Self-similar regimes of wave collapse

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We obtain and study analytically self-similar solutions of supersonic wave collapse. We find the effect of a "density funnel," in the center of which the energy of the high-frequency waves is absorbed. We show that the Langmuir collapse can be modelled by the one-dimensional equations of scalar collapse with a specially chosen self-similarity index. We give the numerically obtained solutions of the equations of the supersonic scalar collapse.

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INTRODUCTION

The Langmuir wave collapse phenomenon[1] plays, in our opinion, a decisive role in the physics of a Langmuir turbulent plasma—the collapse is the main mechanism for energy transfer from the Langmuir waves to the electrons and ions in the plasma. It has gradually become clear[2-5] that the collapse is important not only for strong Langmuir turbulence when the ratio of the energy of the oscillations to the thermal energy $W/nT > (k\lambda_d)^2$ (λ_d is the Debye radius), but even for weak turbulence, $W/nT < (m_e/m_i)^{1/2}k\lambda_d$, the spectra of which become substantially restructured when collapse is taken into account, and even more so for kinds of turbulence which are intermediate as far as their strength is concerned.

Although collapse has by now been successfully used to explain a number of astrophysical phenomena,[6] there is as yet no direct experimental proof of the existence of collapse. Faith in the correctness of the collapse concept is based first of all on the many results of numerical calculations (see, e.g., Refs. 7 to 9 and the surveys in Refs. 5 and 10) and also on some analytical facts. Among the latter, self-similar substitutions play an important role; these describe self-similar collapse regimes as $t \to t_0$ (t_0 is the moment of collapse). Self-similar substitutions were discovered already in the first paper of one of the present authors[1] and subsequently have been discussed in almost every paper devoted to collapse (see, e.g., Ref. 11).

The numerical solution of the problem of the evolution of a packet of Langmuir waves[7,8] demonstrates with reasonable accuracy how a self-similar regime is reached. However, the self-similar solutions themselves which describe the shape of the density wells (cavitons) which occur in the plasma have so far not been found. This is explained partially by the fact that the maximum symmetry for collapse[12,13] is axial. Therefore, the evaluation of the shape of the caviton requires the solution of a rather complex two-dimensional non-linear elliptical problem.

The present paper is devoted to an approximate solution of this problem. It is based upon a numerical solution of a closely similar but simpler problem. Langmuir collapse belongs to a number of physical phenomena which can be called wave collapses. A typical example of wave collapse is self-focusing. However, self-focusing simulates Langmuir collapse only very roughly. Considerably more common features are shared with it by the model which was studied in Ref. 14 and which we shall call scalar collapse. Scalar collapse allows exactly the same self-similar substitutions as Langmuir collapse; however, the cavitons in the case of scalar collapse can have any symmetry, including spherical symmetry.

In Ref. 14 we solved numerically the equations for the scalar spherically symmetric collapse and we fixed with great accuracy the arrival at the self-similar regime. Nonetheless doubts were expressed in Ref. 15, on the basis of numerical calculations, about the existence of the corresponding self-similar solutions.

In the present paper we present the self-similar solutions for scalar wave collapse. These solutions exist for any dimensionality of space and are marked by an additional index α. We show that these solutions simulate with great accuracy the problem of the self-similar solutions of real Langmuir collapse for the special choice $\alpha = \frac{2}{3}$ in the one-dimensional geometry. We show also that the self-similar collapse can exist also when the Langmuir waves are damped, provided the damping rate does not increase too rapidly with increasing k. All results refer to the most interesting case of supersonic collapse.

§1. GENERAL PROPERTIES OF SUPERSONIC COLLAPSE

The Langmuir wave collapse is described in dimensionless variables (see Ref. 1) by the set of equations

$$\nabla^2(i\varphi_t+\nabla^2\varphi)=\operatorname{div}(n\nabla\varphi), \tag{1.1}$$

$$\left(\frac{\partial^2}{\partial t^2}-c^2\nabla^2\right)n=\nabla^2|\nabla\varphi|^2. \tag{1.2}$$

Here φ is the time envelope of the high-frequency electrostatic potential, n the variation in the plasma density, and c the ion-sound speed. (Usually one makes the equations dimensionless in such a way that $c=1$, but it is convenient for us to leave c undetermined.)

The equations for scalar collapse have the form

$$i\psi_t+\nabla^2\psi=n\psi, \tag{1.3}$$

$$n_{tt}-c^2\nabla^2 n=\nabla^2\Phi, \quad \Phi=|\psi|^2. \tag{1.4}$$

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It is clear from Eqs. (1.1), (1.2) and (1.3), (1.4) that, roughly speaking, $\psi \sim \nabla\varphi$.

Both sets of equations are physically universal, they describe a non-linear medium in which high-frequency waves with dispersion $\omega_k = k^2$ (or $\omega_k = \omega_0 + k^2$) interact with low-frequency waves with dispersion $\Omega_k = c|k|$. The systems differ in the details of the interaction mechanism between the high- and the low-frequency waves.

