Passage of ultrarelativistic positronium atoms through matter

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Passage of an ultrarelativistic positronium atom through a thin layer of dense matter is investigated for the conditions in which the layer thickness is $l \ll \hbar c\gamma/\Delta E$, where $\Delta E \sim me^4/4\hbar^2$ is the characteristic difference in the energies of the stationary states of positronium. It is shown that if l is much greater than the mean free path, the probability of observation of the positronium atom in a bound state is inversely proportional to l.

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1. INTRODUCTION

Let a quantum system (elementary particle, nucleus, atom, etc.) in its ground state hit a target. In each collision inside the target the system with a certain probability can change its internal state (be excited or broken up into parts) or can remain in the initial state, changing only its direction and velocity. What is the probability W that after traversal of a target of thickness l the system will remain in its initial state? The answer to this question will depend on the ratio between the characteristic time τ determined by the internal structure of the system and the interval $T = L_{in}/v$ which separates successive inelastic collisions with the particles of the target (here $L_{in} = 1/N\sigma_{in}$ is the mean free path for inelastic interactions, v is the velocity of the system, and N is the number of atoms per unit volume).

The quantity τ satisfies the relation

$$\tau \approx \hbar/\Delta E, \tag{1}$$

where ΔE is the characteristic value of the energy difference of the eigenstates of the system; for ultrarelativistic systems τ must be replaced by $\gamma\tau$, where γ is the Lorentz factor. Usually the time τ is very short, so that

$$\gamma\tau v/L_{in} \ll 1. \tag{2}$$

Then after each collision the fate of the system is decided long before the next collision, i.e., in each interaction the system will be in a single definite stationary state. A cascade process based on the pattern of independent successive collisions develops. Here inverse transitions of the produced excited states to the initial state are possible. If such transitions can be neglected, the answer to the question posed is the ordinary exponential

$$W = \exp(-l/L_{in}). \tag{3}$$

The situation is different for the condition

$$\gamma\tau v/L_{in} \gg 1. \tag{4}$$

Now the behavior of the system is determined not by independent actions of each of the successive collisions individually, but by the combined and essentially simultaneous action of many collisions, and as a result of this, simple expressions like Eq. (3) lose their meaning. A situation which is essentially similar has already been discussed in the literature,[1)] in particular, in connection with the regeneration of neutral K mesons, where τ is very long as a result of the small mass difference of the K_S^0 and K_L^0 mesons.

In a two-level system such as the neutral K mesons, the simplest case is realized. If there are several levels, the behavior becomes more complicated, although its general nature remains as before. Qualitatively new features arise for systems with an infinite number of levels, and the purpose of the present work is to analyze such systems in the specific example of passage of ultrarelativistic positronium through a thin target. As was shown by Nemenov,[3] beams of ultrarelativistic positronium atoms produced in decay of ultrarelativistic π^0 mesons in the reaction $\pi^0 \to \gamma + (e^+e^-)$ can be achieved in existing accelerators (see also Ref. 4). For positronium in the ground state, $\tau \approx 10^{-16}$ sec, and with a Lorentz factor $\gamma \approx 10^3$ the value of $\gamma\tau$ is $\approx 10^{-13}$ sec, which corresponds to a range $\gamma\tau c \approx 3 \cdot 10^{-3}$ cm. On the other hand, the cross section for interaction of positronium with the atoms of the target is so large (for example, in aluminum $\sigma \sim 2 \cdot 10^{-18}$ cm^2 according to Ref. 5) that in condensed matter $L_{in} \sim 10^{-6}$–10^{-5} cm, i.e., the condition (4) can be satisfied with a large margin. In what follows we shall assume that in addition to (4) an even stronger condition is satisfied:

$$l \ll \gamma\tau c. \tag{5}$$

Since in the case of positronium $\Delta E \sim me^4/4\hbar^2$, where m is the electron mass, in view of Eq. (1) this is equivalent to the requirement

$$l \ll 4\left(\frac{\hbar}{mc}\right)\left(\frac{\hbar c}{e^2}\right)^2 \gamma = 3 \cdot 10^{-6}\gamma \text{ cm}. \tag{5'}$$

It follows from Eq. (5′) that for $\gamma \sim 10^3$ the subsequent theory applies only to sufficiently thin targets, the thickness of which does not exceed a few microns.

2. ULTRARELATIVISTIC POSITRONIUM. IMPACT APPROXIMATION

The interaction of a positronium atom with the atoms of the target reduces to the Coulomb interactions of its electron and positron with the electric fields created by the nuclei and electrons of the atoms. Under the conditions considered, these interactions can be analyzed in the framework of the impact approximation. The radius

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of positronium is about twice that of ordinary atoms. If the target is made of a material with a high atomic number Z, the electric field in its atoms is concentrated inside a screening region whose dimensions are small in comparison with those of the atom itself. Therefore, putting off for the moment the analysis of the general case, we shall assume that in each individual collision there actually takes part either only the electron of a positronium atom or the positron, but not both particles together.

