Parametric turbulence and Cherenkov heating of electrons in a spatially inhomogeneous plasma

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Parametric absorption and the Langmuir turbulence in a plasma with a spatially inhomogeneous density are studied. Quantitative criteria for the applicability of the weak-turbulence approximation in the presence of a density gradient, which are connected with the suppression of the aperiodic instability of the Langmuir waves, are obtained. The efficiency of the parametric absorption of the pump waves and the Langmuir-turbulence spectrum are found in the one-dimensional approximation with allowance for the convective transfer of the energy of the waves from the resonance region. The process involving the transfer of the absorbed energy to the plasma electrons is considered, and the spectrum of the hot electrons and the energy flux transported by them are found.

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1. The accumulation of a large quantity of experimental data on the action of high-power electromagnetic radiation on a plasma requires the construction of a nonlinear theory of parametric interaction of waves in a spatially inhomogeneous plasma. Thus far, the theory of parametric turbulence has been developed in sufficiently great detail only in the approximation of a spatially homogeneous plasma (see Ref. 1, as well as Refs. 2-7). As applied to a plasma with an inhomogeneous density, the theory of parametric turbulence is confined to either numerical computer calculations[8] (in the limit of high radiation energy fluxes), or attempts, based on intuitive reasons, to apply the results of the theory of homogeneous-plasma turbulence.[9]

The qualitative peculiarity of the turbulent state in a spatially inhomogeneous plasma is due primarily to the joint action of two factors: the nonlinear interaction of the waves, which occurs also in a spatially homogeneous plasma, and the convective transport of the waves between plasma layers with different densities. In the present paper we take these two effects into account together for the first time, and formulate the bases of a nonlinear theory of convective parametric instabilities in a spatially inhomogeneous plasma.

The entire discussion is based on the weak-turbulence approximation, since it can be asserted on the basis of the content of the present paper that recourse to the concept of strong turbulence is not necessary for a broad range of recent experiments on the action of laser and high-frequency radiations on a plasma. This is due first and foremost to the proposition, established in the present paper, that the aperiodic parametric instability of the Langmuir waves in a plasma with an inhomogeneous density can be so strongly suppressed that the highly turbulent state does not occur. It is precisely because of this that, as shown below, the weak-turbulence approximation corresponds to the conditions that are of practical interest in connection with experiments on the action of moderate laser-radiation fluxes on a plasma.[2] In Secs. 2 and 3 we formulate the basic equations for the nonlinear interaction of the Langmuir and ion—acoustic waves in a plasma with inhomogeneous density. The conditions of applicability of the weak-turbulence approach to an inhomogeneous plasma are obtained in Secs. 3 and 4 of the paper. In Sec. 5 we consider the question of the efficiency of the parametric conversion of the heating electromagnetic radiation into plasma waves, and determine the instability-saturation level. The principal characteristics of the Langmuir-turbulence spectrum in a plasma with a nonuniform density are obtained in Sec. 6. The Cherenkov acceleration and heating of the electrons in a parametrically turbulent inhomogeneous plasma are investigated in Sec. 7.

2. We shall consider a laminarly inhomogeneous plasma in which the density depends on the coordinate x, but the temperature is constant. A pumping wave propagating in such a plasma excites parametric instabilities that turn out in a number of cases to be convective (see, for example, Refs. 10-12). We shall, without specifying the parametric instability completely, assume that there is in the plasma a region of dimension L_1 (comparable to the density-inhomogeneity scale L) where the conditions for parametric resonance are fulfilled. There occurs at each point x of this region an exponential amplification of the Langmuir waves with frequency $\omega_g(x)$ [amplification factor $\varkappa(x)$].

