Collisionless relaxation of phase oscillations of particles trapped by accelerating waves

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Damping of phase oscillations of charged particles trapped by accelerating electromagnetic waves is determined for the specific case of a circularly polarized wave. The universality of the effect is deduced from the nature of the adiabatic invariant of the trapped particles for a broad class of waves. The physical origins of the effect are the moving apart of the walls of a potential well and the relativistic increase in the mass of the particles. The effect results in the acceleration of the particles to high energies and in anomalously strong absorption of the waves by these particles.

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The acceleration of particles trapped by electromagnetic waves traveling at an increasing velocity is self-evident and has been discussed on many occasions.[1-7] A nontrivial aspect of this effect is the retention of the particles in the trapped state, i.e., the problem of a change in the amplitude of phase oscillations of the trapped particles as a result of a change in the wave parameters. This aspect is discussed in Ref. 8 using an adiabatic invariant; it is shown there that such phase oscillations are damped for a moderately fast decrease in the wave amplitude. Moreover, the law of conservation of energy has been used elsewhere[9] to show that the acceleration of particles may continue until a wave transfers almost all its energy to the particles.

We shall show that the strong trapping of particles by accelerating waves is due to a characteristic physical effect which is collisionless relaxation of phase oscillations. In decelerating waves such phase oscillations usually grow, i.e., the trapped particles are not retained by a wave. The effect occurs in waves of any type which can carry trapped particles. They include waves with a phase velocity smaller than the velocity of light c, as well as amplitude-modulated waves, wave packets, and fronts. In the last case the trapping occurs at the group velocity and the confinement is due to an average longitudinal force resulting from an inhomogeneity of the wave amplitude. The physical origin of the effect is the acceleration of the potential wells formed by the waves, so that we shall not consider the influence of some secondary factors such as the deformation of wells because of dispersion and absorption, particle collisions, etc.

We shall consider a circularly polarized wave described by the vector potential

$$A_x=A(z)\cos\psi,\ A_y=-A(z)\sin\psi, \tag{1}$$

$$\psi=\omega t-\int k(z)\,dz,\quad \omega=\text{const}, \tag{2}$$

where $A(z)$ and $k(z)$ are sufficiently slowly varying functions, so that the process of separation of the explicit dependence on the phase ψ is meaningful (we shall use a system of units in which the lengths and velocities are divided by the velocity of light c, whereas the momentum p, energy, and potential are divided by mc, mc^2, and mc^2/e, respectively, where m and e are the rest mass and charge of a particle).

We shall assume that the slow variation of the wave number k and the amplitude A along the longitudinal coordinate z is due to the presence of a weakly inhomogeneous medium whose influence is manifested by the existence of a permittivity $\varepsilon(z)$. We shall assume that the medium is such that the wave described by Eq. (1) is decelerated, $\varepsilon(z)>1$, and can carry trapped particles. It should be pointed out that these particles can also contribute to the wave deceleration.[10]

In the case of particles trapped by the wave of Eq. (1) there is an adiabatic invariant[6,11]

$$I=\left(\frac{A}{k^2-\omega^2}\right)^{1/2}\beta B(\beta), \tag{3}$$

where $\beta>0$ is a parameter representing the intensity of phase oscillations: when $\beta=0$, the phase oscillations disappear, whereas for $\beta=1$ the peak-to-peak amplitude of the oscillations reaches 2π, and for $\beta>1$, the particle passes through; a combination of complete elliptic integrals $B(\beta)$ varies monotonically from $\pi/4$ to 1 throughout the interval $0<\beta<1$.

Considerable acceleration of the trapped particles and a corresponding strong wave attenuation[8,9] follow from the adiabatic invariant of Eq. (3). When there are relatively few trapped particles, their acceleration in the $k\to\omega$ case can be divided arbitrarily into two stages: in the first stage the wave amplitude A varies slightly because the number of particles is small and the change in the trapping parameter β is governed by the reduction in the denominator $(k^2-\omega^2)^{1/2}$, i.e., phase oscillations relax and the particle confinement becomes even stronger; however, the wave energy is finite and we unavoidably have the second stage when the reduction in A becomes so strong that it masks the reduction in the denominator of Eq. (3), the parameter β increases, and for $\beta=1$ the particles are released. In view of the smallness of the denominator the value of A at the moment of release is so small that we can speak of complete absorption of a wave and of the transfer of all its energy to the accelerated particles. The wave attenuation then increases with decreasing trapped-particle concentration N. However, this concentration clearly has a lower limit so that trapped particles form a continuum relative to the wave: $N\gg k^3$.

