymptotic methods of solving nonlinear differential equations are known which make essential use of the very fact of the existence of such oscillations (see, for example, Ref. 8), and in which the solution is sought in the form of a rapidly-oscillating function with slowly varying parameters. As applied to equations of the type (1), these methods are based on the fact that, in the case of a homogeneous structure [i.e., for $d\mu/dx=d\rho/dx=j_e(x)\equiv 0$], Eq. (1) has a solution $\varphi_0(z)$ that describes a periodic vortex lattice:

$$\varphi_0(z+2\pi)=\varphi_0(z)+2\pi, \tag{2}$$

where the function z is a dimensionless coordinate:

$$z=hx, \tag{3}$$

h being the vortex structure's wave number, which is proportional to mean magnetic field. In the case of weak perturbations of the homogeneous equation, the complete solution should be close to φ_0, but now the structure constant $a=2\pi/h$ may vary slowly along the length of the junction.

Several attempts are known to have been made[2,4,9] to apply such an approach to equations formally close to (1). In all these investigations the dynamical case, in which the term $\partial^2\varphi/\partial t^2$ is essentially taken into consideration on the left-hand side of the equation, was analyzed. At the same time it is assumed in Refs. 2 and 9 that the slow variations of h along the length of the junction are caused not by the inhomogeneity of the junction, as in the case of Eq. (1), but by small losses (terms of the type $\partial\varphi/\partial t$) and the extraneous current. In Ref. 4 the term containing $\sin\varphi$ is assumed to be small. Therefore, the approach used in these papers cannot be directly used to solve Eq. (1).

The purpose of the present paper is to construct an approximate solution to Eq. (1) by the asymptotic method, and use this solution to describe the behavior of the vortex structures. It is assumed that the characteristic distance over which the parameters of Eq. (1) vary is much greater than λ_J. This assumption is the basis of the method used in the present paper to solve the formulated problem. The method, which leads to the replacement of Eq. (1) by the "abridged" equation (15), is expounded in Sec. 2; in Sec. 3 we discuss the physical meaning of the parameters of the abridged equation. In Secs. 4 and 5 we find with the aid of this equation the critical current of an inhomogeneous junction for different external-magnetic-field strengths. Section 6 is devoted to the comparison of the results obtained with experiment.

2. DERIVATION OF THE ABRIDGED EQUATION

Let us first consider the auxiliary problem of the homogeneous Josephson junction. The formation in such a junction of a static periodic vortex lattice described by Eq. (1) in which μ^2 and ρ^2 are constants and $j_e=0$, i.e., by the equation

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$$k^2\partial^2\varphi_0/\partial z^2=\sin\varphi_0, \tag{4}$$

where the function $\varphi_0(z,k)$ satisfies the periodicity condition (2), is possible. In this case we can assume that the variables z and x are connected by the equation

$$dz/dx=h, \tag{5}$$

the parameter k being defined by the relation

$$\mu h=\rho k. \tag{6}$$

Let us now consider an inhomogeneous Josephson junction whose specific parameters $\mu(x)$ and $\rho(x)$ vary smoothly along the junction, and in which the distributed extraneous current $j_e(x)$ is weak, i.e., for which

$$d\mu(x)/dx\sim d\rho(x)/dx\sim j_e(x)\sim\varepsilon\to 0. \tag{7}$$

This means that the function φ that describes the inhomogeneous lattice and satisfies Eq. (1) can be sought in the form of the expansion

$$\begin{gathered}\varphi=\varphi_0(z,k)+\varphi_1(z,k)+\varphi_2(z,k)+\ldots,\\ \varphi_1\sim\varepsilon,\quad \varphi_2\sim\varepsilon^2,\ldots.\end{gathered} \tag{8}$$

The quantities h and k satisfy, as before, the relations (5) and (6), although now they can be slowly varying functions of the variable x.