The major part of the following results refer to the set (1.3), (1.4) and in the majority of the cases, unless otherwise stipulated, we can easily transfer to the set (1.1), (1.2). The set (1.3) is a particular case of a more general system consisting of Eq. (1.4) and the equation

$$i(\psi_t + \mathbf{v}_{gr}\nabla\psi) + \nabla^2\psi = n\psi, \tag{1.5}$$

if we put $v_{gr} = 0$. Here v_{gr} is the group velocity of the high-frequency waves.

The system (1.4), (1.5) describes light which is self-focused through striction, and in real cases $v_{gr} \gg c$. This is the case, in particular, for self-focusing short-wavelength, $(k\lambda_d)^2 > m_e/m_i$, Langmuir waves ($v_{gr} \approx 3v_{T_e}(k\lambda_d)^2$, $c = (T_e/m_i)^{1/2}$). In this situation Eq. (1.5) reduces to (1.3), provided $\mathbf{v}\cdot\nabla\psi = 0$, i.e., if the Laplacian in (1.3) is two-dimensional. We shall assume in what follows that the problem is considered in a medium with an arbitrary dimensionality m.

The equations for scalar collapse have as integrals of motion

$$\begin{gathered} I_1 = \int |\psi|^2 d\mathbf{r}, \quad \mathbf{P} = i\int (\psi\nabla\psi^* - \psi^*\nabla\psi)\, d\mathbf{r}, \\ N = \int n d\mathbf{r}, \quad \mathbf{M} = \int n\mathbf{r}\, d\mathbf{r}, \\ I_2 = \int \left\{ |\nabla\psi|^2 + n\Phi + \frac{1}{2}v^2 + \frac{1}{2}c^2n^2 \right\} d\mathbf{r}. \end{gathered} \tag{1.6}$$

Here $\mathbf{v}$ is the velocity of the medium and is determined by the set of equations

$$n_t + \operatorname{div}\mathbf{v} = 0, \quad \mathbf{v}_t + c^2\nabla n = -\nabla\Phi, \tag{1.7}$$

which is equivalent to Eq. (1.4).

Equation (1.3) can be modified by taking into account the frequency wave damping. Let the dispersion law be

$$\omega_k = k^2 + i\gamma(k).$$

Here $\gamma(k)$ is the damping rate. To take damping into account we must substitute in (1.3)

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \hat{\gamma}.$$

Here $\hat{\gamma} = \gamma(-i\nabla)$ is a pseudo-differential operator with symbol $\gamma(k)$.

We consider for the set (1.3), (1.4) a localized initial condition (wave packet) of characteristic size L and characteristic intensity Φ. Assume at the same time an initial density variation n and its time derivative n_t, while initially $\Phi \gg c^2 n$. This means that we can neglect in Eq. (1.4) the term $c^2\nabla^2 n$ in comparison with $\nabla^2\Phi$. This approximation is equivalent to the condition that the sound velocity be negligibly small, and is traditionally called the supersonic approximation, a designation not too felicitous, as the real velocity of the motion of the matter under the conditions of Eq. (1.4) is much smaller than the sound velocity.

The qualitative nature of the process described by Eqs. (1.3), (1.4) in the supersonic approximation is clear from the equation

$$n_{tt} = \nabla^2\Phi. \tag{1.8}$$

Let initially $n = 0$, $n_t = 0$ while the field ψ has a maximum at the origin. We then have at $r = 0$ $\nabla^2\Phi < 0$, $n_{tt} < 0$, and after a time τ there occurs at the origin a density well

$$n \sim \Phi\tau^2/L^2. \tag{1.9}$$

We note that Eq. (1.3) is a non-stationary Schrödinger equation. It describes for sufficiently small n the dispersive spreading of the wave packet, with a characteristic time $\tau \sim L^2$. If the condition

$$\Phi L^4 \gg 1 \tag{1.10}$$

is satisfied, a density well is formed after the time of spreading, and for the well $nL^2 \gtrsim 1$. In such a well there occurs a discrete level, in which part of the energy of the high-frequency wave is trapped. The spreading then stops whereas the deepening of the density well continues. We give an estimate of the dynamics of this process. Let Φ be the density of the "trapped" energy of the high-frequency waves. In the three-dimensional case we have then

$$\Phi L^3 \approx \text{const}, \quad \Delta\Phi \approx -\text{const}/L^5.$$

For the ground state $n \propto 1/L^2$, whence we have an equation for the density variation in the center of the well:

$$n_{tt} + an^{5/2} = 0,$$

with a a positive constant. This equation has the solution

$$n \approx \text{const}/(t_0 - t)^{4/3}, \tag{1.11}$$

in which case

$$\Phi \approx \Phi_0/(t_0 - t)^2. \tag{1.12}$$

Thus, the density variation and the quantity Φ become infinite after a finite time. This is wave collapse. It is clear from a comparison of (1.11) and (1.12) that as the time t_0 of the collapse is approached the condition for the applicability of the supersonic approximation $\Phi \gg c^2 n$ is improved.