We shall write the internal wave function of positronium in the ground state in the form $\varphi_1(\mathbf{r})$, where $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$ is the distance between the electron and the positron. If after collision with some atom the electron obtains a transverse momentum[2] $\mathbf{q}_1$, the center of gravity of the positronium will change its momentum also by an amount $\mathbf{q}_1$, and the internal wave function will go over to the form

$$\varphi(\mathbf{r})=\varphi_1(\mathbf{r})\exp(-i\mathbf{q}_1\mathbf{r}/2\hbar). \tag{6}$$

On the other hand, if the positron took part in the collision considered and received a transverse momentum $\mathbf{q}_2$, then the center of gravity of the positronium will change its momentum by an amount $\mathbf{q}_2$, and the internal wave function will go over to

$$\varphi(\mathbf{r})=\varphi_1(\mathbf{r})\exp(i\mathbf{q}_2\mathbf{r}/2\hbar). \tag{6'}$$

Strictly speaking, the states (6) and (6′) are not stationary, but when the condition (5) is satisfied they have no time to change before the next collision. Therefore after traversing the entire target the center of gravity of the positronium acquires a transverse momentum $\mathbf{Q}_1+\mathbf{Q}_2$ and the positronium wave function is

$$\varphi(\mathbf{r})=\varphi_1(\mathbf{r})\exp(-i(\mathbf{Q}_1-\mathbf{Q}_2)\mathbf{r}/2\hbar), \tag{6''}$$

where $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are the combined transverse momenta acquired by the electron and positron in all collisions inside the target. All collisions in effect merge into a single combined collision, and the positronium interacts at once with the entire target as if with a huge single atom.

By means of Eq. (6′) it is easy to obtain the probability W_{1k} of a transition to any final stationary state of the positronium φ_k:

$$W_{1k}=\left|\int\varphi_1(\mathbf{r})\varphi_k^*(\mathbf{r})\exp(-i(\mathbf{Q}_1-\mathbf{Q}_2)\mathbf{r}/2\hbar)d^3\mathbf{r}\right|^2. \tag{7}$$

In particular, the probability that it remain in the initial state is

$$W_{11}=\left|\int\varphi_1^2(\mathbf{r})\exp(-i(\mathbf{Q}_1-\mathbf{Q}_2)\mathbf{r}/2\hbar)d^3\mathbf{r}\right|^2. \tag{7'}$$

As is well known (see Ref. 6, Sec. 139), it follows from Eq. (7′) that

$$W_{11}=[1+(\mathbf{Q}_1-\mathbf{Q}_2)^2a^2/16\hbar^2]^{-4}, \tag{8}$$

where $a=2\hbar^2/me^2$ is the positronium radius.

Since the quantities $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are random transverse vectors, the expression (8) must still be averaged over the appropriate distribution $U(\mathbf{Q}_1,\mathbf{Q}_2)$. Then

$$\langle W_{11}\rangle=\iint U(\mathbf{Q}_1,\mathbf{Q}_2)\left[1+\frac{(\mathbf{Q}_1-\mathbf{Q}_2)^2a^2}{16\hbar^2}\right]^{-4}d^2\mathbf{Q}_1d^2\mathbf{Q}_2. \tag{9}$$

Note that $\langle W_{11}\rangle$ does not coincide with the probability of absence of collisions inside the target; collisions are not forbidden, and the direction of motion of the positronium can change; it is required only that eventually the positronium turn out to be in its initial internal state.

The number of collisions inside the target is determined by the ratio between l and L_{tot}, where L_{tot} is the mean range corresponding to all possible interactions of positronium, including elastic interactions; in the model considered here, L_{tot} is half the corresponding range for a free electron or positron. If the target thickness l is so small that $l/L_{tot}\ll 1$, then only one collision can occur inside the target and the corresponding probability is l/L_{tot}. Consequently the distribution is given by

$$U(\mathbf{Q}_1,\mathbf{Q}_2)=\left(1-\frac{l}{L_{tot}}\right)\delta^2(\mathbf{Q}_1)\delta^2(\mathbf{Q}_2)+\frac{l}{L_{tot}}u(\mathbf{Q}_1,\mathbf{Q}_2), \tag{10}$$

where the function $u(\mathbf{Q}_1,\mathbf{Q}_2)\equiv u(\mathbf{q}_1,\mathbf{q}_2)$ describes the normalized distribution of the momenta transferred to the electron and positron in an elementary collision. It follows from Eqs. (9) and (10) that

$$\langle W_{11}\rangle=\left(1-\frac{l}{L_{tot}}\right)+\frac{l}{L_{tot}}\iint\frac{u(\mathbf{q}_1,\mathbf{q}_2)d^2\mathbf{q}_1d^2\mathbf{q}_2}{[1+(\mathbf{q}_1-\mathbf{q}_2)^2a^2/16\hbar^2]^4}$$
$$=1-\frac{l}{L_{tot}}\left\{1-\iint\frac{u(\mathbf{q}_1,\mathbf{q}_2)}{[1+(\mathbf{q}_1-\mathbf{q}_2)^2a^2/16\hbar^2]^4}d^2\mathbf{q}_1d^2\mathbf{q}_2\right\}.$$