We shall assume that the plasma is not isothermal (i.e., that $T_e \gg T_i$), and consider the decay interaction of the Langmuir (l) and ion-sound (s) waves to be the mechanism underlying the nonlinear saturation of the instability. (It is shown in Ref. 2 that this nonlinear process is the dominant process under the conditions in question.) The statistical theory of plasma turbulence is based on the kinetic equations for the Langmuir and ion-sound wave numbers $N_l^\sigma(\omega, \mathbf{k}_\perp, x)$ and $N_s^\sigma(\omega, \mathbf{k}_\perp, x)$. These equations have the following form:

$$\frac{\partial}{\partial t}N_l^\sigma(\omega,\mathbf{k}_\perp,x)+\sigma v_{lx}\frac{\partial}{\partial x}N_l^\sigma=-2\gamma_l^\sigma(\omega,\mathbf{k}_\perp,x)N_l^\sigma(\omega,\mathbf{k}_\perp,x)$$
$$+\frac{\pi}{8}\frac{\omega_{Le}^2}{n_eT_e}\sum_{\sigma'\sigma''}\int\frac{d\omega'}{v_{lx}'}\frac{d\mathbf{k}_\perp'}{(2\pi)^3}\int\frac{d\omega''}{v_{sx}''}d\mathbf{k}_\perp''\omega''\left(\frac{\mathbf{k}\mathbf{k}'}{kk'}\right)^2\{\delta(\omega'-\omega-\omega'')$$
$$\times\delta(\mathbf{k}_\perp'-\mathbf{k}_\perp-\mathbf{k}_\perp'')\delta(\sigma'k_{lx}'-\sigma k_{lx}-\sigma''k_{sx}'')[N_l^\sigma(\omega,\mathbf{k}_\perp,x)N_l^{\sigma'}(\omega',\mathbf{k}_\perp',x)$$
$$-N_s^{\sigma''}(\omega'',\mathbf{k}_\perp'',x)(N_l^\sigma(\omega,\mathbf{k}_\perp,x)-N_l^{\sigma'}(\omega',\mathbf{k}_\perp',x))]$$
$$-\delta(\omega-\omega'-\omega'')\delta(\mathbf{k}_\perp-\mathbf{k}_\perp'-\mathbf{k}_\perp'')\delta(\sigma k_{lx}-\sigma'k_{lx}'-\sigma''k_{sx}'')$$

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$$\times [N_l^\sigma(\omega,\mathbf{k}_\perp,x)N_l^{\sigma'}(\omega',\mathbf{k}_\perp',x)-N_s^{\sigma''}(\omega'',\mathbf{k}_\perp'',x)(N_l^{\sigma'}(\omega'\ \mathbf{k}_\perp',x) -N_l^\sigma(\omega,\mathbf{k}_\perp,x))]\},\quad (1)$$

$$\frac{\partial}{\partial t}N_s^\sigma(\omega,\mathbf{k}_\perp,x)+\sigma v_{sx}\frac{\partial}{\partial x}N_s^\sigma=-2\gamma_s^\sigma(\omega,\mathbf{k}_\perp)N_s^\sigma(\omega,\mathbf{k}_\perp,x)$$
$$+\frac{\pi}{4}\frac{\omega_{Le}^2\omega}{n_eT_e}\sum_{\sigma'\sigma''}\int\frac{d\omega'}{v_{lx}'}\frac{d\mathbf{k}_\perp'}{(2\pi)^3}\int\frac{d\omega''}{v_{lx}''}d\mathbf{k}_\perp''\left(\frac{\mathbf{k}'\mathbf{k}''}{k'k''}\right)^2$$
$$\times\delta(\omega-\omega'+\omega'')\delta(\mathbf{k}_\perp-\mathbf{k}_\perp'+\mathbf{k}_\perp'')\delta(\sigma k_{sx}-\sigma' k_{lx}'-\sigma'' k_{lx}'')$$
$$\times[N_l^{\sigma'}(\omega',\mathbf{k}_\perp',x)N_l^{\sigma''}(\omega'',\mathbf{k}_\perp'',x)+N_s^\sigma(\omega,\mathbf{k}_\perp,x)$$
$$\times(N_l^{\sigma'}(\omega',\mathbf{k}_\perp',x)-N_l^{\sigma''}(\omega'',\mathbf{k}_\perp'',x))].$$