It is important to note the following. In the potential well the field ψ oscillates with a frequency ω of order n. The characteristic inverse time for changes in the well is then of the order of

$$\frac{1}{\tau} \sim \frac{n_t}{n} \sim \frac{1}{t_0 - t}.$$

Notwithstanding the tendency of this time to zero as $t \to t_0$, it follows from (1.11) that the condition $1/\tau \ll \omega$ is satisfied near the collapse, which means that the evolution of the shape of the well can be considered to be adiabatically slow. In the adiabatic approximation

$$\psi = \chi \exp(-i\lambda^2(t)). \quad (1.13)$$

Here $-\lambda^2(t)$ is the time-dependent level in the well and χ is the eigenfunction corresponding to it which satisfies the stationary equation

$$-\lambda^2\chi + \Delta\chi = n\chi. \quad (1.14)$$

The condition for the conservation of the integral I_1 becomes now an additional equation

$$\frac{\partial}{\partial t}\int \chi^2\, d\mathbf{r} = 0. \quad (1.15)$$

When we use $\Phi = \chi^2$ Eqs. (1.14) and (1.15) supplement Eq. (1.8), forming a complete set of equations for adiabatic scalar supersonic collapse. When we take damping into account, Eq. (1.15) is replaced by

$$\frac{\partial}{\partial t}\int \chi^2\, dr + \langle\chi|\hat{\gamma}|\chi\rangle = 0. \quad (1.16)$$

Equations (1.8), (1.14), (1.15) conserve the integrals N and $\mathbf{M}$, and also the integral I_2 which now, as one can easily check, has the form

$$I_2 = -\lambda^2(t) I_1 + \frac{1}{2}\int v^2\, d\mathbf{r}. \quad (1.17)$$

The integral P is identically equal to zero. If $I_2 < 0$, it follows at once from (1.17) that

$$\lambda^2 \geqslant |I_2|/I_1. \quad (1.18)$$

Thus, if $I_2 \leqslant 0$ the level certainly exists and, if $I_2 < 0$ this level can not disappear during the evolution.

We note also that the adiabaticity condition $\omega \gg 1/\tau$ can be rewritten in a different form. Using $n \propto 1/L^2$ we find from Eq. (1.8) that $1/\tau \propto \Phi^{1/2}$. Hence, if we take the condition for the applicability of the supersonic approximation into account, we have for the adiabaticity condition

$$c^2 n \ll \Phi \ll n^2. \quad (1.19)$$

There can also be other levels in the density well. If the well is sufficiently deep so that the frequency of a transition between neighboring levels $\Delta\omega \gg 1/\tau$, these levels also satisfy the adiabaticity condition. If the number of these levels is k, we have

$$1/\tau \sim (\Phi/k)^{1/2}, \quad \Delta\omega \sim n/k,$$

so that it is necessary to have

$$c^2 n \ll \Phi \ll n^2/k. \quad (1.20)$$

In this case the wavefunction ψ can be written in the form

$$\psi = \sum_{n=1}^{k} \chi_n \exp(-i\lambda_n^2(t)) + \tilde{\psi}. \quad (1.21)$$

Here χ_n and $-\lambda_n^2$ are the wave functions and eigenvalues of the n-th adiabatic level, $\tilde{\psi}$ is the non-adiabatic part which is expanded in terms of the non-adiabatic levels and the eigenfunctions of the continuous spectrum.

Accurate to terms of order $1/(\Delta\omega\tau)^2$ we can put

$$\Phi = \sum_{n=1}^{k} |\chi_n|^2 + |\tilde{\psi}|^2. \quad (1.22)$$

The function $\tilde{\psi}$ satisfies Eq. (1.3). For χ_n we have

$$-\lambda_n^2(t)\chi_n + \Delta\chi_n = n\chi_n, \quad \frac{\partial}{\partial t}\int \chi_n^2\, d\mathbf{r} = 0, \quad (1.23)$$

which closes the set of equations. When damping is present it is necessary to include it explicitly into the equation for the non-adiabatic part and for χ_n to put

$$\frac{\partial}{\partial t}\int \chi_n^2\, d\mathbf{r} + \langle\chi_n|\hat{\gamma}|\chi_n\rangle = 0. \quad (1.24)$$