The quantity in the curly brackets coincides in meaning with the probability of any excitation of positronium as the result of a single collision which has occurred and is equal to $\sigma_{in}/\sigma_{tot}=L_{tot}/L_{in}$. Therefore in the limit considered

$$\langle W_{11}\rangle\approx 1-l/L_{in}. \tag{11}$$

As expected, Eq. (11) coincides with the first two terms of the expansion of the exponential (3). It can be shown that when the remaining terms are taken into account the quantity $\langle W_{11}\rangle$ always exceeds this exponential (see below). Rather thick targets in which $l/L_{tot}\gg 1$ are especially interesting in this sense. Each of the quantities $\mathbf{Q}_1$ and $\mathbf{Q}_2$ is then distributed according to a Gaussian law, which with allowance for axial symmetry can be written in the form

$$dV(Q)=\frac{2Q}{\langle Q^2\rangle}\exp\left(-\frac{Q^2}{\langle Q^2\rangle}\right),$$

where $Q=|\mathbf{Q}|$. The transverse momentum $\mathbf{Q}$ coincides with the transverse momentum which would be acquired by a free electron (or positron) which had traveled through the same target and had undergone in it multiple Coulomb scattering. Therefore

$$\langle Q^2\rangle=p^2\langle\theta^2\rangle, \tag{12}$$

where $\langle\theta^2\rangle$ is the mean square multiple-scattering angle of an electron possessing longitudinal momentum p equal to half the positronium momentum. The quantity $\langle\theta^2\rangle$ is given by the formula

$$\langle\theta^2\rangle=\left(N\int\sigma_{sc}(\theta)\theta^2d\Omega\right)l, \tag{13}$$

where $\sigma_{sc}(\theta)$ is the differential cross section for scattering of an electron by an atom into an element of solid angle $d\Omega$ without regard to excitation and ionization of the atom; the integration in Eq. (13) is carried out over the entire solid angle.

In the model considered here, the quantities $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are statistically independent. Therefore the random quantity $(\mathbf{Q}_1 - \mathbf{Q}_2)$ can be replaced by the quantity $\mathbf{Q}$, where $\mathbf{Q}$ is the total momentum which a free electron would obtain in double the number of collisions. Consequently Eq. (9) can be rewritten in the form

$$\langle W_{11}\rangle = \frac{1}{\langle Q^2\rangle}\int_0^\infty \frac{Q\exp(-Q^2/2\langle Q^2\rangle)}{(1+Q^2a^2/16\hbar^2)^4}\,dQ, \tag{14}$$

where $\langle Q^2\rangle$ is defined by Eq. (12).

For a sufficiently thick target the quantity $\langle Q^2\rangle$ is very large in comparison with $\hbar^2/a^2$ and the exponential in Eq. (14) can be replaced by unity. Then Eq. (14) goes over to

$$\langle W_{11}\rangle \approx \frac{1}{\langle Q^2\rangle}\int_0^\infty \frac{QdQ}{(1+Q^2a^2/16\hbar^2)^4}. \tag{14'}$$

After integration we eventually obtain

$$\langle W_{11}\rangle = 8\hbar^2/3a^2p^2\langle\theta^2\rangle, \tag{15}$$

where $\langle\theta^2\rangle$ is given by Eq. (13).

As is well known, the coefficient of l in Eq. (13) is approximated with high accuracy by the Williams formula

$$N\int \sigma_{ec}(\theta)\theta^2 d\Omega = \left(\frac{E_s}{pv}\right)^2 \frac{1}{L_{\rm rad}}, \tag{13'}$$

where $E_s = (4\pi\hbar c/e^2)^{1/2} m_e c^2 \approx 21$ MeV and $L_{\rm rad}$ is the radiation length (see for example Ref. 7). It is clear from this that as $v/c \to 1$ the probability $\langle W_{11}\rangle$ does not depend on the positronium energy and is inversely proportional to the layer thickness l.[3)] Substituting the numerical values of the constants into Eq. (15), we obtain

$$\langle W_{11}\rangle = \frac{8}{3}\left(\frac{\hbar c}{E_s a}\right)^2 \frac{L_{\rm rad}}{l} = 2.1\cdot 10^{-8}\frac{L_{\rm rad}}{l}. \tag{15'}$$

For sufficiently large values of $l/L_{\rm in}$ the probability (15′) can be several orders of magnitude greater than the value calculated[4)] with the formula $\exp(-l/L_{\rm in})$.