Here the superscript σ $(=\pm 1)$ distinguishes the waves propagating in the direction of decrease $(\sigma=\pm 1)$ and the direction of increase $(\sigma=-1)$ of the plasma density, the frequencies of the interacting waves are assumed to be positive, n_e and T_e are the density and temperature of the electrons, γ_s^σ is the damping rate of the ion-acoustic waves, and γ_l^σ is the damping rate of the Langmuir waves. In the wave-generation region $\gamma_l^\sigma < 0$. The equations (1) are similar to the equations of nonlinear interaction of waves in a spatially homogeneous plasma,[2,7,13] with the difference, however, that the wave frequency ω has been introduced in place of the wave number k_x as an argument in the functions N, k_x being connected with ω, $\mathbf{k}_\perp$, and x by the eikonal equation $k_x=k_x(\omega,\mathbf{k}_\perp,x)$. For the Langmuir waves

$$k_x\equiv k_{lx}=\{[\omega^2-\omega_{Le}^2(x)]/(3v_{Te}^2)-\mathbf{k}_\perp^2\}^{1/2},\quad (2)$$

where v_{Te} is the thermal velocity of the electrons and $\omega_{Le}(x)$ is the electron Langmuir frequency, while for the acoustic waves $k_x\equiv k_{sx}=[(\omega/v_s)^2-\mathbf{k}_\perp^2]^{1/2}$ (where $v_s=v_{Te}\omega_{Li}/\omega_{Le}$ is the velocity of sound) does not depend on the coordinate. In the equations (1) the numbers N of quanta are normalized in such a way that the average electric field intensity E of the waves is given by the expression

$$\frac{E_{l,s}^2(x)}{8\pi}=\int\frac{d\omega\,\omega}{v_x(\omega,\mathbf{k}_\perp,x)}\frac{d\mathbf{k}_\perp}{(2\pi)^3}[N_{l,s}^+(\omega,\mathbf{k}_\perp,x)+N_{l,s}^-(\omega,\mathbf{k}_\perp,x)],\quad (3)$$

the group velocity of the waves along the inhomogeneity axis x being connected with k_x by the relation $v_x=[\partial k_x(\omega,\mathbf{k}_\perp,x)/\partial\omega]^{-1}$. The integration in the formula (3) is over the high-frequency region $\omega \gtrsim \omega_{Le}(x)$ in the computation of E_l and over the low-frequency region $0<\omega<\omega_{Li}(x)$ (where ω_{Li} is the ion-plasma frequency) in the computation of E_s.

3. To reveal the qualitative effects due to the spatial inhomogeneity of the plasma, we shall consider the one-dimensional turbulence case, in which the spectra of the Langmuir and ion-acoustic oscillations are concentrated in the region of small $k_\perp$ values. Such a situation can be realized, for example, under conditions when the primary-instability growth rate has a maximum along the inhomogeneity direction. Formally, the assumption that the turbulence spectra are one-dimensional can be written in the form $N^\sigma(\omega,\mathbf{k}_\perp,x)=(2\pi)^2\delta(\mathbf{k}_\perp)N^\sigma(\omega,x)$, which allows to perform the integration on the right-hand sides of the equations (1) and write them for the steady state in the following form:

$$\sigma v_l(\omega,x)\frac{\partial}{\partial x}N_l^\sigma(\omega,x)=-2\gamma_l^\sigma(\omega,x)N_l^\sigma(\omega,x)+\Gamma[P^{-\sigma}(\omega+k_lv_s,x)-P^\sigma(\omega-k_lv_s,x)],$$
$$\sigma v_s\frac{\partial}{\partial x}N_s^\sigma(\omega,x)=-2\gamma_s^\sigma(\omega,x)N_s^\sigma(\omega,x)+\Gamma P^\sigma(\omega_l(\omega,x),x).\quad (4)$$