If the number k is large ($k \gg 1$) we can use the quasiclassical approximation to calculate the levels. Neglecting the non-adiabatic part we arrive in that case at the well known Vedenov-Rudakov equations[16] for the collisionless kinetics of Langmuir waves. In the framework of these equations the difference between the scalar and the Langmuir collapse disappears. Breizman[17] has studied the self-similar collapse in this case. We also note that when the conditions for adiabaticity are not satisfied it is impossible to use the Vedenov-Rudakov equations. If there are no decays of high-frequency into low-frequency waves we can use instead Rubenchik's collisionless kinetics.[18] It is valid for Langmuir waves in a plasma when

$$k\lambda_d < (m_e/m_i)^{1/2}, \quad W/nT < (k\lambda_d)^2.$$

§2. SELF-SIMILAR SCALAR COLLAPSE SOLUTIONS

Numerically the set (1.3), (1.4) has been studied earlier in Ref. 14 for the spherically symmetric case. Collapse was observed and to a good accuracy Eq. (1.11) was confirmed. It was also shown that as $t \to t_0$ the collapse becomes self-similar. This is explained by the fact that the set of Eqs. (1.8), (1.14) allows the self-similar substitution[1]

$$\chi = (t_0 - t)^{-1}\varphi(\xi), \quad n = (t_0 - t)^{-2\alpha} V(\xi),$$
$$\xi = (t_0 - t)^{-\alpha}\mathbf{r}, \quad \lambda^2 = (t_0 - t)^{-2\alpha}\lambda_0^2.$$

In that case $\varphi(\xi)$ satisfies the Schrödinger equation

$$-\lambda_0^2\varphi + \Delta\varphi = V\varphi, \quad (2.1)$$

and $V(\xi)$ the equation

$$\alpha^2(2 + \xi\nabla)[(2\alpha + 1)/\alpha + \xi\nabla] V = \Delta\varphi^2. \quad (2.2)$$

In (2.1) and (2.2) α is an arbitrary constant. The conservativeness condition (1.15) enables us to determine it uniquely:

$$\alpha = 2/m.$$

Solutions with $\alpha \neq 2/m$ can be realized only in a nonconservative medium, if the damping has a power-law character $\gamma \propto k^\beta$. In a medium with damping, $\alpha > 2/m$ and we then find from (1.16) that $\beta = 1/\alpha$. Thus, if $\beta < 1/\alpha$ damping cannot halt the collapse, and for $\beta > 1/\alpha$ there are no self-similar damped solutions. Solutions with $\alpha < 2/m$ can be realized only in an amplifying medium and have no physical meaning.

The conditions for the applicability of the supersonic adiabatic approximation (1.19) impose on α the conditions

$$^1/_2<\alpha<1. \tag{2.3}$$

When $\alpha > 1$ self-similar solutions violate the condition for the applicability of the supersonic approximation. In that case they can be used only as intermediate asymptotics under the condition $\Phi \gg c^2 n$. Just such a situation is realized in the one-dimensional case when $m=1$ ($\alpha = 2$). When $\alpha \leq \frac{1}{2}$ self-similar approximations have no physical meaning. Only when $m=3$ do we have $\alpha = \frac{2}{3}$ and conditions (2.3) are satisfied up to the collapse itself. (A numerical experiment[14] gave for that case $\alpha = \frac{2}{3} \pm 0.03$.) Self-similar solutions have a meaning also in a medium with damping when $1 < \beta < \frac{3}{2}$.

The case $m=2$ is very interesting. In that case $\alpha = 1$ and the ratio of the quantities $c^2 n$ and Φ remains constant in the collapse process. However, in that case the self-similar substitution is allowed not only by Eq. (1.8), but also by the more exact Eq. (1.4).

We consider completely symmetric self-similar solutions. They satisfy the equation

$$\frac{\alpha^2}{\xi}\frac{\partial}{\partial\xi}\xi^{1-1/\alpha}\frac{\partial}{\partial\xi}\xi^{2+1/\alpha}V=\frac{1}{\xi^{m-1}}\frac{\partial}{\partial\xi}\xi^{m-1}\frac{\partial\Phi}{\partial\xi}. \tag{2.4}$$

Integrating it we get

$$\alpha^2 V=\frac{\Phi-\Phi_0}{\xi^2}-\frac{1-\alpha(m-2)}{\alpha\xi^{2+1/\alpha}}\int_0^\xi \xi^{-1+1/\alpha}(\Phi-\Phi_0)\,d\xi,\quad \Phi_0=\Phi|_{\xi=0}. \tag{2.5}$$

We must solve Eq. (2.5) together with Eq. (2.1) with the boundary condition

$$\varphi|_{\xi=0}=\Phi_0^{1/2}.$$

The set (2.1) and (2.5) allows solutions with a singularity at some point ξ_0:

$$V\approx\frac{2}{(\xi-\xi_0)^2},\quad \psi\approx\pm\frac{\sqrt{2}\,\alpha\xi_0}{\xi-\xi_0}. \tag{2.6}$$

As the point ξ_0 is arbitrary, the singularity can occur almost for any value of Φ_0. Only those solutions can have a physical meaning for which $\xi_0 = \infty$. The corresponding values of Φ_0 which are determined by solving the equations on a computer are given in §5. We also note that Eqs. (2.1) and (2.5) are invariant under the substitutions

$$\xi\to\gamma\xi,\quad V\to\gamma^{-2}V,\quad \lambda\to\gamma^{-1}\lambda.$$

We can therefore, without loss of generality, assume that $\lambda = 1$. We easily find from (2.5) the asymptotic behavior of V as $\xi \to \infty$:

$$\alpha^2 V\approx-\frac{\alpha(m-2)\Phi_0}{\xi^2}-\frac{1-\alpha(m-2)}{\alpha\xi^{2+1/\alpha}}\int_0^\infty \xi^{-1+1/\alpha}\Phi\,d\xi+\ldots. \tag{2.7}$$

The remaining terms of the asymptotic form are exponentially small as $\xi \to \infty$.

