

Nonlinear acoustic effects in metals near the electron-topological transition point

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(Submitted 14 May 1981)

Zh. Eksp. Teor. Fiz. **81**, 1528-1541 (October 1981)

It is shown that metals near the electron topological transition point (phase transition of order $2/3$) are a principally new type of acoustic medium with a nonanalytic dependence of its elasticity characteristics on the strain. Elasticity-theory equations are derived for such media and it is shown that when the strain is small the nonlinear corrections due to the closeness to the transition point are always large compared with those due to the cubic anharmonicity. Exact solutions of the equations are found in the form of simple waves. Nonlinear distortions of these waves during propagation are investigated. The role of quantum effects, which may be pronounced in the immediate vicinity of the transition point, where the de Broglie wavelength of the "slow" Fermi electrons becomes comparable with the characteristic macroscopic scales, is investigated. These effects lead to nonlocal equations in elasticity theory.

PACS numbers: 62.20.Fe, 03.40.Dz, 62.65.+k, 43.25.Ba

INTRODUCTION

According to the established terminology, an electron-topological transition (ETT), or a phase transition of order $2-\frac{1}{2}$, is the name now given to the phase transition that takes place in metals following a change in the topology of the Fermi surface (formation of a cavity, the breaking of a bridge, etc.). ETT were first investigated in Ref. 1, where it was noted that at sufficiently low temperatures the thermodynamic potentials of the metal acquire near the transition point peculiar increments such that the second derivative of these quantities with respect to the variables connected with the ETT (for example, the derivative of the free energy with respect to pressure) remain finite at the transition point, whereas the third derivatives diverge.

These properties of the metal are due to the appearance near the ETT on the Fermi surface of such sheets (formation of new cavities) or such sections (breaking of a neck, etc.), to which anomalously small values of the electron quasimomentum correspond. Electrons with such a value of the Fermi momenta will be called hereafter slow, to distinguish them from fast electrons whose Fermi momentum is of the order of \hbar/a , where a is the characteristic distance between the atoms. Singular increments to the thermodynamic potential correspond to the contribution made by the slow electrons, so that such increments exist only on one side of the ETT point, namely in the region where the Fermi surface has the smaller connectivity.

The foregoing gives grounds for assuming that when elastic waves propagate in a single crystal of a metal near the ETT point, nonlinear effects should be observed, much stronger than the nonlinearities due to the usual cubic anharmonicity. The present paper is devoted to the investigation of such effects. We shall neglect terms that have the same (or higher) order of smallness than the cubic anharmonicity, and it is this which determine the accuracy of the analysis.

The exposition follows the following sequence. In Sec. 1 we derived nonlinear elasticity-theory equations that

are valid near the ETT point in the "classical" region. In Secs. 2 and 3 we discussed the influence of quantum effects in the region where the de Broglie wavelength of the slow electrons becomes large compared with the characteristic scales. In Sec. 4 we obtain exact solutions for the equations obtained in Sec. 1 for the propagation of the elastic waves in a metal near the ETT point.

The solutions take the form of simple (Riemann) waves. We investigate the nonlinear evolution of such waves. We find the toppling length and the law of the damping of the wave after it topples and the shock wave is produced. Section 5 is devoted to a discussion of the results. For simplicity, all the rigorous results were obtained for the case $T=0$. The influence of a finite metal temperature on the considered effects is discussed in Sec. 5.

1. "CLASSICAL" EQUATIONS OF ELASTICITY THEORY NEAR THE ETT POINT

Before we proceed to writing down the equations of elasticity theory near the ETT point, we must find how the main parameter of the transition $z = \epsilon_c - \epsilon_F$, (ϵ_c is the critical value of the electron energy corresponding to the change of the topology of the Fermi surface and ϵ_F is the Fermi energy) depends on the strain tensor u_{ij} of the crystal. For good metals, the characteristic time in which a restructuring of the Fermi surface takes place is of the order of 10^{-16} – 10^{-15} sec. This is much less than the characteristic time of variation of u_{ij} on account of the propagation of an elastic wave in the metal. It can therefore be assumed in the considered problem that the electron subsystem responds without delay to the change of the strain field, so that when the contribution introduced by the free electrons to the elastic characteristics of the metal is determined, the instantaneous distribution of the strains can be regarded as a static field.

In addition, it is assumed throughout this section that the micro- and macro-scales of the problem do not overlap, i.e., the de Broglie wavelength of the Fermi electrons is small enough even for small particles. In this

case the strain field can be regarded as locally homogeneous relative to the electron system. The change of the Fermi-electron energy following application of the strain field and following a simultaneous change of the electron momentum is

$$\delta\varepsilon_F = (\mathbf{v}_F, \delta\mathbf{p}) + \lambda_{ij}(\mathbf{p}_{F0}) u_{ij} = v_{F0} \delta p_{\perp} + \lambda_{ij}^0 u_{ij}, \quad (1)$$

where $\mathbf{v}_F = \partial\varepsilon_F / \partial\mathbf{p}$, λ_{ij} is the tensor of the deformation potential ($\lambda_{ij} = \lambda_{ji}$, $|\lambda_{ij}| \sim \varepsilon_F$), δp_{\perp} is the projection of $\delta\mathbf{p}$ on \mathbf{v}_{F0} . The zero label here and below denotes the characteristics of the metal at $u_{ij} = 0$.