Here $\Gamma=\omega_{Le}^2\omega_{Li}/24n_eT_ev_{Te}$ is the nonlinear-wave-interaction constant, $k_l(\omega,x)$ is given by the formula (2) with $k_\perp=0$,

$$\omega_l(\omega,x)=\left[\omega_{Le}^2(x)+\frac{3}{4}\frac{v_{Te}^2}{v_s^2}\omega^2\right]^{1/2},$$

and the function $P^\sigma(\omega,x)$ is given by the relation

$$P^\sigma(\omega,x)=N_l^\sigma(\omega+k_l(\omega,x)v_s,x)N_l^{-\sigma}(\omega-k_l(\omega,x)v_s,x)$$
$$+N_s^\sigma(2k_l(\omega,x)v_s,x)[N_l^\sigma(\omega+k_l(\omega,x)v_s,x)-N_l^{-\sigma}(\omega-k_l(\omega,x)v_s,x)]$$

and is the energy flux of the Langmuir waves with frequency $\sim\omega$ over the spectrum [i.e., along the frequency (ω) axis in Fig. 1].

To simplify the system (4), let us make a change of variable by introducing in place of the coordinate x the new variable $\Omega=2k_l(\omega,x)v_s$, which is equal to the frequency of the acoustic wave generated in the decay of the Langmuir wave of frequency ω at the point x. Furthermore, assuming that the inhomogeneity scale $L=-|\partial\ln\omega_{Le}^2(x)/\partial x]^{-1}>0$ is a constant quantity, we go over to the dimensionless functions

$$l^\sigma(\omega,\Omega)=(2\Gamma L/v_s)N_l^\sigma(\omega,x),\ s^\sigma(\omega,\Omega)=(2\Gamma L/v_s)N_s^\sigma(2k_l(\omega,x)v_s,x),$$
$$p^\sigma(\omega,\Omega)=(2\Gamma L/v_s)^2P^\sigma(\omega,x).$$

Taking into account the fact that $\omega_{Li}\ll k_lv_{Te}\ll\omega_{Le}$, we obtain the following system of equations:

$$2\sigma\omega\frac{\partial l^\sigma(\omega,\Omega)}{\partial\Omega}=-2\lambda_l^\sigma(\omega,\Omega)l^\sigma+p^{-\sigma}\left(\omega+\frac{\Omega}{2},\Omega\right)-p^\sigma\left(\omega-\frac{\Omega}{2},\Omega\right)\quad (5)$$

$$-\sigma\omega\frac{\partial s^\sigma(\omega,\Omega)}{\partial\omega}=-2\lambda_s^\sigma(\omega,\Omega)s^\sigma+p^\sigma(\omega,\Omega),\quad (6)$$

where

$$p^\sigma(\omega,\Omega)=l^\sigma(\omega+\Omega/2,\Omega)l^{-\sigma}(\omega-\Omega/2.\ \Omega)$$
$$+s^\sigma(\omega,\Omega)[l^\sigma(\omega+\Omega/2,\Omega)-l^{-\sigma}(\omega-\Omega/2.\ \Omega)].$$

and the $\lambda_l^\sigma=2L\gamma_l^\sigma/v_s$; $\lambda_s^\sigma=2L\gamma_s^\sigma/v_s$ are the dimensionless damping constants for the waves.

The system of equations (5) and (6) should be sup-

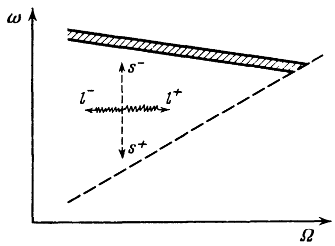

FIG. 1. Direction of propagation of free Langmuir and acoustic waves. The dashed straight line indicates the boundary of the region of Cherenkov absorption of the Langmuir waves. The hatched band is the region of parametric amplification of the waves.

plemented by boundary conditions describing the generation, reflection, and damping of the interacting waves. Without allowance for the nonlinear interaction, the function l^+ describes the Langmuir waves traveling in the direction of increase of Ω in the (ω, Ω) plane, while the function l^- describes the waves traveling in the opposite direction (see Fig. 1). The acoustic perturbations propagate along the ω axis; the function s^+ describes the waves traveling in the direction of decrease of the frequency ω, while the function s^- describes the waves traveling in the opposite direction.