It is appropriate here to elucidate one of the apparent paradoxes of self-similar solutions. From the self-similar substitution (2.1) we find for the quantity

$$N=\int n\,d\mathbf{r}$$

the relation

$$N(t)=(t_0-t)^{\alpha(m-2)}N(0),\quad N(0)=N|_{t=0}. \tag{2.8}$$

However, N is an integral of the motion. For finite N this is possible only if $m=2$. Therefore, if $m \neq 2$ the quantity N must either be equal to zero or be infinite. From (2.7) it is clear that when $m=1$,

$$V\approx\Phi_0/\alpha\xi^2,$$

and N is given by an integral which diverges as $\xi \to \infty$. When $m=1$ we have thus $N=0$. This agrees with the fact that $n(\xi) > 0$ when $\xi \to \infty$ whereas clearly $n(\xi) < 0$ for sufficiently small ξ. When $m=3$,

$$V\approx-\Phi_0/\alpha\xi^2.$$

The integral N now diverges, and $N=\infty$. It is then unnecessary to require that $n(\xi)$ have a different sign in different parts of space.

The fact that the integral N is infinite does not at all mean that in the real problem the density variation is infinite. Indeed, a self-similar solution can be realized only in a finite region of the physical space $r<R$. Since

$$\xi=R(t_0-t)^{-\alpha},\quad \alpha>0,$$

as $t\to t_0$ for any finite point of space $\xi \to \infty$ and in it the asymptotic behavior (2.7) is established. Written in the variables r and t Eq. (2.7) has the form

$$\alpha^2 V=-\frac{\alpha(m-2)\Phi_0}{r^2}-\frac{S_0(t_0-t)^{1/\alpha}}{r^{2+1/\alpha}}+\ldots,$$
$$S_0=[1-\alpha(m-2)]\int_0^\infty \xi^{-1+1/\alpha}\Phi(\xi)\,d\xi=\text{const}. \tag{2.9}$$

We see that in any finite point r as $t \to t_0$ the quantity V tends to a finite limit

$$V\approx-(m-2)\Phi_0/\alpha r^2,$$

i.e., as $t \to t_0$ there occurs in the medium an integrable $1/r^2$-type singularity. There is in this case no density influx from the external region—the singularity is formed from "the own resources" of the density well. This is particularly clear if we note that the second term in the asymptotic expression (2.9) is negative and growing as compared to the first term as $r \to 0$.

In the conservative case $\alpha = 2/m$ the integral I_1 is conserved. Like the integral N, the integral I_2 is not consistent with the self-similar substitution, but as one can see easily, it always converges. Hence it follows that in the conservative case $N_2 \equiv 0$. This fact has been noted several times in the past (see Refs. 7, 8). There is also no paradox here as it is not at all presupposed that $I_2 = 0$ for any initial wave packet from which the self-similar singularity "grows."

As we have already indicated, when $\alpha = 1$ the self-similar substitution is allowed also by the "sound" Eq. (1.2). We give the result of integrating the equation for the density V in that case, restricting ourselves to the most interesting variant $m=2$. In that case

$$V=\frac{1}{(\xi^2-c^2)^{3/2}}\int_0^\xi (\xi^2-c^2)^{1/2}\,\Phi_\xi\,d\xi.$$

§3. EFFECT OF THE "FUNNEL"

In the symmetric case Eq. (2.1) has the form

$$\varphi_{\xi\xi}+\frac{m-1}{\xi}\varphi_\xi-(V+1)\varphi=0. \tag{3.1}$$

We expand φ and V in series in even powers of ξ as $\xi \to 0$:

$$\varphi=\varphi_0(1+\alpha_1\xi^2+\alpha_2\xi^4+\ldots), \tag{3.2}$$

$$V=V_0+V_1\xi^2+\ldots. \tag{3.3}$$

From Eqs. (2.5), (3.1) we get

$$V_0=\left(\frac{\alpha(2\alpha+1)}{\Phi_0}-1\right)^{-1}, \tag{3.4}$$

$$V_1=4\alpha_1^2(1+m)\left(\frac{2\alpha(4\alpha+1)}{\Phi_0}-1\right)^{-1}, \tag{3.5}$$

$$\alpha_1=\left[2m\left(1-\frac{\Phi_0}{\alpha(2\alpha+1)}\right)\right]^{-1},$$
$$\alpha_2=\frac{\alpha_1^2}{2}\left(\frac{m}{2+m}+\frac{\Phi_0}{2\alpha(4\alpha+1)}\right)\left(1-\frac{\Phi_0}{2\alpha(4\alpha+1)}\right)^{-1}. \tag{3.6}$$