Although up to this time we have been considering only the probability $\langle W_{11}\rangle$, the approach used above is completely applicable also to calculation of the probabilities $\langle W_{1k}\rangle$ corresponding to the transition from the positronium ground state to any discrete state [see Eq. (7)]; this applies also to transitions between excited discrete states. In all cases for very thin targets the result is described by ordinary formulas containing effective cross sections, while for sufficiently thick targets the transition probability is

$$\langle W_{mk}\rangle = \beta_{mk}\frac{\hbar^2}{a^2p^2\langle\theta^2\rangle} = \beta_{mk}\left(\frac{\hbar c}{E_s a}\right)^2\frac{L_{\rm rad}}{l}, \tag{16}$$

where β_{mk} is a numerical coefficient which depends on the initial and final levels. As we have seen, $\beta_{11} = 8/3$. It can be shown that the combined probability of transition of positronium from the ground state to the nearest excited states with principal quantum number $n = 2$ is determined by Eq. (16) with $\beta_{12} \approx 0.47$. In the case of excitation of levels with a principal quantum number $n = 3$ we have the coefficient $\beta_{13} = 0.156$. For very large n we have $\beta_{1n} \approx 128/5n^3$. It is clear that after traversal of a sufficiently thick layer of matter by the positronium (under the conditions $L_{\rm in} \ll l \ll \gamma c\tau$) the total probability that it remain in a bound state is proportional to $1/l$ and is small in comparison with unity, while the probability of breakup is close to unity. However, it must be emphasized again that the probability of remaining in a bound state can be many times greater than the "customary" value $\exp(-l/L_{\rm br})$, where $L_{\rm br} = 1/N\sigma_{\rm br}$ is the mean range corresponding to breakup of the positronium.

As we have mentioned above, after traversal of the target the center of gravity of the positronium acquires a transverse momentum $\mathbf{Q}_1 + \mathbf{Q}_2$; accordingly, the direction of motion of the positronium changes by an angle

$$\alpha = (\mathbf{Q}_1 + \mathbf{Q}_2)/2p.$$

For a thick target, when the statistically independent vectors $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are distributed according to a Gaussian law, the angle α is also distributed according to a Gaussian law, and the quantity $\langle\alpha^2\rangle$ is half the mean square angle for multiple scattering of an electron with momentum p. In the Gaussian approximation the random quantities $\mathbf{Q}_1 + \mathbf{Q}_2$ and $\mathbf{Q}_1 - \mathbf{Q}_2$ are statistically independent. Therefore the angular distribution of the positronium atoms will not depend on what internal state they turn out to be in after passing through the target.[5]

3. GENERAL DISCUSSION. THE EIKONAL APPROACH

It was assumed above that the size of the positronium is large in comparison with the screening radius of the target atoms. In order to understand what occurs when this condition is not satisfied, we shall give a purely technical discussion of the opposite limiting case, which would be realized if the size of positronium were very small in comparison with atomic dimensions.[6)] Then the electron and positron would be almost together and the same electric field would act on them inside an atom. This would lead to a complete correlation of the transverse momenta of the electron and positron. As a result of the opposite signs of the charge, we would have the equalities $\mathbf{Q}_1 = -\mathbf{Q}_2$, $\langle(\mathbf{Q}_1 - \mathbf{Q}_2)^2\rangle = 4\langle Q^2\rangle$, where $\langle Q^2\rangle$ is defined by Eq. (12) (for the case of "large" positronium discussed in Sec. 2 $\langle(\mathbf{Q}_1 - \mathbf{Q}_2)^2\rangle = 2\langle Q^2\rangle$). As a result, values one-half those of Eqs. (15) and (16) would be obtained and, as we can easily understand, this determines the lower limit of the effect. In fact, in the general case we always have $\langle(\mathbf{Q}_1 - \mathbf{Q}_2)^2\rangle \leq 4\langle Q^2\rangle$. Consequently, for any real ratio of the sizes of positronium and the target atoms we have the inequalities

$$\langle W_{11}\rangle > \frac{4}{3}\left(\frac{\hbar c}{E_s a}\right)^2\frac{L_{\rm rad}}{l}, \quad \langle W_{mk}\rangle > \frac{1}{2}\beta_{mk}\left(\frac{\hbar c}{E_s a}\right)^2\frac{L_{\rm rad}}{l}, \tag{16'}$$

where β_{mk} are the coefficients entering into Eq. (16).

We note that treatment of the probabilities $\langle W_{mk}\rangle$ as mean squares of the transition form factors [see Eqs. (7)–(9)] is valid only if the correlation between the transverse momenta of the electron and positron either is absent or does not depend on the distance between the electron and positron. For an arbitrary ratio between the value of a and the screening radius R this is not the case. More accurate formulas which take into account this circumstance can be obtained by the eikonal method, which is widely used for description of processes at high energies (in particular, in the Glauber theory[8]). The condition of applicability of this method to the problem of interest here coincides precisely with Eq. (5); it can also be written in the form

$$l \ll pa^2/\hbar, \tag{5''}$$

where $p = |\mathbf{p}|$ and $2\mathbf{p}$ is the momentum of the positronium.