The $\Omega=0$ line (the ordinate axis) is the turning line for the Langmuir oscillations. If we neglect the nonlinear interaction of the Langmuir waves near their turning line, then we should require the equality of the number of quanta arriving at, and the number of quanta departing from, the $\Omega=0$ line:

$$l^+(\omega, 0)=l^-(\omega, 0). \qquad (7)$$

The condition (7) has been written under the assumption that the system is free from the aperiodic parametric instability of the Langmuir waves. This, generally speaking, limits the applicability of the relation (7) to low turbulence levels. Indeed, the excitation of the aperiodic instability of the Langmuir waves is possible in the case in which $W_l/n_eT_e > (r_{De}\delta k)^2$ (Ref. 14), or in the case in which $W_l/n_eT_e > (r_{De}/L)^{2/3}$ (Ref. 15), where W_l is the energy density of the Langmuir oscillations, δk is the spectral width, and $r_{De}=v_{Te}/\omega_{Le}$. In our one-dimensional model $W_l \approx \omega_{Le}\delta k N_l/2\pi$, and

$$\omega_{Le}(r_{De}\delta k)^2 \approx \delta\omega > (r_{De}/L)^{2/3}\omega_{Le},$$

where $\delta\omega$ is the frequency width of the turbulence spectrum. Therefore, the neglect of the aperiodic instability is justified when

$$\Gamma N_l \leqslant \omega_{Li}(\delta\omega/\omega_{Le})^{1/2}, \qquad (8)$$

or when $l(\omega, 0) \lesssim (\delta\omega/\omega_{Le})^{1/2}L/r_{De}$. Below we shall show that this inequality is fulfilled under the practically interesting conditions.

At sufficiently high $\Omega > \Omega_m = 2k_m(\omega)v_s$ (k_m is the highest wave number for which the Cherenkov absorption of the Langmuir waves is still weak), the generation of the waves is impossible because of the presence of strong Landau damping. Therefore, the amplitudes of the Langmuir waves coming from the high-Ω region should be small:

$$l^-(\omega, \Omega)|_{\Omega\gg\Omega_m} \to 0. \qquad (9)$$

The boundary conditions for the acoustic waves can be formulated in similar fashion. The turbulence region is bounded along the ω axis ($\omega \sim \omega_0$, the frequency of the pumping wave). Therefore, it is necessary to require that the amplitudes of the acoustic waves going into the turbulence region be small, i.e., that

$$s^+(\omega\gg\omega_0, \Omega)\to 0, \quad s^-(\omega\ll\omega_0, \Omega)\to 0. \qquad (10)$$

The Langmuir-wave generation occurs in the neighborhood of the decay line $\omega=\omega_d(\Omega)$, where the damping constant λ_l is negative. Since the decay processes lead to the decrease of the frequencies of the interacting waves, it is clear that the turbulence region should lie in the region $\omega<\omega_d(\Omega)$; at the same time, the convective transport of the waves can, generally speaking, lead to some broadening of the turbulence spectrum toward the high-frequency region.

4. Let us, to begin with, consider the problem of the parametric instability of a Langmuir-wave packet with a prescribed energy flux density q and a spectral width $\Delta\omega$. Let this packet propagate in the direction of increase of the plasma density, and be completely absorbed at the point $\Omega=0$. The assumption about the prescribed pump field corresponds to the requirement that the distribution function for the waves in the packet $l^-(\omega, \Omega)=l_0 f([\omega-\omega_0]/\Delta\omega)$, where $f(x)\sim 1 (\int_{-\infty}^{+\infty} dx f(x)=1)$ describes the shape of the packet and $l_0=4\pi\Gamma Lq/v_s\omega_0\Delta\omega$ is the dimensionless energy flux density for the waves in the packet, be Ω independent.