From the condition for the conservation of the electron neutrality of the metal when the strain field is applied, it follows that the change of the Fermi surface must not lead to a change in the value of the phase volume bounded by this surface, i.e.,

$$\oint_{\varepsilon_F} \delta p_{\perp} d\sigma_F = 0. \quad (2)$$

From this, using relation (1), we easily obtain that

$$\delta\varepsilon_F = u_{ij} \oint_{\varepsilon_F} \frac{\lambda_{ij}}{v} d\sigma_F / \oint_{\varepsilon_F} \frac{d\sigma_F}{v} = \overline{\lambda_{ij}^0} u_{ij}. \quad (3)$$

If the surface ε_{F0} is multiply connected, then the integration in (3) does not extend over all its sheets. It is important, however, that if such a multiply connected Fermi surface consists of a cavity containing a large number of electronic states, and a small cavity¹⁾ whose volume is proportional to $\sim |z|^{3/2}$, then the integration over the small cavity introduces into the fundamental equation corrections $\sim |z|^{1/2} u_{ij}$, which can be disregarded in the considered approximation. A similar result is easily obtained also in the case when a neck is broken. We arrive at the following important conclusion: slow electrons make no contribution to the Fermi-energy correction due to the crystal deformation.

Since ε_c is determined from the condition $(\partial\varepsilon / \partial\mathbf{p})_c = 0$, it follows that $\delta\varepsilon_c = \lambda_{ij}^0 u_{ij}$ upon deformation of the crystal. We find thus that in a deformed crystal

$$z = z_0 + \Lambda_{ij} u_{ij}, \quad (4)$$

where $z_0 = \varepsilon_{c0} - \varepsilon_{F0}$ and $\Lambda_{ij} = \lambda_{ij}^0 - \overline{\lambda_{ij}^0}$. The transition point is then determined by the condition $z = 0$.

Assume for simplicity that the ETT corresponds to formation in the elementary Brillouin cell of g new spherical electron surfaces at $\varepsilon_F > \varepsilon_c$ (i.e., at $z < 0$). Then the Fermi momentum p_{FS} of the slow electrons is connected with z by the obvious relation

$$p_{FS}^2 / 2m^* = |z|, \quad (5)$$

where m^* is the effective mass of the electron in a small zone. The singular free-energy increment due to the slow electrons is equal to¹

$$\delta\mathcal{F} = \int_{\varepsilon_c}^{\varepsilon_F} \int_{\mathbf{v}} (\varepsilon - \varepsilon_F) \delta\nu(\varepsilon) dV d\varepsilon = - \frac{2gV}{15\pi^2 \hbar^3} (2m^*)^{3/2} |z|^{1/2} \theta(-z), \quad (6)$$

where $\delta\nu(\varepsilon)$ is the change of the state density on account of the ETT, $\theta(z)$ is the Heaviside unit function [$\theta(z) = 0$ at $z \leq 0$ and $\theta(z) = 1$ at $z > 0$] and its presence reflects the fact that $\delta\mathcal{F} \neq 0$ on only one side of the transition point.

The variational derivative $\delta\mathcal{F} / \delta u_{ij}$, which reduces in

this case to ordinary differentiation with account taken of (4), determines the nonlinear increment to the stress tensor σ'_{ij} , after which we obtain the elasticity-theory equations in standard fashion. In the general case, at an arbitrary form of the ETT, these equations take the form

$$\rho \partial^2 u_i / \partial t^2 = \partial \sigma_{ij} / \partial x_j, \quad (7)$$

$$\sigma_{ij} = \sigma_{ij}^0 + \sigma'_{ij}, \quad (8)$$

$$\sigma_{ij}^0 = \lambda_{ijm} u_{im}, \quad (9)$$

$$\sigma'_{ij} = B \Lambda_{ij} \theta(\pm z) |z|^{1/2}; \quad (10)$$

here λ_{ijm} is elastic-moduli tensor (not to be confused with the tensor λ_{ij} of the strain potential). The constant B is of the order of $|B| \sim a^{-3} \varepsilon_F^{3/2}$; the exact value of B , its sign, as well as the sign of the argument of the θ function are determined by the specific form of the ETT. Thus, in the example considered above,

$$\theta = \theta(-z), \quad B = -g(2m^*)^{3/2} / 3\pi^2 \hbar^2.$$

Let us note some important properties of Eqs. (7)–(10). First, we found that the value of the nonlinear increment to the stress tensor at a given point is expressed in terms of the value of the strain tensor at the same point. This locality of the equations of motion is a direct consequence of the assumption that the slow electrons are classical and, as will be shown below, it takes place only at not too small values of $|z|$. In addition, certain important conclusions can be drawn from the very form of the function $\sigma'_{ij}(u_{ij})$ (10). A distinction must be made here between two cases:

A. The metal is above the transition point, i.e., the transition has already occurred and the values of u_{ij} are such that the condition $|\Lambda_{ij} u_{ij}| \ll z_0$ is satisfied. It is possible then to expand in (10) in terms of the small parameter $|\Lambda_{ij} u_{ij}| / |z_0|$. Confining ourselves to terms of second order of smallness and including the terms linear in σ_{ij}^0 , which leads to a correction of the order of ²⁾ $(|z_0| / \varepsilon_F)^{1/2} u_{ij}$, we find that in the case considered the tensor σ'_{ij} turns out to be quadratic in the strain, i.e., it coincides in form with the nonlinear increments due to the cubic anharmonicity. It is important, however, that the contribution made to σ'_{ij} by the cubic anharmonicity turns out to be smaller by a factor $(\varepsilon_F / |z_0|)^{1/2}$ than the contribution due to the proximity to the ETT point, i.e., it is precisely the latter which determines all the nonlinear phenomena. Formally, our case can be treated as a considerable increase [by $(\varepsilon_F / |z_0|)^{1/2}$ times] of the anharmonic constants near the point of the ETT.