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It follows that the effect under discussion is due to relaxation of phase oscillations of trapped particles in accelerating waves. We shall consider this effect in greater detail. We note particularly that in the case of a particle interacting with plane waves there is an integral of motion

$$\mathbf{p}_\perp + \mathbf{A}_\perp = \mathbf{C} = \text{const}, \tag{4}$$

which allows us to eliminate the transverse momentum from all the equations. The longitudinal equation of motion of a particle in the wave described by Eq. (1)

$$\gamma \frac{dp_z}{dt} = kAC \sin\varphi + \frac{dA}{dz}(C\cos\varphi - A) \tag{5}$$

can be transformed with the aid of the equation for the energy γ

$$\gamma \frac{d\gamma}{dt} = \omega AC \sin\varphi \tag{6}$$

and with the aid of the identity

$$\gamma^2 = p^2 + 1 \tag{7}$$

to the following equation describing the longitudinal velocity:

$$\frac{1+(\mathbf{C}-\mathbf{A}_\perp)^2}{2(1-u_z^2)}\frac{du_z^2}{dz} + \omega ACu_z \sin\varphi = (C\cos\varphi - A)\frac{dA}{dz} + kAC\sin\varphi. \tag{8}$$

In Eqs. (5)–(8) the symbol φ is the angle between the vectors **A** and **C**:

$$\varphi = \psi - \psi_0, \quad C_x = C\cos\psi_0, \quad C_y = -C\sin\psi_0. \tag{9}$$

Differentiating Eq. (2), we obtain

$$\frac{d\varphi}{dz} = \frac{d\psi}{dz} = \omega - ku_z. \tag{10}$$

The exact equation for $\varphi(z)$ follows from Eqs. (8) and (10):

$$\frac{1+C^2+A^2-2AC\cos\varphi}{(k+\varphi')^2-\omega^2}\frac{\omega^2}{k+\varphi'}(k'+\varphi'') + A'(C\cos\varphi - A) + AC\sin\varphi\left(k - \frac{\omega^2}{k+\varphi'}\right) = 0. \tag{11}$$

The primes denote differentiation with respect to z.

If we substitute $\varphi' = \varphi'' = 0$, we obtain the following equation for the equilibrium values of $\varphi = \varphi_0$:

$$\frac{1+C^2+A^2-2AC\cos\varphi_0}{k^2-\omega^2}\frac{\omega^2 k'}{k} + A'(C\cos\varphi_0 - A) + \frac{AC}{k}(k^2-\omega^2)\sin\varphi_0 = 0. \tag{12}$$

The relaxation of the phase oscillations is important when A' is small; in accelerating waves we have $k' < 0$; and $k^2 - \omega^2 > 0$, so that $\sin\varphi_0 > 0$.

The existence of equilibrium states can be understood physically. In these states the longitudinal velocity and acceleration of a particle are identical with the phase velocity and acceleration of the wave:

$$u_z = \frac{\omega}{k}, \quad \frac{du_z}{dz} = \frac{d}{dz}\frac{\omega}{k}. \tag{13}$$

The above equality applies if a particle experiences a longitudinal force

$$F_z = u_\perp B \sin\sphericalangle(\mathbf{u}_\perp, \mathbf{B}), \tag{14}$$

corresponding to a specific value of the angle φ ($\mathbf{B} = \operatorname{curl}\mathbf{A}$ is the magnetic field of the wave).