The function φ_1 and the subsequent terms of the expansion (8) describe the vortex deformation caused by the inhomogeneity of the vortex structure. Let us require that the function φ given by the expansion (8) satisfy a condition similar to (2), i.e., that φ_1 be a periodic function of z:

$$\varphi_1(z+2\pi,k)=\varphi_1(z,k). \tag{9}$$

Substituting the expansion (8) into Eq. (1), linearizing it with respect to φ_1, and taking the relations (4)–(6) into account, we obtain in the first approximation in ε the following equation for φ_1:

$$k^2\frac{\partial^2\varphi_1}{\partial z^2}-\varphi_1\cos\varphi_0=f(z), \tag{10}$$

$$\rho^2 f(z)=-\frac{d(\mu^2h)}{dx}\frac{\partial\varphi_0}{\partial z}-2\mu^2h\frac{dk}{dx}\frac{\partial^2\varphi_0}{\partial z\partial k}-j_e. \tag{11}$$

The general solution to the inhomogeneous linear equation (10) can be written in the form

$$\varphi_1=\varphi_1^{(0)}+\frac{1}{k^2}\frac{\partial\varphi_0}{\partial z}\int_{z_0}^{z}dz'\int_{z'}^{z}dz''\frac{\partial\varphi_0}{\partial z'}f(z')\left(\frac{\partial\varphi_0}{\partial z''}\right)^{-2}, \tag{12}$$

where the second term is the particular solution to Eq. (10), while $\varphi_1^{(0)}$ is the general solution of the corresponding homogeneous equation:

$$\varphi_1^{(0)}=C_1\frac{\partial\varphi_0}{\partial z}+C_2\frac{\partial\varphi_0}{\partial z}\int_{z_0}^{z}dz'\left(\frac{\partial\varphi_0}{\partial z'}\right)^{-2}, \tag{13}$$

where C_1, C_2, and z_0 are arbitrary constants, any two of which can be considered to be independent.

Substituting the expressions (12) and (13), written for the point $z=z_0$, into the periodicity condition (9) and the analogous condition, following from it, on the derivative $\partial\varphi_1/\partial z$, and subtracting the resulting equalities from each other, we obtain the condition

$$\langle f(z)\partial\varphi_0/\partial z\rangle=0, \tag{14}$$

where the brackets $\langle\ldots\rangle$ denote averaging over the period 2π of the argument z. Thus, Eq. (10) possesses the required periodic solution φ_1 only upon the fulfillment of the condition (14), which assumes, when the explicit form (11) of the function $f(z)$ is used, the form

$$\frac{d}{dx}[\mu^2hA(k)]=-j_e, \tag{15}$$

$$A(k)=\langle(\partial\varphi_0/\partial z)^2\rangle. \tag{16}$$

Equation (4) allows us to represent the function $A(k)$ in the parametric form:

$$A=\frac{4}{\pi\gamma k}E(\gamma),\quad k=\frac{\pi}{\gamma K(\gamma)}, \tag{17}$$

where $K(\gamma)$ and $E(\gamma)$ are the complete elliptic integrals of the first and second kinds respectively.

The condition (15) is the sought "abridged" equation that describes together with the relation (6) the slow variation of the parameters h and k of the vortex lattice along the junction.

3. THE PHYSICAL MEANINGS OF THE VARIABLES h AND k AND THE FUNCTION $A(k)$

As has already been indicated, h is the wave number of the vortex lattice, and is proportional to the local magnetic field, averaged over the local lattice quasiconstant $a=2\pi/h$, in the junction. The variable k, which is defined by the equality (6), differs from the quantity h by the factor μ/ρ, which has the meaning of a local value of the Josephson penetration depth $\lambda_J(x)$, i.e.,

$$k(x)=2\pi\lambda_J(x)/a(x). \tag{18}$$

It is well known (see, for example, Ref. 5) that the properties of the vortex lattice change significantly when we go over from the case $a\gtrsim\lambda_J$ (thin lattice) to the case $\lesssim\lambda_J$ (close lattice). It is precisely because of this that the correct criterion for transition from the thin to the close lattice is a condition for k (namely, $k\sim 1$), and not for h.