B. Greatest interest, however, attaches to the case when $|\Lambda_{ij} u_{ij}|$ is comparable with or large compared to $|z_0|$. In this case, as seen from (10), the value of σ'_{ij} due to the proximity to the transition point turns out to be large compared with the corrections due to the cubic anharmonicity not because of the numerical factor, but because it contains the strain raised to a smaller power. It is precisely this case which will be investigated by us hereafter. We emphasize the non-analytic dependence of σ'_{ij} on the strain (the presence of the θ function and of the fractional power). This non-analyticity is not washed out by the dispersion, owing to the already noted "instantaneous" character of the ETT, and has a real physical meaning.

It follows from (10) that the phenomena connected with the non-analyticity of the function $\sigma'_{ij}(u_{ij})$ should manifest themselves most strongly at $z=0$, i.e., when the propagating elastic wave takes the metal through the transition point. It turns out, however, that in the vicinity of the point $z=0$ expression (10) itself is no longer valid. An important role is assumed in this region by quantum effects, allowance for which leads to a non-local connection between the tensor σ'_{ij} and the strain field. We proceed now to investigate these effects.

2. QUANTUM EFFECTS. GENERAL CASE

In the derivation of (7)–(10) we made use in effect of the quasiclassical character of the motion of the Fermi electrons, since p_{FS} in (5) was regarded simply as a number, and the electron system as a whole was regarded as located in a uniformly deformed crystal. In other words, it was assumed that the de Broglie wavelength of the slow electrons is small compared with the characteristic scale of variation of the tensor u_{ij} , viz., $p_{FS}\Delta x \gg \hbar$. But $p_{FS} \sim |z|^{1/2}$, and at sufficiently small $|z|$ the motion of the slow electrons is no longer quasiclassical. Therefore Eqs. (7)–(10) cannot be used in the region $p_{FS}\Delta x \lesssim \hbar$, where the slow electrons are in essence quantum particles. The characteristic dimension of the “quantum” regions with respect to each of the spatial coordinates can be easily estimated from the condition

$$|2m_\beta^*z(\Delta x_\beta)|^{1/2} \Delta x_\beta \sim \hbar, \quad \Delta x_i = 0, \quad i \neq \beta. \quad (11)$$

Here $\beta=1, 2, 3$, no summation is carried out over β , and m_β^* are the principal values of the effective-mass tensor in the vicinity of the transition point. In a number of cases (breaking of a neck) certain or even all (formation of a hole cavity) values of m_β^* can be negative. The origin in Eq. (11) coincides with the ETT point, so that $z(0)=0$.

In typical cases, when $z(x)$ takes the form of a sufficiently deep potential well, we obtain from (11)

$$\Delta x_\beta \sim \left| \hbar^2 / 2m_\beta^* \left(\frac{\partial z}{\partial x_\beta} \right)_0 \right|^{1/2} \ll L_\beta, \quad (12)$$

$$\Delta z_\beta = \max_\beta \{ |z(\Delta x_\beta)| \} \sim \max_\beta \left\{ \left| \hbar^2 \left(\frac{\partial z}{\partial x_\beta} \right)_0^2 / 2m_\beta^* \right|^{1/2} \right\}, \quad (13)$$

where L_β is the width of the well along a given coordinate axis.

We derive now elasticity-theory equations that are valid in the quantum region. We discuss first the influence of the quantum effects on the value of $\delta\varepsilon_F$. Just as before, $\delta\varepsilon_F$ is determined from the condition that the electroneutrality of the metal be conserved when the metal is strained; this condition now takes the quantum form

$$\langle 0 | \hat{\rho} | 0 \rangle = \text{const}, \quad (14)$$

where $\hat{\rho}$ is the particle-number density operator and $|0\rangle$ is the multiparticle wave function that describes the ground state of all the electrons of the metal and is a complicated functional of the strain field. It is clear that this approach leads us to a dead end and makes the problem practically unsolvable. It was shown in the preceding section, however, that slow Fermi electrons make

no contribution to the change of ε_F on account of the deformation of the crystal even if their motion is quasiclassical. This is all the more true in the quantum case, since the statistical weight of the slow electrons is smaller the smaller $|z|$, i.e., the more “quantum” the problem. The faster electrons, on the other hand, are quasiclassical, so that the quantum condition (14) can be replaced by the classical (2), and $\delta\varepsilon_F$ is then determined as before by condition (3).

We find now the free-energy increment due to the contribution made by the slow electrons. It is clear from the foregoing that to take this contribution into account it suffices to solve the single-particle quantum-mechanical problem of the motion of a slow electron in an effective potential produced by the strain field. It is important that in the approximation considered this effective potential is independent of the quantum state of the slow electrons.

In the quasiclassical case, the energy (i.e., the Hamiltonian) of the slow electron can be represented in the form

$$\varepsilon(\mathbf{p}) - \varepsilon_F \approx \varepsilon_c - \varepsilon_F + (\mathbf{p} - \mathbf{p}_c)^2 / 2m_i^* = z + (\Delta \mathbf{p})^2 / 2m_i^*,$$

where $z(\mathbf{x})$ and $(\Delta \mathbf{p})^2 / 2m_i^*$ play respectively the roles of the potential and kinetic energy. Going to the quantum limit, we should replace the Hamiltonian by the corresponding operator, and the classical equation of motion of the slow electron by the Schrödinger equation

$$-\frac{\hbar^2}{2m_i^*} \frac{\partial^2 \psi}{\partial x_i^2} + z(x)\psi = \varepsilon \psi.$$

Next, if a change in the topology of the electron Fermi surface takes place at the ETT, then the singular increment to the free energy of the metal, just as in Eq. (6), is determined from

$$\delta \mathcal{F} = 2g \int \left\{ \sum_{\varepsilon_n < 0} \varepsilon_n \right\} dV, \quad (15)$$

where ε_n are the eigenvalues of the effective Hamiltonian introduced above and are certain functionals of the strain field: $\varepsilon_n = \varepsilon_n(\mathbf{x}, \{u_{ij}\})$. The summation in (15) is over all the states that lie below the Fermi level, i.e., over $\varepsilon_n < 0$, since the energy in the effective Hamiltonian is reckoned from ε_F . On the other hand, if the transition point corresponds to a change in the topology of the hole Fermi surface, then it is necessary to place a minus sign in front of the right-hand side of (15), and carry out the summation over states with $\varepsilon_n > 0$. For the sake of argument we shall consider below the ETT connected with the change of the electron Fermi surface.