We shall represent the normalized wave function of positronium in its ground state and moving in vaccum as

$$\psi_{2\mathbf{p},1}(\mathbf{R}, t, \mathbf{r}) = \chi_{2\mathbf{p}}(\mathbf{R}, t)\varphi_1(\rho, z). \tag{17}$$

Here $\chi_{2\mathbf{p}}(\mathbf{R}, t)$ is the wave packet which describes the motion of the center of mass of the positron, $2\mathbf{p}$ is the average momentum of the positronium, $\varphi_1(\rho, z)$ is the wave function of relative motion in the rest system of the positronium, and ρ and z are the components of the vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ perpendicular and parallel to the momentum $\mathbf{p}$.

Let the positronium pass through a layer of matter of thickness l. We shall assume that the center of the packet crosses the boundary of the material at the time $t = 0$. Since the velocities of the nuclei and atomic electrons are small in comparison with the velocity of ultrarelativistic positronium, we can discuss the motion of the latter in a medium with fixed locations of the nuclei and electrons of the target. If the condition (5) is satisfied, immediately after the positronium leaves the layer of matter the wave function of the system (electron + positron + macroscopic target) is the product of the wave function (17) taken at $t = l/c$, the wave function characterizing the state of all atoms of the target, and the eikonal phase factor

$$\Phi(\mathbf{R}^{(\perp)}, \rho, \{\mathbf{b}_k; \mathbf{l}_{k1}, \mathbf{l}_{k2}, \ldots\}) = \exp\left\{ i \sum_k \left[\delta\left(\mathbf{b}_k - \mathbf{R}^{(\perp)} - \frac{\rho}{2}; \mathbf{l}_{k1}, \mathbf{l}_{k2}, \ldots\right) - \delta\left(\mathbf{b}_k - \mathbf{R}^{(\perp)} + \frac{\rho}{2}; \mathbf{l}_{k1}, \mathbf{l}_{k2}, \ldots\right)\right]\right\} \tag{18}$$

where $\{\mathbf{b}_k; \mathbf{l}_{k1}, \mathbf{l}_{k2}, \ldots\}$ is the set of transverse coordinates of the nuclei and electrons of the target, $\mathbf{l}_{kn} = \mathbf{r}_{kn}^{(\perp)} - \mathbf{b}_k$, the subscript k numbers the nuclei, and the subscripts kn number the electrons in the atoms (cf. Ref. 6, §152). The phase δ in Eq. (18) is expressed in terms of the potential of the Coulomb interaction by means of the usual relation

$$\delta(\mathbf{a}, l_1, l_2, \ldots) = \frac{1}{\hbar v}\int_{-\infty}^{\infty}\left(\frac{Ze^2}{(\mathbf{a}^2+u^2)^{1/2}} - \sum_{n=1}^{Z}\frac{e^2}{[(\mathbf{a}+\mathbf{l}_n)^2+u^2]^{1/2}}\right)du = \frac{e^2}{\hbar v}\ln\frac{|\mathbf{a}|^Z}{\prod_{n=1}^{Z}|\mathbf{a}+\mathbf{l}_n|}. \tag{19}$$

The total probability that after traversing the layer of matter the positronium will remain in the ground state, regardless of the change of its momentum and without regard to the excitation ionization of the atoms of the target, is given by the expression

$$\langle W_{11}\rangle = \int d^3R \left|\chi_{2\mathbf{p}}\left(\mathbf{R}, \frac{l}{c}\right)\right|^2 \times \overline{\left\langle\!\left\langle \left| \int \Phi(\mathbf{R}^{(\perp)}, \rho, \{\mathbf{b}_k; \mathbf{l}_{k1}, \mathbf{l}_{k2}, \ldots\})\varphi_1^2(\rho, z)\, d^2\rho\, dz \right|^2 \right\rangle\!\right\rangle}, \tag{20}$$

in which the symbol $\langle\langle\ldots\rangle\rangle_{(e)}$ indicates averaging over the electronic state of the target atoms and the bar indicates averaging over the locations of the nuclei. Here we have used the completeness condition for the wave functions of the target atoms and the wave functions which describe the free motion of the center of gravity of the positronium. Equation (20) can be rewritten in the form

$$\langle W_{11}\rangle = \int \overline{\left(\prod_k S(\mathbf{b}_k - \mathbf{R}^{(\perp)}, \rho, \rho')\right)}\varphi_1^2(\rho, z) \times \varphi_1^2(\rho', z')\left|\chi_{2\mathbf{p}}\left(\mathbf{R}, \frac{l}{c}\right)\right|^2 d^2\rho\, dz\, d^2\rho'\, dz'\, d^3\mathbf{R}, \tag{21}$$