In order to elucidate the physical meaning of the function $A(k)$, let us consider the homogeneous (μ=const, ρ=const) portion of a long Josephson junction containing a stationary vortex structure acted upon by a distributed extraneous current j_e (Ref. 3). Using Eqs. (6) and (15), we obtain

$$-j_e=\mu^2\frac{dh}{dx}\frac{d(Ak)}{dk}. \tag{19}$$

The right-hand side of the equality (19) can be regarded as an elastic force exerted by the slightly deformed ($dh/dx\neq 0$) vortex lattice, and balancing the distributed impressed Lorentz force ($\propto j_e$). Thus, the quantity

$$E=d(Ak)/dk \tag{20}$$

gives the specific modulus of elastic deformation of the static lattice. From the relations (17) we have in the thin-lattice limit ($k\ll 1$)

$$A\approx\frac{4}{\pi k}\left(1+\frac{8\pi}{k}e^{-2\pi/k}\right),\quad E\approx\frac{64\pi}{k^3}e^{-2\pi/k}, \tag{21}$$

and for the close lattice ($k\gg 1$)

$$A\approx 1,\quad E\approx 1. \tag{22}$$

Thus, in the case of a thin vortex lattice the elastic modulus E is exponentially small, a fact which is naturally explained by the exponential character of the interaction between the individual vortices (see, for example, Ref. 10).

4. FORMATION OF A STATIC DOMAIN IN AN INHOMOGENEOUS JUNCTION

Let us use the abridged equation (15) to describe the penetration of a magnetic field h into the junction. The penetration of a magnetic field through the edge of a homogeneous junction has been well studied (see, for example, Ref. 11). Of greatest interest in the case of inhomogeneous junctions are those situations in which the linear critical-junction-current density tends to zero as we approach the junction edges ($x=0$ and $x=L$), i.e., in which

$$\rho(x)\to 0 \quad \text{as} \quad x\to 0, L. \tag{23}$$

In this case Eq. (15) is applicable throughout the junction, including its edges.[1)]

The function $\mu(x)$, which describes the specific inductance of the junction, normally remains finite near the junction boundaries in this case. Then in the presence of an external magnetic field (i.e., for $h\neq 0$) the product μh and, on account of the equality (6), the product ρk are finite at the junction boundaries, and we have near a junction boundary

$$\frac{d\varphi}{dx}\approx h+(k\rho)^{-2}\frac{d\rho^2}{dx}\sin z(x), \quad x\approx 0, L, \tag{24}$$

where the second term should be small compared to the first. The relation (24) determines the relation between the magnetic field $d\varphi/dx$ at a given point and its lattice-quasiconstant-averaged value h.

Let there be applied at the junction edge $x=0$ an external field $h_0=(d\varphi/dx)_{x=0}$. Then we have from the equality (24) in the zeroth approximation in ε [see (7)] the following boundary condition for the abridged equation (15):

$$h(0)=h_0. \tag{25}$$

In the absence of a distributed extraneous current ($j_e=0$), Eq. (15) with allowance for the equality (6) yields

$$\mu(x)\rho(x)A(k)k=C, \tag{26}$$

where the constant C of integration should be determined from the boundary condition (25). Using the asymptotic form (22), we have

$$C=\mu^2(0)h_0. \tag{27}$$

It follows from Eq. (26) and the relations (6) and (17) that, as we go into the interior of the junction, where the linear critical-current density (ρ^2) increases, the magnetic field h decreases together with the parameter k. Thus, the external magnetic field penetrates into the inhomogeneous junction, gradually decreasing until it vanishes at some point x_0. In other words, there is formed over the segment $[0, x_0]$ a static domain—a region filled with interacting vortices.

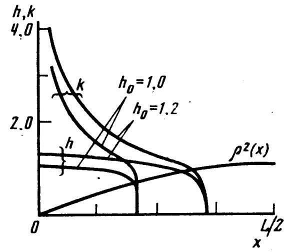

FIG. 1. Entry of a static vortex domain into an inhomogeneous junction ($\mu\equiv 1$, $\rho^2=\sin\pi X/L$) as the field h_0 at the boundary increases. The variables h and k, which characterize the vortex density, were computed for h_0 values equal to 1.0 and 1.2 with the aid of the relations (6), (26), and (27). The right domain boundary reaches the middle, $x=L/2$, of the junction when $h_0=(h_0)_{\max}=4/\pi\approx 1.27$. The vortices begin to move when $h_0\geqslant 4/\pi$.