Calculating the variational derivative of the free energy with respect to u_{ij} , we obtain the nonlinear increment to the stress tensor:

$$\sigma'_{ij} = \frac{\delta \mathcal{F}}{\delta u_{ij}} = \frac{\delta \mathcal{F}}{\delta z} \frac{dz}{du_{ij}} = 2g \Lambda_{ij} \sum_{\varepsilon_n < 0} \frac{\delta \varepsilon_n}{\delta z}.$$

But $\delta \varepsilon_n = \int \psi_n^* \delta z \psi_n dV$, therefore $\partial \varepsilon_n / \partial z = \psi_n^* \psi_n = |\psi_n|^2$. We thus obtain

$$\sigma'_{ij} = 2g \Lambda_{ij} \sum_{\varepsilon_n < 0} |\psi_n(u_{ij})|^2. \quad (16)$$

The equations for the elastic waves have formally, as before, the form (7)–(9), but in place of the simple ex-

pression (10) for σ'_{ij} , we obtain (16), making the equations of motion nonlocal, since $|\psi_n|^2$ in (16) is a complicated functional of u_{ij} .

3. QUANTUM EFFECTS. PLANAR STRAINS

In the general case of an arbitrary three-dimensional strain field, further simplification of the expression for σ'_{ij} is impossible and relation (16) is the end result. In the important particular case of planar strains and relatively deep penetration into the region beyond the ETT point, when the tensor u_{ij} depends on a single spatial coordinate x , and the effective potential $z(x)$ forms a well containing a large number of levels with $\epsilon_n < 0$ (Fig. 1), the problem becomes much simpler, so that an explicit expression can be obtained for σ'_{ij} as a function of the strain field, and the transition from the quantum expression to classical formula (10) can be tracked.

Let us consider this case in greater detail. The quantum regions are located near the turning points of the Fermi electron, and the characteristic dimension of these regions is determined by relations (12) and (13), where $L = a - b$. The distance between two neighboring electron energy levels is $\Delta\epsilon_n \sim \hbar^2/2m_x^* L^2$, so that the total number of levels that land in the quantum region N is of

$$N \sim \frac{\Delta\epsilon_n}{\Delta\epsilon} \sim L^2 \left| \frac{\partial^2 z}{\partial x^2} \right| \sim \left(\frac{L}{\Delta x} \right)^2 \gg 1. \quad (17)$$

In the yz plane, the motion of the electron is free and infinite, so that its wave function is of the form

$$\psi_n(x) = \exp(i p_{\perp} x / \hbar) \psi_n(x),$$

where p_{\perp} is a number and not an operator. Substituting this value of ψ_n in (16) and going over from summation to integration, in view of the large congestion of the levels, we obtain

$$\sigma'_{ij} = C \Lambda_{ij} \int_{x=a}^{x=b} |\psi_n|^2 v(\epsilon_n) d\epsilon_n d^2 p_{\perp}, \quad (18)$$

$$\epsilon_n = \epsilon_0 + \hbar^2 p_{\perp}^2 / 2m_x^*.$$

Later on we shall renormalize C and include in it all the constant factors that arise in the course of the calculation; it is therefore more convenient for us not to specify its value at present. The correct value of the multiplicative constant in the expression for σ'_{ij} is easy to determine by comparing the quantum expression with Eq. (10) in the region of the classical motion of the Fermi electron.

Since there are no special reasons why the state density $\nu(\epsilon_n)$ should vanish at $\epsilon_n = 0$, and we are interested in small values of ϵ_n , we can put in (18) $\nu(\epsilon_n) \approx \nu(0)$ and take $\nu(0)$ outside the integral sign and include it in C .

We consider for simplicity the situation $m_x^* = m_y^* = m_z^* > 0$. In this case

$$d^2 p_{\perp} = dp_x dp_y = 2\pi p_{\perp} dp_{\perp} = 2\pi m_x^* d\epsilon_{\perp},$$

and since ψ_n does not depend on ϵ_{\perp} , the integral with respect to $d\epsilon_{\perp}$ can easily be evaluated. Recognizing that $0 \leq \epsilon_{\perp} \leq -\epsilon_n$, we obtain ultimately

$$\sigma'_{ij} = C \Lambda_{ij} \int_a^b |\psi_n|^2 d\epsilon_n. \quad (19)$$

The convergence of the integral on the lower limit is ensured by the rapid damping of the wave functions in the region of the classically possible values of the energy ($\epsilon_x < z$ at $m_x < 0$ —formation of a cavity; $\epsilon_x > z$ at $m_x^* < 0$ —breaking of a neck).