$$S(\mathbf{b}, \rho, \rho') = \left\langle\!\left\langle \exp\left\{ i\left[\delta\left(\mathbf{b} - \frac{\rho}{2}; \mathbf{l}_1, \mathbf{l}_2, \ldots\right) - \delta\left(\mathbf{b} + \frac{\rho}{2}; \mathbf{l}_1, \mathbf{l}_2, \ldots\right) - \delta\left(\mathbf{b} - \frac{\rho'}{2}; \mathbf{l}_1, \mathbf{l}_2, \ldots\right) + \delta\left(\mathbf{b} + \frac{\rho'}{2}; \mathbf{l}_1, \mathbf{l}_2, \ldots\right)\right]\right\}\right\rangle\!\right\rangle_{(e)}. \tag{22}$$

In Eq. (22) the averaging is now over the electronic state of one atom.

We shall assume that the locations of the nuclei are distributed according to a Poisson law. For a Poisson distribution of points ξ_k in three-dimensional space the following relation is valid[9]:

$$\overline{\prod_k a(\xi_k)} = \exp\left(N\int (a(\xi) - 1)\, d^3\xi\right), \tag{23}$$

where N is the average number of points per unit volume. Using Eq. (23) we verify that, with allowance for the smallness of the atomic size, the result of averaging of the quantity $\Pi_k S(\mathbf{b}_k - \mathbf{R}^{(\perp)}, \rho, \rho')$ over the locations of the nuclei in a macroscopic target will not depend on $\mathbf{R}^{(\perp)}$. The final expression for $\langle W_{11}\rangle$ takes the form

$$\langle W_{11}\rangle = \int \exp(-NlK(\rho, \rho'))\varphi_1^2(\rho, z)\varphi_1^2(\rho', z')\, d^2\rho\, dz\, d^2\rho'\, dz', \tag{24}$$

where

$$K(\rho, \rho') = \int [1 - S(\mathbf{b}, \rho, \rho')]\, d^2\mathbf{b}. \tag{25}$$

It follows from Eq. (22) that $K(\rho, \rho')$ is a negative real function which is symmetric with respect to the permutation $\rho \rightleftharpoons \rho'$ and which satisfies the conditions[7])

$$K(\rho, \rho) = 0, \quad \left.\frac{\partial K(\rho, \rho')}{\partial \rho'}\right|_{\rho' = \rho} = 0. \tag{26}$$

We note that in the eikonal approximation the combined cross section for excitation and breakup also is expressed in terms of the function $K(\rho, \rho')$:

$$\sigma_{in} = \int K(\rho, \rho')\varphi_1^2(\rho, z)\varphi_1^2(\rho', z')\, d^2\rho\, dz\, d^2\rho'\, dz'. \tag{27}$$

Thus, in the limit of very small l we have

$$\langle W_{11}\rangle \approx 1 - N\sigma_{in} l,$$

i.e., Eq. (24) goes over into Eq. (11). In the general case with allowance for the fact that $K(\rho, \rho')$ is real we have the inequality[8])

$$\langle W_{11}\rangle > e^{-N\sigma_{in}l}. \quad (28)$$

Let us now calculate $\langle W_{11}\rangle$ for the condition $N\sigma_{in}l \gg 1$. In this limiting case the argument of the exponential in Eq. (24) is large at $\rho \neq \rho'$ and we can use the method of steepest descent. In view of Eqs. (22), (25), and (26) the function $K(\rho, \rho')$ for small $|\rho' - \rho|$ is described by the expression

$$K_0(\rho,\rho') = \frac{1}{8}\left\langle\!\!\left\langle \int \left[\left(\frac{\partial\delta(\mathbf{b}-\rho/2, \mathbf{l}_1, \mathbf{l}_2, \ldots)}{\partial \mathbf{b}} + \frac{\partial\delta(\mathbf{b}+\rho/2, \mathbf{l}_1, \mathbf{l}_2, \ldots)}{\partial \mathbf{b}} \right)(\rho'-\rho)\right]^2 d^2\mathbf{b} \right\rangle\!\!\right\rangle_{(e)}, \quad (29)$$

which has the structure

$$K_0(\rho,\rho') = A(\rho)(\rho'-\rho)^2 + C(\rho)\frac{(\rho(\rho'-\rho))^2}{\rho^2}, \quad (30)$$

where $\rho = |\rho|$. Substituting Eq. (30) into Eq. (24) and recognizing that the exponential $\exp[-NlK(\rho, \rho')]$ is a rapidly decreasing function, we obtain

$$\langle W_{11}\rangle = \frac{\pi}{Nl}\int \frac{\varphi_1^2(\rho, z)\varphi_1^2(\rho, z')\, d^2\rho\, dz\, dz'}{[A(\rho)(A(\rho)+C(\rho))]^{1/2}}. \quad (31)$$

Here, as in the preceding section, we have neglected terms $\sim \exp(-N\sigma_{in}l)$ (see footnote 4).