To find the domain boundary x_0, let us note that, as $x\to x_0$, not only h, but also k, tends to zero, since μ and ρ are then finite. Taking the relation (21) into account, we obtain for x_0 from the equalities (26) and (27) the equation

$$4\pi^{-1}\mu(x_0)\rho(x_0)=\mu^2(0)h_0. \tag{28}$$

From this it follows that, in the case of weak external fields (i.e., for $h_0\to 0$), the domain boundary x_0 is located in the immediate neighborhood of the junction boundary $[\rho(x_0)\to 0,\ x_0\to 0]$. An increase in the external-magnetic-field intensity leads to the growth of the domain: its boundary x_0 shifts into the region of x-coordinate values corresponding to ever increasing values of the function $\mu(x)\rho(x)$.

The process involving the entry of the domain into the junction is depicted in Fig. 1, where the functions $h(x)$ and $k(x)$ were computed with the aid of the relations (6), (26), and (27) for the case in which $\mu(x)\equiv 1$ and $\rho^2(x)=\sin(\pi x/L)$. The sharp decrease in the field $h(x)$ and in $k(x)$ near the domain boundary x_0 is explained by the exponential character of the interaction between neighboring vortices in a thin ($k\ll 1$) lattice.

5. THE CRITICAL FIELD OF AN INHOMOGENEOUS JUNCTION

The growth of the domain with increasing field intensity will continue until x_0 attains the point at which the function $\mu(x)\rho(x)$ has it maximum value. No static solutions exist at higher-field-intensity values, i.e., the vortices begin to move steadily through the junction. From Eq. (28) we have for the determination of this critical value of the h_0 field the relation

$$\mu^2(0)(h_0)_{\max}=4\pi^{-1}(\mu\rho)_{\max}. \tag{29}$$

For the case of a homogeneous junction (i.e., for μ, $\rho=$const), $(h_0)_{\max}$ coincides with the well-known "thermodynamic" critical field, in which the entry of vortices into the junction becomes energetically advantageous.[5] At the same time, the actual critical field of a homogeneous junction is $\pi/2$ times higher than the thermodynamic critical field because of the pinning of vortices at the potential barrier formed by the sharp junction edge. The formula (29) shows that, in the zeroth approximation in ε, there is generally no difference between these critical fields in an inhomogeneous junction with the "smooth" boundary (23), which is due to the significantly weaker force with which vortices are pinned to the smooth boundary.

Let us now consider the case in which the external field is applied to both ends of the junction:

$$h=h_0 \text{ for } x=0, \quad h=h_L \text{ for } x=L \,. \tag{30}$$

Then a similar domain is formed at the other ($x=L$) junction edge. Let us construct the region of static domain states in the plane of the external magnetic fields (in the same way as is done in Ref. 1 for a homogeneous junction with sharp edges). Until the two domains merge into one domain filling the entire junction, they are independent, and each of them is characterized by its own constant C determined from the corresponding boundary condition. This means that the static states fill at least a square half of whose side is equal to the critical value of the magnetic field:

$$(\mu^2)_{0,L}(h_{0,L})_{max}=4\pi^{-1}(\mu\rho)_{max}. \tag{31}$$

At sufficiently high values of h_0 and h_L the domains merge into a single domain characterized by a single constant C in Eq. (26). Comparing in this case the boundary conditions (27) written for the points $x=0$, L, we obtain the condition

$$\mu^2(0)h_0=\mu^2(L)h_L, \tag{32}$$

the fulfillment of which is necessary for the existence of a single static domain in the junction. Physically, the relation (32) implies the equality of the dimensional magnetic fields H_0 and H_L at the junction edges, i.e., the absence of transport current through the junction. Thus, in the zeroth approximation in ε the region of static states of the junction consists of the square (31) supplemented by the straight line $H_0=H_L$, which continues one of the diagonals of the square (see Fig. 2). Thus, in the considered zeroth approximation the combined static state of the entire junction is possible only when the transport current through the junction is exactly equal to zero, a fact which is explained by the smallness of the potential barrier at the junction boundary. To allow for the pinning of vortices at this barrier, we must