It follows from (5) and (6), when the nonlinear interaction of the generated secondary Langmuir waves is neglected, that the intensity of the parametrically excited l^+ and s^- oscillations depends virtually on the variable $\eta=(\omega+\Omega-\omega_0)/\Delta\omega$ alone. Taking this fact into account, we find that

$$l^+(\eta)\propto s^-(\eta)\propto \exp\left[\varkappa_a\int_{-\infty}^{\eta} d\eta' f(\eta')\right].$$

Consequently, the functions l^+ and s^- are finite for all η, i.e., the parametric instability of the Langmuir-wave packet is convective, and the amplification factor

$$\varkappa_a=\frac{3}{2}\frac{\Delta\omega}{\omega_0}l_0=\frac{\pi}{4}\frac{L}{r_{De}}\frac{q}{n_eT_ev_{Te}} \qquad (11)$$

coincides up to a constant factor with the corresponding expression obtained in the theory of the convective parametric instability $(l\to l+s)$ of a monochromatic Langmuir pumping wave.[10,11] At the same time, the spatial increment growth rate

$$\xi=\varkappa_a|d\eta/dx|=\pi\Gamma q/k_lv_{Te}^2\Delta\omega\propto q \qquad (12)$$

of the perturbations differs from the corresponding expression ξ_{mon} in the case of a monochromatic pump:

$$\xi_{mon}=\gamma_{ls}(v_lv_s)^{-1/2}=(2\Gamma q/3k_lv_{Te}^2v_s)^{1/2}\propto q^{1/2} \qquad (13)$$

(here γ_{ls} is the increment of the ion-acoustic decay instability in a homogeneous plasma). The condition $\xi<\xi_{mon}$ imposes a limitation on the spectral width of the pumping wave in which the weak turbulence theory is applicable, namely,

$$\Delta\omega>\Delta\omega_{min}\approx\omega_{Li}v_E/v_{Te}, \qquad (14)$$

where $v_E=(2q\omega_0/3k_ln_eT_e)^{1/2}$ is the oscillator velocity of the electrons in the pump field. For $\Delta\omega<\Delta\omega_{min}$ the pumping wave should be treated as a monochromatic wave.

The physical meaning of the inequality (14) consists in the fact that the quantity $\Delta\omega_{\min}$ is the spectral width of the amplification band for a monochromatic pumping wave. The fulfillment of (14) leads the the broadening of the spectral width of the amplification band and, consequently, the decrease of the spatial increment.

The amplification factor (11) was derived from above conditions of total neglect of the sound attenuation, i.e., under the assumption that $\gamma_s \ll (\Delta\omega/\omega)\Gamma N_l$. Allowance for the attenuation does not lead to a change in the relations (11) and (12) provided $\Delta\omega > \gamma_s$.

5. Let us now consider the question of the saturation level of the Langmuir turbulence excited in the plasma by some convective instability characterized by the spatial increment ξ_0^σ, the spectral width $\Delta\omega_0(x)$ of the instability band, and the excited-Langmuir-wave frequency $\omega_d(x)$. Here we shall assume that the quantity $\Delta\omega_0$ is small compared to the nonlinear-transfer step $\Omega = 2k_l(\omega, x)v_s$. The Eq. (5) inside the excitation region can be simplified if we take into account the fact that the energy transfer is accompanied by a decrease in the frequency, and therefore there is no flow of energy into the excitation region from the region of higher frequencies [i.e., $p(\omega_d + \Omega/2, \Omega) = 0$]. Furthermore, we shall consider the case of strong ion-acoustic wave attenuation, in which case the contribution of the acoustic waves in the expression for the energy flux across the spectrum can be neglected. Then Eq. (5) assumes the form

$$2\sigma\omega \frac{\partial}{\partial\Omega} l^\sigma(\omega, \Omega) = 2\lambda_0^\sigma l^\sigma(\omega, \Omega) - l^\sigma(\omega, \Omega) l^{-\sigma}(\omega - \Omega, \Omega), \qquad (15)$$

where $\lambda_0^\sigma = 2L\xi_0^\sigma v_l/v_s$ is the dimensionless spatial increment.