It is easy to see that the probabilities of transitions to excited bound states φ_k are calculated with formulas similar to Eqs. (24) and (31) with the substitutions

$$\varphi_1^2(\rho, z) \to \varphi_k^*(\rho, z)\varphi_1(\rho, z), \qquad \varphi_1^2(\rho', z') \to \varphi_k^*(\rho', z')\varphi_1(\rho', z').$$

We emphasize that the asymptotic behavior $1/l$, which was obtained previously on the assumption that the positronium radius is much greater than the screening radius of the atom, has a general significance.

In the case of "large" positronium discussed in the preceding section, the crossing term in Eq. (29), which corresponds to the correlation between the electron and the positron, disappears; here $C(\rho) \simeq 0$, and the quantity A is no longer dependent on ρ, namely:

$$A = \frac{1}{8}\left\langle\!\!\left\langle \int \left(\frac{\partial\delta(\mathbf{b}, \mathbf{l}_1, \mathbf{l}_2, \ldots)}{\partial \mathbf{b}} \right)^2 d^2\mathbf{b} \right\rangle\!\!\right\rangle_{(e)}. \quad (32)$$

It can be shown that in the framework of the eikonal approach Eq. (32) has the equivalent form

$$A = \frac{1}{8}\frac{p^2}{\hbar^2}\int \sigma_{sc}(\theta)\theta^2 d\Omega, \quad (32')$$

where $\sigma_{sc}(\theta)$ is the cross section for scattering of an electron with momentum p by a target atom into an element of solid angle $d\Omega$ regardless of the final state of the atom. Consequently the quantity NlA is proportional to the mean square angle of multiple Coulomb scattering of an electron in a layer thickness l:

$$NlA = \frac{1}{8}\frac{p^2}{\hbar^2}\langle\theta^2\rangle = \frac{1}{8}\left(\frac{E_s}{\hbar c}\right)^2 \frac{l}{L_{\mathrm{rad}}}. \quad (33)$$

We then have

$$\int \varphi_1^2(\rho, z)\varphi_1^2(\rho, z')\, d^2\rho\, dz\, dz' = \frac{1}{(2\pi)^2}\int \mathcal{F}^2(\varkappa^2)\, d^2\varkappa = \frac{1}{3\pi a^2}. \quad (34)$$

Here

$$\mathcal{F}(\varkappa^2) = \int \varphi_1^2(\rho, z) e^{-i\varkappa\rho} d^2\rho\, dz = \left(1 + \frac{\varkappa^2 a^2}{4}\right)^{-2}$$

is the form factor of the ground state. With inclusion of Eqs. (33) and (34), the result (15) follows from Eq. (31). In the general case, as is clear from Eq. (29), we have the inequality

$$NlK_0(\rho,\rho') < \frac{Nl}{4}\left[\left\langle\!\!\left\langle \int \left(\frac{\partial\delta(\mathbf{b}, \mathbf{l}_1, \mathbf{l}_2, \ldots)}{\partial \mathbf{b}} \right)^2 d^2\mathbf{b} \right\rangle\!\!\right\rangle_{(e)}\right](\rho'-\rho)^2 = \frac{1}{4}\left(\frac{E_s}{\hbar c}\right)^2 (\rho'-\rho)^2.$$

This leads immediately to the lower bound (16′).

It must be emphasized that our discussion in this and the preceding section applies equally to parapositronium and orthopositronium. As for the probability of change of the spin state of positronium on passage through a layer of matter, it is negligible ($\sim \langle W_{11}\rangle\langle\theta\rangle^2$).

4. THE CASE OF TWO SEPARATED TARGETS. QUALITATIVE ANALYSIS

After traversal of a sufficiently thick plate by the positronium, its probability for remaining in a bound state is very small, and the probability of breakup is close to unity. On the other hand, the probability of remaining in a bound state falls off rather slowly with increase of l, in proportion to $1/l$, so that when l is doubled it decreases by only a factor of two. The situation is different if instead of a target of double the thickness we have two successive targets of thickness l separated by a rather large distance Λ.

Let us consider the case in which the quantity Λ satisfies the inequality inverse to (5″), i.e.,

$$\Lambda \gg pa^2/\hbar. \quad (35)$$

Then a positronium atom which has broken up after the first target cannot recombine into a bound state inside the second target, since in the path Λ the wave packets of the electron and positron move apart in the transverse direction to a distance $\Delta \gg a$. In fact, it follows from relations similar to Eqs. (15) and (16) that for $\langle W_{mk}\rangle \ll 1$ the angle $\langle\theta^2\rangle^{1/2} \gg \hbar/ap$ and the quantity $\Delta \gg \hbar\Lambda/ap$, i.e., when Eq. (35) is taken into account we have $\Delta \gg a$.