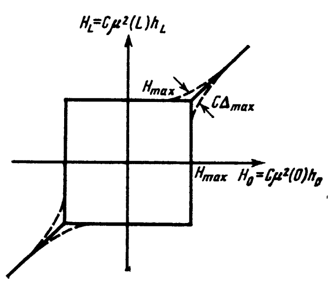

FIG. 2. Region of static states of the junction in the plane of the boundary magnetic fields H_0 and H_L. In the zeroth approximation in the small parameter ε, this region consists of a square with side $2H_{max}$, where $H_{max}=C\mu^2(0)(h_0)_{max}=C\mu^2(L)(h_L)_{max}$, and the continuation of its diagonal $H_0=H_L$. Allowance in the first approximation in ε for the effect of the potential barrier at a junction boundary leads to the expansion, indicated by the dashed lines, of the region of static states.

retain in the relation (24) the second (small) term, which takes account of the specific vortex structure near the points $x=0$ and $x=L$. When the boundary field values are higher than the critical values (31), we have in the first approximation in ε at the points $x=0, L$

$$\mu^2(0)h_0=C+\left[\frac{\mu^4}{C^2}\left(\frac{d\rho^2}{dx}\right)\sin z\right]_{x=0}, \tag{33}$$

$$\mu^2(L)h_L=C+\left[\frac{\mu^4}{C^2}\left(\frac{d\rho^2}{dx}\right)\sin z\right]_{x=L}. \tag{34}$$

Now the difference between the dimensional magnetic fields H_0 and H_L and, consequently, the transport current through the junction, which are proportional to the quantity

$$\Delta=\mu^2(0)h_0-\mu^2(L)h_L, \tag{35}$$

can be nonzero. The definition (35) of the quantity Δ_{max} assumes, when Eqs. (33) and (34) are taken into account, the form

$$\Delta=\frac{1}{\mu^4(0)h_0^2}\left\{\left[\mu^4\left(\frac{d\rho^2}{dx}\right)\sin z\right]_{x=0}-\left[\mu^4\left(\frac{d\rho^2}{dx}\right)\sin z\right]_{x=L}\right\}. \tag{36}$$

For long junctions (i.e., for $L\gg\lambda_J$) the right-hand side of the expression (36) oscillates rapidly as the field h_0 increases, since an insignificant change in the quantity h_0 leads to a significant change in the argument z, which is defined by Eq. (5). Let us construct the envelopes (the dashed curves in Fig. 2) for the oscillating curves (36). In order to determine the distance between the envelopes, let us introduce the Δ-like quantity Δ_{max} (see Fig. 2) characterizing the amplitude of the oscillations. From the equality (36) we have

$$\Delta_{max}=\frac{1}{\mu^4(0)h_0^2}\left\{\left|\mu^4\left(\frac{d\rho^2}{dx}\right)\right|_{x=0}+\left|\mu^4\left(\frac{d\rho^2}{dx}\right)\right|_{x=L}\right\}, \tag{37}$$

where $|\Delta|\leq\Delta_{max}$. The quantities $|\Delta|$ and Δ_{max} are equal at the points of contact of the oscillating curves and their envelopes. It follows from the formula (37) that the amplitude of the oscillations falls off more rapidly ($\sim h_0^{-2}$) in such a junction than in a homogeneous junction with sharp edges ($\sim h_0^{-1}$) (see Refs. 11–13).