We shall consider the stability of equilibrium states in the presence of small phase oscillations. We shall represent the angle φ in the form

$$\varphi = \varphi_0 + \delta\varphi, \tag{15}$$

where $\delta\varphi$ is so small that we can linearize with respect to it:

$$\frac{1+C^2+A^2-2AC\cos\varphi_0}{k^2-\omega^2}\delta\varphi'' + \left(\frac{1+C^2+A^2-2AC\cos\varphi_0}{(k^2-\omega^2)^2}\frac{k'(\omega^2-3k^2)}{k} + \frac{AC}{k}\sin\varphi_0\right)\delta\varphi' + \left(\frac{k^2-\omega^2}{\omega^2}AC\cos\varphi_0 + \frac{2ACk'}{k^2-\omega^2}\sin\varphi_0 - \frac{kCA'}{\omega^2}\sin\varphi_0\right)\delta\varphi = 0. \tag{16}$$

These equilibrium states are stable if the coefficient of $\delta\varphi$ is positive (the coefficient of $\delta\varphi''$ is always positive). The stability of the equilibrium states implies the presence of phase oscillations, i.e., the possibility of particle trapping.

The main result which follows from Eq. (16) is the positive nature of the coefficient in front of $\sigma\varphi'$ in the case of an accelerating wave characterized by $k' < 0$. This implies a relaxation of phase oscillations. It should be noted that the coefficient of $\delta\varphi'$ may be greater than that of $\delta\varphi$, i.e., the angle φ may approach monotonically its equilibrium value. It follows that the confinement of particles by accelerating waves is very stable, which is identical with the result obtained above from the adiabatic invariant.

It should be noted that a similar qualitative result is obtained also on investigation of a change in the adiabatic invariant because of the finite nature of dk/dz (Ref. 12).

We shall now show that the relaxation of phase oscillations of particles trapped by accelerating waves is a fairly common effect. This follows from the nature of the adiabatic invariant of a broad class of weakly inhomogeneous steady-state waves with $k = k(z)$ and $\omega = \text{const}$:

$$I = \oint \gamma\, d\psi, \tag{17}$$

where the integration is carried out during one period of the particle motion. This adiabatic invariant follows from the earlier[11] exact equation

$$\frac{dY}{dz} = \gamma\frac{dk}{dz} - \omega\frac{\partial A_z}{\partial z} - \frac{\mathbf{C}-\mathbf{A}_\perp}{p_z}\omega\frac{\partial \mathbf{A}_\perp}{\partial z}. \tag{18}$$

The differentiation is carried out with respect to an explicit slow variable z and the quantity

$$Y = k\gamma - \omega(p_z + A_z) \tag{19}$$

is the integral of motion of waves with constant parameters.

Bearing in mind that

$$\gamma = \frac{k(Y+\omega A_z)}{k^2-\omega^2} + \frac{\omega\,\mathrm{sign}(ku_z-\omega)}{k^2-\omega^2}\left((Y+\omega A_z)^2 - (k^2-\omega^2)\left((\mathbf{C}-\mathbf{A}_\perp)^2+1\right)\right)^{1/2}, \tag{20}$$

if it is assumed that the selected gauge makes the scalar potential zero (this does not limit the generality of the treatment because the longitudinal wave field is described fully by the longitudinal component of the vector potential A_z), it can easily be shown that

$$dI/dz=0. \tag{21}$$

In the case of trapped particles the adiabatic invariant of Eq. (17) is

$$I=\frac{\omega}{k^2-\omega^2}\int_{\psi_1}^{\psi_2}((Y+\omega A_z)^2-(k^2-\omega^2)((C-A_\perp)^2+1))^{1/2}d\psi, \tag{22}$$

where ψ_1 and ψ_2 are the values of the phase which cause the integrand to vanish. The factor $(k^2-\omega^2)^{-1}$ in front of the integral indicates a reduction in the swing amplitude of phase oscillations $|\psi_2-\psi_1|$ on increase in the phase velocity of the wave $k(z)\to\omega$, as long as the amplitudes A_z and $A_\perp$ decrease moderately rapidly.

The universality of the relaxation of phase oscillations follows from the universality of the adiabatic invariant (22): the latter is valid in the case of relativistic particles in transverse, longitudinal, and mixed waves of arbitrary form, with all the parameters (wave number and, consequently, phase velocity, as well as wave intensity and profile) varying slowly along the direction of propagation.

A similar result is obtained for particles in amplitude-modulated waves[13] and waves propagating along a static magnetic field.[14] It should be noted that the results of a direct numerical modeling of the behavior of trapped particles in accelerating waves traveling along a static magnetic field demonstrate that phase oscillations are damped even for $\omega/k<1$,[15] which also implies relaxation of these oscillations.