When the nonlinear interaction is neglected, Eq. (15) predicts the exponential growth of the number of plasmons with amplification factor $\varkappa_a^\sigma \approx (\Delta\omega_0/\omega)\lambda_0^\sigma$. The last term on the right-hand side of (15) can be neglected when $N_t \exp\varkappa_a^\sigma \ll \xi_0^a v_l/\Gamma$, or when $\varkappa_a < \ln(\xi_0 v_l/\Gamma N_t)$, where N_t is the thermal fluctuation level of the Langmuir waves at the point of entry into the amplification region. This condition establishes the possibility of using the linear theory of convective wave amplication in an inhomogeneous plasma. In the opposite case of large increments, the growth of the waves is terminated by the nonlinear interaction at the level

$$l^\sigma \approx 2\lambda_0^\sigma \qquad (16)$$

or $N_l^\sigma \approx 2\xi_0^\sigma v_l/\Gamma$. This estimate differs from the result of the theory of the homogeneous plasma[7] in that it contains $\xi_0 v_l$ instead γ_0, the increment of the primary instability. The region of applicability of the formula (16) is also different in our case. Let us demonstrate this in the particular case of the ion-acoustic decay instability.

According to the condition (14), ξ_0 should satisfy the inequality $\xi_0 \lesssim \Delta\omega_0/v_s$. In the case of the parametric instabilities the spectral width of the amplification band is proportional to the spatial increment $\Delta\omega_0 \approx \xi_0(\partial\Delta k_x/\partial\omega)^{-1}$, where $\Delta k_x(x, \omega)$ is the detuning of the wave numbers of the interacting waves. The use of this relation yields the relation

$$v_s \partial\Delta k_x/\partial\omega < 1. \qquad (17)$$

For the ion-acoustic parametric instability $\partial\Delta k_x/\partial\omega \approx v_s^{-1}$. Consequently, the turbulence theory is suitable only for the estimation of the saturation level of this instability inside the amplification band, it being suitable to the same extent as the theory that treats the secondary Langmuir waves as monochromatic waves. For this reason, the formula (16) for the ion-acoustic instability coincides up to a numerical factor with the result obtained in Ref. 9, in which the nonlinear-saturation level is estimated in the course of the description of the process of energy transfer across the spectrum in the quasimonochromatic satellite approximation (i.e., with the use for the secondary-instability increment of an expression corresponding to a monochromatic pumping wave).

Indeed, the power $Q(x)$ dissipated in the plasma by the instability-excitation source at a given point x is given by the integral over the wave-excitation region of the product of the spectral energy density of the Langmuir waves and the instability increment:

$$Q(x) = (1/2\pi)\int d\omega\omega\xi_0 N_l \approx \Delta\omega_0\omega_{Le}\xi_0^2 v_l/2\pi\Gamma.$$

In the case of the ion-acoustic instability, whose spatial growth rate is given by the relation (13), we obtain for the effective collision rate the expression

$$\nu_{eff} = 4\pi Q/E_0^2 = (\sqrt{3}/8\pi)k_l v_E, \qquad (18)$$

which was obtained by us in an earlier paper[9] by another method. The assumption that the amplification band is narrow compared to the energy-transfer step (i.e., that $\Delta\omega_0 < k_l v_s$) limits the region of applicability of the formula (18) through the condition $v_E/v_{Te} < k_l r_{De}$.