6. COMPARISON OF THE OBTAINED RESULTS WITH EXPERIMENT

Let us compare the results obtained in Sec. 5 with the experimental data reported in Ref. 7. Figure 3 shows the configuration of the inhomogeneous distributed Josephson junction used in the experiment. By writing the equation for the function, φ, describing the static behavior of the vortex structure in such a junction in the usual fashion,[5,7] we can verify that this equation coincides with Eq. (1) if as the units of length and linear current density we respectively take Josephson penetration depth

$$\lambda_J=(c\Phi_0/8\pi^2 j_c\Lambda_1)^{1/2},$$

and the quantity j_cW_0. Here $\Phi_0=hc/2e$ is the magnetic-flux quantum, j_c is the critical Josephson current density, $\Lambda_1=2\lambda+d_1$ is the magnitude of the magnetic gap inside the junction, λ is the depth of penetration of the magnetic field into the superconducting films (the electrodes), and d_1 is the thickness of the dielectric layer in the junction. In this case

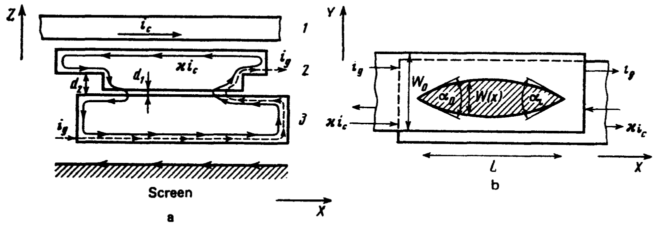

FIG. 3. Diagrammatic representation of the experimental structure used by Broom *et al.*[7]: a) the XY plane of the Josephson junction; the junction area is the hatched region; b) the shape of the XZ plane. A "control" current i_c flowing through the superconducting electrode 1 induces in the electrode 2 a current $\varkappa i_c$ ($\varkappa < 1$) whose distribution over the electrodes 2 and 3 is indicated by the continuous curves. The "gate" current i_g (the dashed curve) determines the transport current flowing through the junction.

$$\rho^2(x)=W(x)/W_0, \tag{38}$$

$$\mu^2(x)=m+(1-m)\rho^2(x). \tag{39}$$

The parameter $m \leq 1$ is equal to the ratio of the inductance per unit area of the junction to the corresponding quantity outside the junction, or, alternatively,

$$m=\Lambda_1/\Lambda_2=(2\lambda+d_1)/(2\lambda+d_2), \tag{40}$$

where Λ_2 and d_2 are the magnitude of the magnetic gap and the thickness of the dielectric layer outside the junction.

For the specific junction configuration used in the experiment, the junction width $W(x)$ near the edges behaves linearly (see Fig. 3a):

$$W(x)=\begin{cases}\alpha_0 x & \text{for} \quad x\to 0\\ \alpha_L(L-x) & \text{for} \quad x\to L.\end{cases} \tag{41}$$

Let us consider the circuit for prescribing the currents in the junction.[7] A "control" current i_c flows along the film 1 and induces in the upper electrode 2 a current $\varkappa i_c$, where $\varkappa$ is the coupling coefficient (equal to 0.82 in the experiment in question[7]). This current, flowing through the junction edges and the lower film 3 (as depicted in Fig. 3a by the continuous curve), produces an external magnetic field at the junction edges. Besides this current, the magnetic field at the junction edges also induces a "gate" (transport) current i_g, which is passed through the junction, and is distributed over the electrodes as shown by the dashed curve in Fig. 3a.

To find the magnetic field produced by these currents, let us use the well-known relation (see, for example, Ref. 5) connecting $d\varphi/dx$ with the strength of the current i flowing through the film:

$$\frac{d\varphi}{dx}=\frac{2\pi}{\Phi_0}\frac{1}{c}\mathscr{L}i, \tag{42}$$

where $\mathscr{L}$ is the linear inductance density of the system shown in Fig. 3. Taking account of the above-considered distributions of the currents $\varkappa i_c$ and i_g, we can write Eq. (42) at the junction edges (i.e. for $\rho=0$) in the form

$$h_0=2\varkappa i_c/mi^*, \tag{43}$$

$$h_L=2(\varkappa i_c+i_g)/mi^*, \tag{44}$$

where the current $i^*=2j_c\lambda_J W_0$ is the critical current of a homogeneous distributed junction with sharp edges in zero magnetic field.[11]