The relaxation of phase oscillations is easy to understand physically. In the case of nonrelativistic values of the phase velocity $k\gg\omega$ the acceleration of a wave increases greatly the wavelength; in the case of trapped particles this is equivalent to the motion of the walls of a potential well away from one another and it is to these walls that a particle gives up its vibrational energy. In the relativistic case of $k\sim\omega=\text{const}$ an increase in the wavelength is unimportant but an increase in the mass of the trapped particles becomes significant and it is proportional to $(k^2-\omega^2)^{-1/2}$, so that the amplitude of phase oscillations also decreases.

We shall conclude by noting that the confinement of particles in waves is governed by the parameter $eE/mc\omega$ (E is the electric field of the wave) and the effect is important in the case of sufficiently low wave frequencies. Therefore, it follows from our mechanism that we can expect a dip of the electromagnetic radiation spectrum at low frequencies.

The effect under consideration gives rise to a relatively small number of particles of high energies, typical of cosmic rays. Therefore, the mechanism applies also to the generation of cosmic rays and the associated astrophysical phenomena.[16] Moreover, such acceleration causes strong absorption of the wave energy and, therefore, it may be of interest for radiative plasma heating. We can thus see that trapped particles are well entrained by the carrier waves if the velocity of the latter increases; this is fairly universal and causes particle acceleration and anomalously strong wave absorption.

[1] A. V Gaponov and M. A. Miller, Zh. Eksp. Teor. Fiz. **34**, 751 (1958) [Sov. Phys. JETP **7**, 515 (1958)].
[2] E. Asseo, G. Laval, R. Pellat, R. Welti, and A. Roux, J. Plasma Phys. **8**, 341 (1972).
[3] Ya. N. Istomin, V. I. Karpman, and D. R. Shklyar, Zh. Eksp. Teor. Fiz. **69**, 909 (1975) [Sov. Phys. JETP **42**, 463 (1975)].
[4] V. I. Karpman, Ya. (J.) N. Istomin, and D. R. Shklyar, Phys. Lett. A **53**, 101 (1975).
[5] H. H. Klein and W. M. Manheimer, Phys. Rev. Lett. **33**, 953 (1974).
[6] V. I. Karpman, Ya. (J.) N. Istomin, and D. R. Shklyar, Phys. Scr. **11**, 278 (1975).
[7] L. M. Kovrizhnykh and A. S. Sakharov, Fiz. Plazmy **2**, 97 (1976) [Sov. J. Plasma Phys. **2**, 54 (1976)].
[8] V. Ya. Davydovskiĭ and E. M. Yakushev, Zh. Eksp. Teor. Fiz. **52**, 1068 (1967) [Sov. Phys. JETP **25**, 709 (1967)].
[9] V. Ya. Davydovskiĭ and E. M. Yakushev, Zh. Tekh. Fiz. **39**, 2236 (1969) [Sov. Phys. Tech. Phys. **14**, 1687 (1970)].
[10] V. Ya. Davydovskiĭ, V. G. Sapogin, and A. S. Ukolov, Zh. Tekh. Fiz. 50, 1175 (1980) [Sov. Phys. Tech. Phys. **25**, 673 (1980)].
[11] V. Ya. Davydovskiĭ, Zh. Eksp. Teor. Fiz. **77**, 519 (1979) [Sov. Phys. JETP **50**, 263 (1979)].
[12] N. A. Davydovskaya, V. Ya. Davydovskiĭ, and A. S. Ukolov, Zh. Tekh. Fiz. **43**, 1132 (1973) [Sov. Phys. Tech. Phys. **18**, 722 (1973)].
[13] V. Ya. Davydovskiĭ and Yu. S. Filippov, Zh. Tekh. Fiz. **47**, 897 (1977) [Sov. Phys. Tech. Phys. **22**, 538 (1977)].
[14] V. Ya. Davydovskiĭ and A. S. Ukolov, Izv. Vyssh. Uchebn. Zaved. Fiz. No. 10, 117 (1976).
[15] V. Ya. Davydovskiĭ and A. S. Ukolov, Izv. Vyssh. Uchebn. Zaved. Fiz. No. 11, 78 (1974).
[16] V. Ya. Davydovskiĭ and Yu. S. Filippov, Pis'ma Astron. Zh. **6**, 282 (1980) [Sov. Astron. Lett. **6**, 156 (1980)].

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