It is easily seen that in the region of the classical motion of the Fermi electron Eq. (19) coincides with Eq. (10). Indeed, using the quasiclassical expression for the wave function, we obtain from (19)

$$\sigma'_{ij} \approx C \Lambda_{ij} \int_a^b \frac{1}{p} \cos^2 \left(\frac{1}{\hbar} \int_a^x p dx' - \frac{\pi}{4} \right) \epsilon d\epsilon$$

$$\approx \frac{1}{2} C \Lambda_{ij} \int_a^b \frac{\epsilon d\epsilon}{[2m_x^*(\epsilon - z)]^{1/2}} = B \Lambda_{ij} |z|^{3/2}. \quad (20)$$

We now investigate in greater detail the influence of the quantum effects on the elastic properties of the metal near the turning points of the Fermi electrons. We choose for the sake of argument $m_x^* > 0$ and consider the vicinity of the point a (see Fig. 1), so that $z(x)$ can be represented in the form

$$z(x) = z_a' x, \quad z_a' = (dz/dx)_a < 0, \quad (21)$$

and the Schrödinger equation for the slow electrons is written as

$$\frac{d^2 \psi}{dx^2} + \frac{2m_x^*}{\hbar^2} (\epsilon + |z_a'| x) \psi = 0. \quad (22)$$

A solution of (22) satisfying the condition $\psi(x) \rightarrow 0$ as $x \rightarrow -\infty$ is expressed in terms of the Airy function Φ (Ref. 2). Assuming the normalization constant to be equal to unity, we obtain

$$\psi = \Phi(-\bar{\eta}), \quad \bar{\eta} = (x + \epsilon |z_a'|^{-1}) / \Delta x_a, \quad (23)$$

$$\Phi(-\bar{\eta}) = \frac{1}{\sqrt{\pi}} \int_0^{\bar{\eta}} \cos \left(\frac{u^3}{3} - u \bar{\eta} \right) du,$$

where Δx_a is given by (12) in which the order-of-magnitude symbol should be replaced by an equal sign.

Substituting (23) in (19), we arrive at the relation

$$\sigma'_{ij} = C \Lambda_{ij} |z_a' \Delta x_a|^2 \int_{-\infty}^{x/\Delta x_a} \Phi^2(-\bar{\eta}) (\bar{\eta} - x/\Delta x_a) d\bar{\eta}. \quad (24)$$

From the form of (24) follows directly an important result: in the case of planar strains the nonlinear corrections to the stress tensor can be represented, even when quantum effects are taken into account, by the tensor components Λ_{ij} , which do not depend on the proximity of the state of the metal to the ETT point, all multiplied by the same scalar factor, i.e., all the tensor compo-

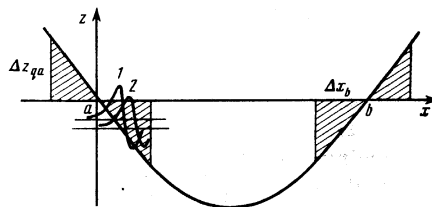


FIG. 1. Potential of slow electrons (schematic): a and b —turning points for an electron with $\epsilon = \epsilon_F$. The regions of the influence of the quantum effects are shown shaded. 1 and 2—wave functions for two closely lying levels.

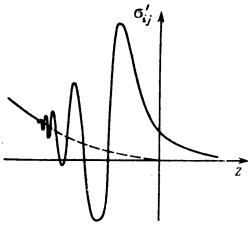


FIG. 2. Influence of quantum effects on the dependence of the components of the nonlinear increment to the strain tensor on the strain at a certain fixed nonzero value of $\partial z/\partial x$; $z = z_0 + \Lambda_{ij} u_{ij}$. The transition regions correspond to the half-space $z < 0$; the dashed line shows the "classical" curve $\sigma'_{ij} = B \Lambda_{ij} \theta(-z) |z|^{3/2}$.

nents σ'_{ij} vary in the same manner on moving away from the transition point.

Using next the asymptotic representations of the Airy function at large positive and large negative values of its argument, and using the correspondence principle in the form (20), we obtain after simple transformations

$$\sigma'_{ij} = \frac{3}{32} B \Lambda_{ij} |z|^{3/2} \frac{\exp(-4|\eta|^{3/2}/3)}{|\eta|^3} \quad \text{as } \eta \rightarrow -\infty, \quad (25)$$

$$\sigma'_{ij} = B \Lambda_{ij} |z|^{3/2} \left\{ 1 - \frac{3}{16} \frac{\sin 4\eta^{3/2}/3}{\eta^2} \right\} \quad \text{as } \eta \rightarrow \infty, \quad (26)$$

where $\eta = x/\Delta x_a$. Alternately, expressing x in terms of z and z'_a in accordance with (21), and taking (4) and (12) into account, we obtain

$$\eta = \left| \frac{2m_z}{\hbar^2} \Lambda_{kl} \frac{\partial u_{kl}}{\partial x} \right|^{1/2} (z_0 + \Lambda_{ij} u_{ij}) / \Lambda_{mn} \frac{\partial u_{mn}}{\partial x}. \quad (27)$$

In terms of our variables, the ETT point corresponds to $\eta=0$. As seen from (25) and (26), Δx_a is actually that characteristic scale which determines the width of the quantum region. Allowance for the quantum effects, leads first, to a nonzero nonlinear increment to the stress tensor below the transition point (as $\eta \rightarrow -\infty$), this being due to the finite depth of penetration of the Fermi electrons into the classically accessible region $z > 0$. Second, on advancing into the region of classical motion ($\eta \rightarrow \infty$), the quantum oscillations of the quantity σ'_{ij} not only increase in frequency but decrease in amplitude.

The nonlocality of the interaction in the quantum region causes σ'_{ij} to depend not only on the strain but also on higher derivatives of the displacement vector [in the case considered, $\sigma'_{ij} = \sigma'_{ij}(u_{ij}, \partial u_{ij}/\partial x)$]. The dependence of σ'_{ij} on the strain at a fixed value of $\partial u_{ij}/\partial x$, with allowance for quantum effects, is shown in Fig. 2.