Using Eqs. (43) and (44), let us reconstruct in the plane of the currents i_c and i_g the static-state region shown in Fig. 2, limiting ourselves to the $i_g \geq 0$ case. In the zeroth approximation in ε, this region consists of a triangle (see, Fig. 4a) and the straight line $i_g=0$. The relations (31) and (44) allow us to determine the critical value i_0 of the transport current i_g is zero external magnetic field (i.e., for $i_c=0$):

$$i_0=\frac{2}{\pi}\left\{\frac{W_{max}}{W_0}\left[m+(1-m)\frac{W_{max}}{W_0}\right]\right\}^{1/2} i^*. \tag{45}$$

In the next, first approximation in ε, there exists outside the triangle a "residual" critical current that oscillates rapidly as the control current i_c is varied. Subtracting Eq. (43) from Eq. (44), and using the definition (37) of the quantity Δ_{max}, we arrive at the value of the ordinate of the envelope of the oscillating critical current:

$$(i_g)_{max}=\left(\frac{\pi}{4}\right)^3\left(\frac{i_0}{\varkappa i_c}\right)^2 m^2(\alpha_0+\alpha_L)\frac{\lambda_J}{W_0}\left\{\frac{W_{max}}{W_0}\left[m+(1-m)\frac{W_{max}}{W_0}\right]\right\}^{-3/2} i_0. \tag{46}$$

To compare the obtained results (45) and (46) with experiment, let us use the experimental parameter values given in Ref. 7: $j_c=1.6\times 10^3$ A/cm^2, $L\approx 6\lambda_J=60$ μm, $W_{max}=32$ μm, $W_0=35.5$ μm, $d_1 \ll 2\lambda$, and $d_2=200$ nm. Using the well-known expression for λ_J (see, for example, Ref. 11) and the known value of j_c, and taking account of the uncertainty in the experimental parameter values, we find that $\lambda=80\pm 15$ nm. Thus, the values of the magnetic gaps inside and outside the junction (Λ_1 and Λ_2) are respectively equal to 160 ± 30 nm and 360 ± 30 nm.

Substituting the numerical values of the parameters into the formula (45), we find that the strength of the current $i_0=6.6\pm 0.7$ mA, whereas the experimental value for $i_0=7.7\pm 0.2$ mA.

It follows from experiment[7] that the side oscillations first come in contact with their envelope when $i_c=(1.6\pm 0.1)i_0$, at which value $i_g=0.60\pm 0.05$ mA. For the second and third contacts the ratio i_c/i_0 is equal to 2.1 $\pm$ 0.1 and 2.4 $\pm$ 0.1, while the current i_g is respectively equal to 0.30 ± 0.05 and 0.25 ± 0.05 mA. Taking account

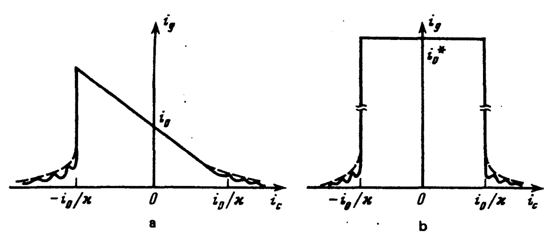

FIG. 4. Region of static states of the junction in the plane of the currents i_c and i_g: a) the transport current is fed into the junction from one edge; b) the transport current is injected into an interior region of dimension $L' < L$ (Refs. 1 and 3). The critical current i_0 is given in the zeroth approximation in ε by the formula (45); the current $i_c^*=f_c L' W_{max}$ is significantly higher than i_0 if $L \gg \lambda_j$. The rapid critical-current oscillations that occur when $|i_c| > i_0\varkappa$ are represented diagrammatically; the dashed line depicts their envelope, and is given by the formula (46).

of the fact that $\alpha_0 \approx \alpha_L \approx 1.20 \pm 0.5$, we find from the expression (46) the value $i_\varepsilon = 0.4 \pm 0.1$ mA for the first contact and the values 0.20 ± 0.08 and 0.17 ± 0.06 mA for the second and third.