Therefore we should be interested only in those cases in which after passing through the first target the positronium atom turns out to be bound. The masses m_k of the discrete stationary states φ_k differ from each other. Therefore after traversing the path Λ, there arises between any pairs of states φ_k and φ_j a phase difference $\alpha_{kj} = (m_k - m_j)\Lambda c/\gamma\hbar$. If the distance Λ between the targets is so large that the inequality (35) is satisfied, then the phases $\alpha_{kj} \gg 1$, i.e., an incoherent mixture of states φ_m with weights proportional to $1/l$ arrives at the second target. Each of these states passes through the second target independently of the others and, on leaving it, results in a bound state of positronium with a small probability, which is also proportional to $1/l$. Altogether, after traversing the two targets, the probability of formation of bound positronium turns out to be negligible ($\sim 1/l^2$). There is a great contrast to the case of a target of double thickness (or of two targets located close together).

The present work was initiated by L. L. Nemenov, who called our attention to the possibility of nonexponential damping of ultrarelativistic positronium passing through a thin layer of matter and who emphasized in this connection the important role of the condition (5). We express our sincere gratitude to him. We also thank G. F. Drukarev for taking part in the discussions.

Note added in proof (25 September 1981). An analysis shows that for validity of the results obtained by us the condition (5) is necessary but not sufficient. It is necessary also to require that the transverse displacements of the electron and positron as the result of multiple scattering be small in comparison with the size of the positronium. This leads to a somewhat more severe condition $l \ll 10^{-6}\gamma^{2/3}L_{\rm rad}^{1/2}$, where l and $L_{\rm rad}$ are expressed in centimeters. A preprint has recently been published on a similar subject (L. L. Nemenov, JINR preprint R2-81-263, Dubna, 1981).

1)We can mention the analysis of interactions of γ rays of rather high energy with nuclei, taking into account the possibility of mutual conversions of γ rays and secondary hadrons.[1] In its more general aspect the role of the characteristic time τ is discussed in connection with the problem of so-called young particles (see for example the review by Feĭnberg[2]).

2)The change of longitudinal momentum in the problem of interest here can be neglected.

3)To avoid possible confusion we note that in experimental studies for purely technical reasons in determining $\langle\theta^2\rangle$ a cutoff of some type is used in the upper limit of the integration over the scattering angles, and the magnitude of the limiting angle itself is considered dependent on the target thickness l. This leads to a violation of the linear relation between $\langle\theta^2\rangle$ and l. However, in the present work we are discussing the quantity $\langle\theta^2\rangle$ without any artificial cutoffs, so that $\langle\theta^2\rangle \sim l$ and Eqs. (13) and (13′) are valid.

4)Strictly speaking, in the right-hand side of Eq. (15) we must add terms proportional to $\exp(-l/L_{\rm tot})$ which are due to those cases in which positronium passes through the target without interaction or with a small number of interactions. For $l \gg L_{\rm tot}$ these terms are rapidly damped; in what follows we shall discuss those thicknesses l for which the relative contributions of the exponential terms to $\langle W_{11}\rangle$ is negligible. We note that results such as Eq. (15), of course, concern not only positronium. In particular, if conditions (4) and (5) are satisfied, then in passage of ultrarelativistic atoms or ions through sufficiently thick films the probability $\langle W_{11}\rangle$ also is proportional to $1/l$. For the hydrogen atom $\langle W_{11}\rangle = (16/3)(\hbar c/E_s a)^2 L_{\rm rad}/l$, where $a = 2\hbar^2/me^2$ is twice the Bohr radius. In contrast to the case of positronium, this result does not depend on the ratio between the radii of the hydrogen atom and the atoms of the medium.

5)An analysis shows that this result applied only to the limiting case $a \gg R$ considered here (a is the radius of postronium and R is the screening radius). For $a/R \sim 1$ this property of independence is destroyed.

6)Besides, just this situation can occur for the bound system $(\mu^+\mu^-)$, which is similar to positronium.

7)According to Eq. (22), ${\rm Re}K(\rho, \rho') \geq 0$, $K(\rho, \rho') = K^*(-\rho, -\rho') = K^*(\rho', \rho)$. In view of the axial symmetry K can depend only on the scalars ρ^2, ρ'^2, and $\rho\rho'$. With allowance for this, we have ${\rm Im}K(\rho, \rho') = 0$.

8)The result (28) follows from the general relation

$$\int \mathcal{F} e^H d\tau > \exp\left(\int \mathcal{F} H\, d\tau \Big/ \int \mathcal{F} d\tau\right) \int \mathcal{F} d\tau,$$

which is valid for positive-definite functions $\mathcal{F}$ and for real functions H of any number of variables. In the case considered here

$$H = -NlK(\rho, \rho'),\ \mathcal{F} = \varphi_1^2(\rho, z)\,\varphi_1^2(\rho', z'),\ \int \mathcal{F} d\tau = 1 \quad (d\tau = d^3\mathbf{r}\, d^3\mathbf{r}').$$

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