4. SIMPLE WAVES

In this section we investigate the propagation of elastic waves in a metal, neglecting the influence of the above-discussed quantum effects and using Eqs. (7)–(10). This approach remains correct so long as the characteristic values of $|z|$ (for the considered wave) remain large compared with Δz_0 , as will in fact be assumed hereafter.

In addition, we confine ourselves to a study of cases wherein the wave produces in the crystal a planar

single-component strain field, i.e., the displacement vector depends not only on a single spatial coordinate x , but is directed along one of the coordinate axes. This situation arises, for example, when elastic waves of a definite polarization propagate along a crystal symmetry axis.

It is convenient to represent the value of z in this case in the form

$$z = z_0 + \Lambda \frac{\partial u}{\partial x} = \Lambda \left\{ \frac{\partial u}{\partial x} - \left(\frac{\partial u}{\partial x} \right)_c \right\} = \Lambda \zeta, \quad (28)$$

where $(\partial u/\partial x)_c$ has the meaning of the critical strain, and in our formulation of the problem it is defined uniquely by (28), while Λ denotes the only independent component of the tensor Λ_{ij} , remaining now after convolution of Λ_{ij} with u_{ij} .

In our notation, Eqs. (7)–(10) take the form

$$\frac{\partial^2 u}{\partial t^2} = s^2(\zeta) \frac{\partial^2 u}{\partial x^2}, \quad (29)$$

$$s(\zeta) = s_0 \{ 1 + \alpha \theta(\pm \zeta) |\zeta|^{1/2} \}, \quad (30)$$

where $s_0 = \text{const}$ is the "linear" speed of sound for the given type of elastic waves;

$$\alpha = \frac{3}{4} \frac{B}{\rho s_0^2} |\Lambda|^{1/2} \text{sign } \Lambda \zeta, \quad |\alpha| \sim 1.$$

Equation (29) is the standard equation of elastic waves in a medium in which the speed of sound depends on the strain.³ In a number of cases such an equation can be integrated in general form without specifying the $s(\zeta)$ dependence. In particular, if we replace u by a new independent variable $v = \partial u/\partial t$, connected with ζ by the obvious relation

$$\partial v/\partial x = \partial \zeta/\partial t,$$

and seek for Eq. (29) singular solutions that satisfy the additional condition

$$\partial(\zeta, v)/\partial(x, t) = 0,$$

then (29) is easily reduced to the form

$$\frac{\partial \zeta}{\partial t} \pm s(\zeta) \frac{\partial \zeta}{\partial x} = 0,$$

whose general solution is

$$\zeta = F(\xi), \quad \xi = x \pm s(\zeta)t, \quad (31)$$

or to another equivalent form

$$x = f(\zeta) \pm s(\zeta)t, \quad (32)$$

where F and f are arbitrary functions. Solutions of type (31) and (32) are called in fluid mechanics simple or Riemann waves.

We investigate now the nonlinear distortions produced in a simple propagating wave by a concrete $s(\zeta)$ dependence in the form (30). For the sake of argument, we confine ourselves to a C_+ wave propagating along the positive x axis, for which $\xi = x - s(\zeta)t$, and choose positive values of α and of the argument of the θ function. It follows from (30) and (31) that in the region $\zeta < 0$ a simple wave goes over into an ordinary traveling wave, so that the sections of the wave lying in the region $\zeta < 0$ propagate without distortion at a velocity s_0 up to the onset of nonlinear damping due to the toppling of the wave

in the region $\zeta > 0$. The evolution of the wave after its toppling will be investigated below. We consider now the region $\zeta > 0$. We choose for the sake of argument $F(\xi)$ in the form (see Fig. 3)

$$\zeta = A \{1 - |\xi|/L\}, \quad |\xi| \leq L, \quad A > 0. \quad (33)$$

Solving Eq. (33) with respect to ζ , taking (30) and (31) into account, we obtain

$$\begin{aligned} \zeta &= A \{ [L - (x - s_0 t)] / [L - \alpha s_0 t A^{1/2}] \}^2 \\ \text{at } L &\geq x - s_0 t \left\{ 1 + \alpha A^{1/2} \frac{L - (x - s_0 t)}{L - \alpha s_0 t A^{1/2}} \right\} \geq 0; \\ \zeta &= A \{ [L + (x - s_0 t)] / [L + \alpha s_0 t A^{1/2}] \}^2 \\ \text{at } -L &\leq x - s_0 t \left\{ 1 + \alpha A^{1/2} \frac{L + (x - s_0 t)}{L + \alpha s_0 t A^{1/2}} \right\} \leq 0. \end{aligned} \quad (34)$$

The length and time x^* and t^* of the toppling are obtained from the conditions

$$(\partial x / \partial \zeta)_{x, t} = 0, \quad (\partial^2 x / \partial \zeta^2)_{x, t} = 0,$$

which must be satisfied simultaneously. For expression (34) these conditions lead to the formulas

$$x^* = \frac{L}{\alpha A^{1/2}} (1 + \alpha A^{1/2}), \quad t^* = \frac{L}{\alpha s_0 A^{1/2}}. \quad (35)$$

It is easily seen that in the considered example the wave crest becomes aligned with the forward base of the through at $x = x^*$ and $t = t^*$.