V_0 must be a negative quantity so that

$$\Phi_0>\alpha(2\alpha+1). \tag{3.7}$$

For the case $m=3$ there follow important conclusions from (3.7). As $\xi\to\infty$ the asymptotic behavior of V is

$$V\to-c_1/\xi^2,\quad c_1=\Phi_0/\alpha>2\alpha+1>{}^1/_4.$$

This means that the potential well in the self-similar solution has an infinite number of levels which condense towards zero. This fact was noted in a paper by Fraĭman[19] where it was used as the basis for a hypothesis that supersonic collapse is unstable (see also Ref. 20).[1)] In actual fact from Eq. (3.7) completely different consequences flow.

As the self-similar solution is valid only in a finite region of space there will for any $t<t_0$ be only a finite number of levels which increases as $t\to t_0$ as $\frac{2}{3}\ln[t_0/(t_0-t)]$ while the depth of the lower levels will become infinite.

In a real physical situation the singularity in the amplitude φ is, of course, not reached. When the wave pulse is compressed to rather small dimensions r_0, some dissipation mechanism or other is switched on and leads to the absorption of the pulse energy. After that there remains in the medium a "density funnel"

$$V=-\Phi_0/\alpha r^2, \tag{3.8}$$

with $\Phi_0/\alpha>\frac{1}{4}$.

This density funnel is a potential well inside of which a depression to the center must occur (see Ref. 21). The funnel starts to "draw in" the energy of the high-frequency waves which are absorbed in the funnel center when they reach it from the surrounding space. This process can be described as follow. Equations (1.3), (1.8) have exact solutions

$$n=-n_0/r^2,\quad n_0>{}^1/_4,\quad \varphi=\varphi_0 r^{-1/2+i\varepsilon},\quad \varepsilon=(n_0^2-{}^1/_4)^{1/2}, \tag{3.9}$$

and in this case

$$|\varphi|^2=|\varphi_0|^2/r,\quad \Delta|\varphi|^2=0,\quad n_{tt}=0.$$

These solutions are invalid when $r<r_0$ where $r_0\approx c^2n_0/|\varphi_0|^2$ is a quantity determined by the sound speed. In the funnel there occurs absorption of the wave energy, but the funnel itself is not a stationary formation since a spherical wave is formed in it and diverges with a velocity c.

If φ_0 is a characteristic value of the field at the edge of the funnel and τ a characteristic lifetime for it, the following value of the integral I_1 is additionally absorbed in the funnel:

$$\delta I_1\approx|\varphi_0|^2\varepsilon\tau\approx n_0\varphi_0^2\tau. \tag{3.10}$$

If c is sufficiently small, the quantity δI_1 can be comparable with and even surpass the decrease in the integral I_1 due to the self-similar collapse which forms the funnel. Finally we can say that the appearance of an infinite number of levels in the well does not only prevent the collapse, but even increases the absorption of energy due to that mechanism.

§4. SELF-SIMILAR LANGMUIR COLLAPSE SOLUTIONS

All considerations given in Sec. 1 equally pertain to Langmuir collapse. Equations (1.1) and (1.2) also admit of a transition to supersonic and adiabatic approximations and the self-similar substitution

$$n=(t_0-t)^{-2\alpha}V(\xi),\quad \xi=\mathbf{r}(t_0-t)^{-\alpha}.$$

In this case

$$\varphi=f\exp(-i\lambda_0^2t),\quad f=g(\xi)(t_0-t)^{\alpha-1/2}. \tag{4.1}$$

For g and V we get the equations

$$\Delta(-\lambda_0^2g+\Delta g)=\operatorname{div}(V\nabla g), \tag{4.2}$$

$$\alpha^2(2+\xi\nabla)\left(\frac{2\alpha+1}{\alpha}+\xi\nabla\right)V=\Delta|\nabla g|^2. \tag{4.3}$$

Of course, in the three-dimensional case we must put $\alpha=\frac{2}{3}$.

One shows easily (see, e.g., Ref. 8) that Eqs. (4.2) and (4.3) can not have spherically symmetric solutions as in that case the quantity $|\nabla g|^2$ has a minimum in the center. Langmuir collapse has a dipole character—the density well for it has axial symmetry around the z-axis and the potential g is antisymmetric:

$$g(-z,r)=-g(z,r).$$

This fact can be used to simplify Eqs. (4.2) and (4.3).

It is known from numerical experiments[7,8] that the Langmuir caviton is very flat—its radial size $l_\perp$ is three to five times larger than its thickness l_z. The quantity $\mu=l_z/l_\perp$ can thus be considered to be a small parameter.