We should, in estimating the correlation between the theory and experiment, take the following facts into account: First, the theoretical approach used presupposes that a fairly large ($\gg 1$) number of vortices are concentrated in the Josephson junction, whereas the experimental relation $L \approx 6\lambda_J$ admits of the existence in the junction of only two-to-three vortices for those i_c/i_0 values at which the comparison was performed. Secondly, in computing the current i_0 from the formula (45), we did not take into consideration the pinning of vortices at the junction edges, an effect which can, according to the formula (46), give corrections of the order of several tenths of a milliampere, and thus place the difference between the theoretical and experimental values within the confidence interval.

Thus, the foregoing comparison of the results obtained in the present paper with the experimental data indicates a good agreement between them even for quite short junctions.

In conclusion of the present section, let us mention the interesting situation (see Refs. 1 and 3) in which the transport current is fed to the junction along the x axis in a distributed fashion and the magnetic field, as in the above-considered case, is produced by a control current i_c. Then the function $j_e(x)$ in the right-hand sides of Eqs. (1) and (15) is nonzero in some region of length $L' < L$. We shall assume that the linear critical Josephson current density $\rho^2(x)$ is this region is practically equal to its maximum value.

So long as the current i_c satisfies the condition $\varkappa|i_c| < i_0$, the domains existing near the junction boundaries do not extend to the transport-current-injection region. Therefore, the distributed current $j_e(x)$ has no effect on the vortices in the domains, and the critical transport-current value $i_0{}^*$ is equal to $j_c W_{\max} L'$, which, for $L' \gg \lambda_J$, is significantly higher than the current i_0 found earlier (see Fig. 4b). When the current i_c attains the value $i_0/\varkappa$, the domain boundaries come up to the boundaries of the injection region. Then the transport current j_e begins to exert on the vortices of one of the domains a Lorentz force that leads to the movement of the vortices through the Josephson junction. Consequently, in the zeroth approximation in ε, the critical current $(i_\varepsilon)_{\max}$ is equal to zero when $\varkappa|i_c| > i_0$. But in this case also allowance for the barrier at a junction boundary leads to a finite critical current $(i_\varepsilon)_{\max}$ that is again given by the formula (46). As a result, there arises a dependence, depicted in Fig. 4b, of the critical current on the control current. Such a quasirectangular characteristic may be quite advantageous for a number of applications of distributed Josephson junctions.

7. CONCLUSION

In the present paper we have described a procedure for constructing an approximate solution φ to the nonlinear differential equation (1) on the basis of the slowness of the variation of the coefficients of the equation. The problem then reduces to the abridged equation (15), which is essentially a recipe for the correct construction of the function $\varphi(x)$ from the known solution to Eq. (4). The abridged equation advantageously differs from the original equation (1) in that it is a simple differential relation for the variable k, which is a slowly varying function of the coordinate x. It then turns out that the main characteristics of the vortex structure can be easily obtained in their analytic forms with the aid of Eq. (15). A comparison of the obtained results [the formulas (45) and (46)] with the experiment of Broom *et al.*[7] shows that the expounded approach describes very well the properties of even relatively short Josephson junctions.

Finally, let us note that Eq. (15) allows us to analyze a number of other properties of inhomogeneous distributed Josephson junctions. For example, we can easily find with its aid the magnetization curves of junctions of complex shape, the remanent magnetization due to the capture of vortices on the relief of the junction. Furthermore, the approach developed in the present paper can be used to describe the nonstationary processes that occur in inhomogeneous junctions even when the processes are strongly damped.

[1)]If the edges are sharp (i.e., if ρ falls abruptly to zero), then Eq. (15) is not directly applicable at the points $x = 0$, L. But even in this case it is possible to describe the properties of the junction with the aid of Eq. (15) after an appropriate scaling of the boundary conditions.

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