We shall show that Eqs. (35), which determine the toppling length and time, are universal and are not connected with the specific choice of $F(\xi)$, provided the propagating wave is such that at the initial instant its shape is completely described by specifying a single scale with dimension of length. To this end we write down the solution (31) in the form

$$\zeta = A G^2(\xi/L),$$

or, equivalently

$$x = s_0 t (1 + \alpha \zeta^{1/2}) + L g((\zeta/A)^{1/2}), \quad (36)$$

where L is the characteristic "length" of the wave, G is an arbitrary function, g is a function inverse to G , and both satisfy the condition $G(1) \sim g(1) \sim 1$. Applying to (36) the toppling criterion $(\partial x / \partial \zeta)_{x^*, t^*} = 0$ and recognizing that $g'((\zeta^*/A)^{1/2}) \sim 1$, we obtain

$$t^* \sim L / |\alpha| s_0 A^{1/2} \sim L / s_0 A^{1/2}.$$

Substituting this value of t^* in (36) and discarding small corrections, we find that

$$x^* \sim s_0 t^* \sim L / A^{1/2} \gg L. \quad (37)$$

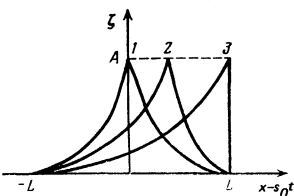


FIG. 3. Profile of the wave (33) at various instants of time ($\alpha > 0$): 1) $t = 0$; 2) $0 < t < t^*$; 3) $t = t^*$. An opposite change of the shape of the wave, namely stretching of the leading edge and shortening of the trailing one, takes place at $\alpha < 0$.

We note that the toppling length due to the cubic anharmonicity is $x_3^* \sim L/A \gg L/A^{1/2}$, i.e., in accordance with the statements made in the introduction, the nonlinearities due to the proximity to the ETT point manifest themselves much earlier than the "classical" forms of nonlinearity.

We consider now a simple wave localized in a finite region of space, i.e.,

$$(\partial u / \partial x)_{x \rightarrow \pm \infty} \rightarrow 0,$$

or, equivalently, the function F in (31) satisfies the conditions

$$F(\xi)_{\xi \rightarrow \pm \infty} \rightarrow -(\partial u / \partial x)_e.$$

We shall show that such a wave has an integral of motion namely, the displacement vector behind its trailing edge, i.e., $u(x, t)$ as $x \rightarrow -\infty$ for a C_+ wave and as $x \rightarrow \infty$ for a C_- wave. We have

$$\begin{aligned} u(\mp \infty) &= \mp \int_{-\infty}^{\infty} \frac{\partial u}{\partial x} dx = \pm \int_{\zeta(-\infty)}^{\zeta(\infty)} x d\zeta = t \int_{\zeta(-\infty)}^{\zeta(\infty)} s(\zeta) d\zeta \\ &\quad \pm \int_{\zeta(-\infty)}^{\zeta(\infty)} f(\zeta) d\zeta = I_1 + I_2, \end{aligned}$$

where the upper sign pertains to the C_+ wave and the lower to the C_- wave [we have evaluated the first integral by parts and use expression (32)].

But $s(\zeta)$ is a single-valued function, and

$$\zeta(-\infty) = \zeta(\infty) = -(\partial u / \partial x)_e,$$

therefore $I_1 = 0$. We for I_2 , for a wave occupying a finite region of space the function $f(\zeta)$ is multiply valued (see Fig. 3), therefore I_2 must be broken up into a sum of integrals from $\zeta_m = \min\{\zeta(\xi)\}$ to $\zeta_M = \max\{\zeta(\xi)\}$ and into integrals that are evaluated in the opposite direction. One branch of $f(\zeta)$ is substituted in the integrals from ζ_m to ζ_M , and the other in the integrals from ζ_M to ζ_m . The net result is a constant I_2 equal numerically to the area bounded by the $f(\zeta)$ curve and by the ξ axis, and generally speaking different from zero. It is important, however, that I_2 does not depend on t , i.e., it is an integral of the motion.

At $x > x^*$ the continuous solution (31), (32) ceases to satisfy the uniqueness condition, therefore at $x > x^*$ the solution becomes discontinuous, i.e., a shock wave is produced. The location of the break is easily determined by stipulating that the displacement vector behind the wave front remain an integral of the motion also after the toppling. This requirement leads to the well-known law of equal areas: the break is located in such a way that the total area under the $\zeta(x)$ curve remains unchanged. We note that from the equal-area law it follows that after the toppling the profile of the wave begins to be distorted also at $\zeta < 0$ (see Fig. 4). The physical reason for this distortion is that the formation of the discontinuity leads to a strong damping of the wave, which affects also the linear part of its profile.

The law governing the damping of the wave after the formation of the break depends on the shape of the wave. Thus, for example, for a wave specified by expressions (33) and (34) and having no linear part in the region ζ

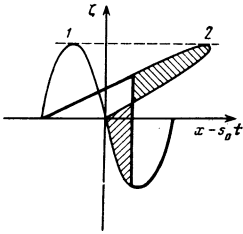


FIG. 4. Damping of the wave after the toppling: 1—initial shape of the wave at $t=0$; 2—non-single-valued solution at $t > t^*$; the thick line shows the true discontinuous solution. The areas of the shaded sections are equal.

< 0 , the coordinate x^* of the break point is displaced in accordance with the law

$$x^*(t) = s_0 t - L + [2L(L + \alpha s_0 t A^{1/2})^2]^{1/2}, \quad t \geq t^*, \quad \alpha > 0,$$

and its amplitude decreases like

$$\xi_M(t) = \xi(x^*) = A \left(\frac{2L}{L + \alpha s_0 t A^{1/2}} \right)^{2/3}, \quad t \geq t^*, \quad \alpha > 0.$$

For a wave in the form of a right triangle at $t=0$, with one side lying on the $\xi=0$ axis and equal to L , and the other the leading front with $\xi_M = \xi|_{\xi=L} = A$, we have

$$x^*(t) = s_0 t + L(1 + \alpha s_0 t A^{1/2}/L)^{1/2}, \quad t \geq 0, \quad \alpha > 0;$$

$$\xi_M(t) = A(1 + \alpha s_0 t A^{1/2}/L)^{-1/2}, \quad t \geq 0, \quad \alpha > 0,$$

etc.