We shall solve the set (4.2) and (4.3) in the vicinity of $r=0$ by expanding in a series in r. One checks easily that we need then consider only even powers of r:

$$g=g_0+g_1r^2+g_2r^4+\ldots,\quad V=V_0+V_1r^2+V_2r^4+\ldots. \tag{4.4}$$

By virtue of what we have said above

$$g_1\approx\mu^2\frac{\partial^2g_0}{\partial z^2},\quad g_2\approx\mu^4\frac{\partial^4g_0}{\partial z^4}$$

and so on. Substituting (4.4) into (4.2) and (4.3) we consider terms of zeroth order in r and in them put $\mu=0$. This yields a closed set of equations for the junctions $g_0(z)$ and $V_0(z)$ which, if we put

$$\varphi=\frac{\partial}{\partial z}g_0(z),$$

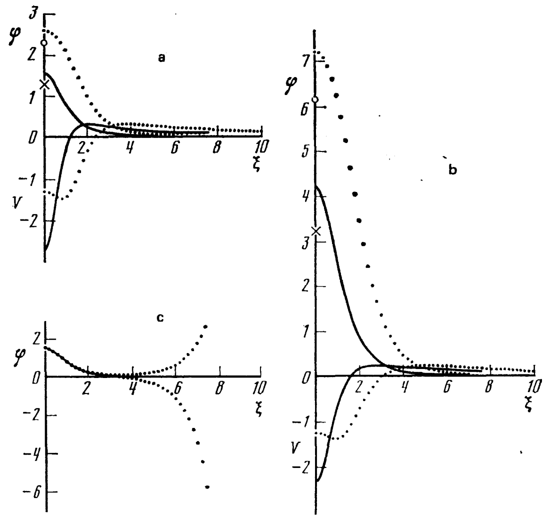

FIG. 1. Legend: Solutions $\varphi(\xi)$ and $V(\xi)$ of Eqs. (2.1), (2.4): Solid lines—quantities Φ_{01}^* from the interval $\alpha(2\alpha+1)<\Phi_{01}<2\alpha(4\alpha+1)$; dots—quantities Φ_{02}^* from the interval $2\alpha(4\alpha+1)<\Phi_{02}<+\infty$. The cross × indicated the value $\varphi_0=[\alpha(2\alpha+1)]^{1/2}$, the circle ○ the value $\varphi_0=[2\alpha(4\alpha+1)]^{1/2}$. a) $m=1$, $\alpha=2/3$; b) $m=1$, $\alpha=2/m$; c) $m=1$, $\alpha=2/3$ with the upper curve given by the value $\Phi_0=\Phi_{01}^*+\Delta\Phi_0$, and the lower one by $\Phi_0=\Phi_{01}^*-\Delta\Phi_0$, $\Delta\Phi_0/\Phi_{01}^*\approx 0.03$.

are the same as the set (2.1), (2.5) for $m=1$, $\alpha=\frac{2}{3}$, i.e., as the problem of one-dimensional collapse with an anomalous value ($\frac{2}{3}$ instead of 2) of α. The terms which we have dropped are of order μ^2 so that the zeroth approximation which we considered guarantees an adequate accuracy.

As $t\to t_0$ there arises in the Langmuir case an axially symmetric density variation

$$n=S(\theta)/(r^2+z^2). \tag{4.5}$$

Here θ is the angle with the z-axis. From the results of §3 it follows that $S(0)>0$. It is at the present time unknown whether there is an infinite number of levels in the well (4.5), i.e., whether for the plasma collapse a funnel effect is possible. At any rate, for this it is necessary that $S(\pi/2)<0$. An elucidation of this problem would be important for an estimate of the absorption which is produced in plasma turbulence by the collapse.

§5. NUMERICAL RESULTS

To check on the existence of a localized ground state

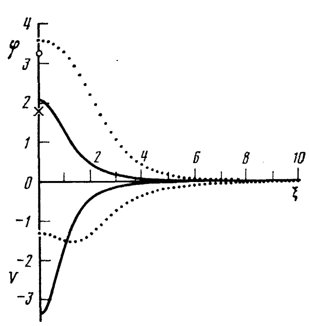

FIG. 2. Legend: See legend to Fig. 1. $m=2$, $\alpha=2/m$.

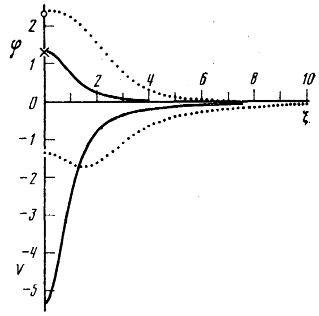

FIG. 3. Legend: See legend to Fig. 1. $m=3$, $\alpha=2/m$.

of the function φ in the potential well V, we numerically integrated Eqs. (3.1) and (2.4) which had been integrated once:

$$\frac{\partial V}{\partial \xi}=\frac{1}{\alpha^2\xi^3}\left[(m-2)(\Phi-\Phi_0)+\xi\frac{\partial\Phi}{\partial\xi}\right]-\frac{(2+1/\alpha)}{\xi}V. \tag{5.1}$$