5. DISCUSSION OF RESULTS

We discuss first those assumptions on which the results of the preceding sections are implicitly or explicitly based. First, we have neglected all the linear dissipative effects. This neglect is inessential in the elasticity-theory equations, since the dissipative terms can be easily separated in these equations in standard fashion.⁴ The resultant equations, however, no longer have solutions of the type (31), (32), since none of the results concerning this aspect of the problem can be trivially generalized to the case of a dissipative medium. This means physically that the conclusions concerning the evolution of simple waves remain valid if the toppling length turns out to be small compared with the characteristic dissipative length: $\Gamma x^* \ll 1$, where Γ is the logarithmic damping decrement. It might seem that this condition contradicts the other initial assumption, that the temperature of the metal be low, inasmuch as sound is strongly damped in metals at low temperatures.⁵ Actually there are no contradictions, but a condition is imposed on the characteristic value of $|z|$ that is reached in the elastic wave³: $T \ll |z| \ll \epsilon_F$, where T is determined from the condition $\Gamma(T)x^* \ll 1$.

In addition, in the derivation of (31) and (32) we used essentially the fact that the sound velocity can be represented as a function of a single dynamic variable, namely the strain: $s = s(\xi)$. From the results of Sec. 3 it follows, however, that in the region $|z| \lesssim \Delta z_q$ the $s(\xi)$ dependence goes over into a dependence on two dynamic variables, $s(\xi) \rightarrow s(\xi, \partial \xi / \partial x)$. Therefore (31) and (32) are likewise inapplicable in the region where quantum effects have an influence. The conditions for the applicability of (31) and (32) take thus ultimately the form

$$\max\{T, \Delta z_q\} \ll |z| \ll \epsilon_F. \quad (38)$$

Two cases must be distinguished here. If the wave is such that $z(x)$ does not vanish anywhere, so that the condition (38) is satisfied at all points of the wave, then the results of the preceding section are perfectly rigorous. If, however, (38) is satisfied only for a characteristic values of z and there exist in the wave regions for which $|z| \lesssim \max\{T, \Delta z_q\}$, then the structure of the wave in these regions becomes smeared out either by temperature ($T \gg \Delta z_q$) or by quantum ($\Delta z_q \gg T$) effects. This smearing, however, does not influence the evolution of the basic part of the wave, so that the results of Sec. 4 turn out to be applicable in this case, too.

We present now some estimates. Since the proximity to the ETT point does not change the order of magnitude of Γ in typical cases,⁶ it follows that at $T \ll \theta_D$, where θ_D is the Debye temperature, the damping decrement of the long-wave sound in pure metals is determined, for scattering of electrons by phonons, by the expression⁵

$$\Gamma \sim \frac{1}{a} \left(\frac{\theta_D}{T} \right)^3 \left(\frac{\omega_s}{\omega_D} \right)^2, \quad (39)$$

where $\omega_{s,D}$ is the characteristic frequency of the sound or of the Debye phonon. In this case the condition $\Gamma x^* \ll 1$ with allowance for (37) leads to the limitation

$$a(LA^{1/2})^{-1}(\theta_D/T)^3 \ll 1.$$

In particular, at $T \approx 0.2\theta_D$, $L \sim 10^{-2}$ cm, $a \approx 10^{-8}$ cm, and $A \sim 10^{-3}$, which corresponds to a pressure jump ~ 100 atm in the elastic-wave front, we get from (35) and (39) $x^* \sim 0.3$ cm and $\Gamma x^* \sim 0.1$. We note that in order to obtain under these conditions the same toppling length as a result of cubic anharmonicity, the strains required would be $A_3 \sim 3 \cdot 10^{-2}$, i.e., the jumps of the pressure in the wave front would have to be of the order of several thousand atmospheres.

Next, since $\theta_D \sim 10^{-3}\epsilon_F$ for good metals, at $A \sim 10^{-3}$ we obtain $z \sim A\epsilon_F \sim \theta_D$. Finally, the width of the quantum region is

$$\Delta z_q \sim (\hbar^2 z^2 / L^2 m^*)^{1/2} \sim \epsilon_F (aA/L)^{1/2} \sim 10^{-6}\epsilon_F,$$

i.e., $z \gg T \gg \Delta z_q$ in the case considered.

In conclusion it must be emphasized that metals near the ETT points are, from the viewpoint of nonlinear acoustics, a principally new medium with a very peculiar dependence of its elastic properties on the strain. In the present paper we investigated only the simplest nonlinear phenomena that take place in such a medium, but these obviously are not the only problems. In particular, interesting effects can be expected in nonlinear interaction of waves, in propagation of low-amplitude waves when quantum effects become substantial, etc.

¹Usually there exists not one such cavity but several identical cavities located at symmetrical points of the Brillouin cell.

²On going to infinitesimal strains, we verify that this correction corresponds to a renormalization of the dispersion law of the long-wave phonons (i.e., in fact to a change in the speed of sound) because of their interaction with slow electrons. The change of the dispersion law of a phonon of arbitrary wavelength, due to such an interaction, can be easily obtained in second-order perturbation theory from Eq. (19) below.

We, however, solve a more difficult problem, namely we are interested in allowance for nonlinear phenomena due to the propagation of elastic waves of finite rather than infinitesimal amplitude.

³The restriction $|z| \ll \varepsilon_F$ follows from the very formulation of the problem, since the ratio $|z|/\varepsilon_F$ is that dimensionless parameter which determines the proximity to the ETT point. On the other hand, it follows from (4) that $|u_{ij}| \sim |z|/\varepsilon_F$, so that smallness of this parameter guarantees also smallness of the crystal strain, i.e., applicability of elasticity theory.

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Translated by J. G. Adashko