The expansion (3.2) to (3.6) leaves, when integrating from $\xi=0$, a single parameter Φ_0 in the problem. The solution of Eqs. (3.1), (5.1) is thus determined by the value Φ_0^* for which φ and V tend to zero as $\xi\to\infty$. For small deviations $\Delta\Phi_0$ from Φ_0^* the function φ goes to infinity in accordance with (2.6) (see Fig. 1c). In that case, when $\Delta\Phi_0>0$, the function $\varphi\to+\infty$ whereas for $\Delta\Phi_0<0$ we have $\varphi\to-\infty$. Such a behavior of the function φ is typical of Eqs. (3.1) and (5.1) and was observed for all values of m and α.

Somewhat unexpected was the presence of two solutions for each spatial dimensionality. One can easily obtain an explanation of this fact from the expansion of φ and V, (3.2) to (3.6), as $\xi\to 0$ and the asymptotic behavior (2.7) as $\xi\to\infty$. Indeed, for the correct behavior of φ as $\xi\to 0$ we have $\Phi_0>\alpha(2\alpha+1)$. In that case the function $V_\xi>0$ when $\Phi_0<2\alpha(4\alpha+1)$ at small $\xi\to 0$ and we have a monotonic function V when $V<0$, whereas when $\Phi_0>2\alpha(4\alpha+1)$ and $\xi\ll 1$ we have $V_\xi<0$ and to join it to its asymptotic behavior we must have an extremum of the function V when $\xi\neq 0$ and $V<0$.

The solutions are given in Figs. 1 to 3. The values Φ_{01}^* and Φ_{02}^* are given in Table I with a relative accuracy of 10^{-3}. (We note that our result contradicts the paper by Gol'tsman and Fraiman[15] where on the basis of a numerical calculation it was stated that there are no supersonic self-similar collapse regimes in the case of scalar collapse with $m=2$, $\alpha=1$.) The solution with $m=1$, $\alpha=\frac{2}{3}$ simulates the Langmuir collapse problem (§4). The solution with $m=1$, $\alpha=2$ is the intermediate asymptotic behavior for the formation of a large amplitude one-dimensional soliton.

There are also solutions for values of the self-similarity index $\alpha\neq 2/m$. In Fig. 4 we give the corresponding values of Φ_0^* for $m=3$. It is clear that when α

TABLE I.

m	α	Φ_{01}^*	Φ_{02}^*	m	α	Φ_{01}^*	Φ_{02}^*
1	$^2/_3$	2,506	6.750	2	1	4.277	12.95
1	2	17.88	52.19	3	$^2/_3$	1.913	5,90

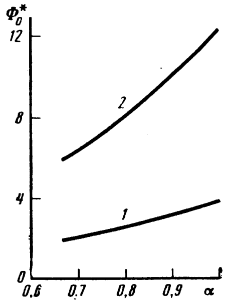

FIG. 4. Legend: The intensity Φ_0^* as function of the self-similarity constant α ($m = 3$): 1: $\Phi_{01}^*(\alpha)$; 2: $\Phi_{02}^*(\alpha)$.

changes from $2/m$ to 1 the amplitude changes by an amount $\Delta\Phi_0/\Phi_0^* \sim 1$.

In order to increase the accuracy of the calculations we used the expansion (3.2) to (3.6) up to $\xi \sim 10^{-1}$ to 10^{-2}, after that the calculation was performed using the Runge-Kutta method of fourth order of accuracy and in a number of cases for comparison using the sixth-order Milne formula.[22] The accuracy of the calculations was monitored by the approach to the asymptotic behavior and in the case of $m=1$ by the condition $N=0$. In all calculations the error was less than 10^{-3}.

In conclusion the authors express their gratitude to E. B. Bogomol'nyi for useful discussions.

Note added in proof (29 October 1981). In fact, Eqs. (3.1) and (5.1) admit an infinite number of solutions. This is connected with the fact that when $\Phi_{0n} = n\alpha(2n\alpha + 1)$ the terms α_k and V_{k-1} in the expansion (3.2), (3.3) tend to infinity at $k \geq n$. In other words, when $\Phi_0 \approx \Phi_{0n}$ the terms $\alpha_n^{\xi 2n}$ and $V_{n-1}\xi^{2(n-1)}$ in the expansion make an appreciable contribution, irrespective of how small ξ is. We are grateful to L. M. Degtyarev for discussing this problem.

1) The problem of the additional levels is not at all connected with the problem of instability. The latter must be solved on the basis of an analysis of the linearized equations. However, the analogy induced by Fraǐman and Litvak[19,20] with the one-dimensional instability of a soliton is inadmissible if only because the collapse, in constrast to a soliton, is an essentially non-stationary effect which takes place during a finite time. The higher levels in the density well correspond to slower processes which at the instant of collapse turn out to be in fact "frozen in" and do not affect the collapse.

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