The formula (18) is also inapplicable in relatively weak pump fields, in which the amplification length ξ_0^{-1} turns out to be greater than the acoustic-wave-attenuation distance v_s/γ_s. In this case $\gamma_{ls} < \gamma_s(v_l/v_s)^{1/2}$, and, as is well known,[12] the formula for ξ_m is no longer valid, and should be replaced by another: $\xi_0 = \gamma_{ls}^2/\gamma_s v_l$, which takes into account the sound attenuation inside the amplification band. The use of this relation leads to the following expression for ν_{eff}:

$$\nu_{eff} = \frac{1}{8}\frac{\gamma_{ls}^2}{\gamma_s} = \frac{1}{8}\omega_{Le}\frac{\omega_s}{\gamma_s}\frac{v_E^2}{v_{Te}^2}, \qquad (19)$$

which coincides with the result obtained in the theory of the homogeneous plasma.[7] But the formula (19) is valid not for $\gamma_{ls} < \gamma_s$, as is the case in the homogeneous plasma,[7] bur in the broader pump field amplitude region where $\gamma_{ls} < \gamma_s(v_l/v_s)^{1/2}$. This means that, for $\gamma_{ls} > \gamma_s$, the absorption is stronger in the inhomogeneous-plasma model than in the homogeneous case.

A different situation obtains in the case of the two-stream instability, for which $\partial\Delta k_x/\partial\omega \sim v_l^{-1}$ and, consequently, the condition (17) is strictly fulfilled. Here the quasimonochromatic approximation used in Ref. 9 to describe the secondary decays is no longer applicable, and from (16) we have for ν_{eff} the estimate

$$\nu_{eff}=(3/4\pi)\,\omega_{Le}(v_{Te}/c)^2(\gamma_{ll}/k_l v_s)$$

($\gamma_{ll}=\frac{1}{4}k_0 v_E$ is the increment of the two-stream instability and $k_0=3^{1/2}\omega_0/2c$ is the wave number of the pumping wave), which coincides with the result of the theory of the homogeneous plasma. Let us emphasize that the assumption that the amplification band is narrow compared to the energy-transfer step limits the region of applicability of the above-given formula for ν_{eff} through the condition $\gamma_{ll}>k_l v_s$.

The quantity ν_{eff} can similarly be estimated for other instabilities, parametric or other.

6. The pump-field energy transferred to the Langmuir waves in the parametric-interaction region is subsequently redistributed among other Langmuir and ion-acoustic waves. The combined influence of the nonlinear transfer over the spectrum as a result of the decay interaction and the convective transport of the wave energy along the inhomogeneity direction leads to the formation of a quasistationary turbulence spectrum.

Let us, to begin with, consider the case of strong acoustic-wave attenuation, when we can obtain the expression for the turbulence spectrum in its explicit form. Neglecting the convection of sound and its effect on the flow of energy over the spectrum, we find from Eqs. (5) and (6) that[16]

$$2\sigma\omega\partial l^\sigma(\omega,\Omega)/\partial\Omega=l^\sigma(\omega,\Omega)\,[l^{-\sigma}(\omega+\Omega,\Omega)-l^{-\sigma}(\omega-\Omega,\Omega)],$$
$$2\lambda_s s^\sigma(\omega,\Omega)=l^\sigma(\omega+\Omega/2,\Omega)\,l^{-\sigma}(\omega-\Omega/2,\Omega). \qquad (20)$$

Since under conditions of nonlinear saturation of the instability the width of the turbulence spectrum is at least greater than the transfer step (i.e., $\delta\omega>\Omega$), the right-hand side of the first equation in (20) can be written in the differential approximation $l(\omega+\Omega)-l(\omega-\Omega)\approx 2\Omega\partial l(\omega)/\partial\omega$. Then the system of two differential equations for the functions l^σ reduces to a single equation for the function $\psi(y,z)=l^+(\omega,\Omega)l^-(\omega,\Omega)$:

$$\frac{\partial^2}{\partial y^2}\psi+\frac{\partial^2}{\partial z^2}\ln\psi=0,\quad y=\omega^2,\quad z=\Omega^2.$$

The solutions to this equation that are of interest to us should satisfy the boundary conditions (7) and (9). Furthermore, the intensities of the waves $l^\pm(\omega_d,\Omega)$ are prescribed at the boundary of the amplification band at $\omega=\omega_d$.

The particular solution for ψ satisfying the conditions (7) and (9) can be found by separating the variables. Then we have for l^σ

$$l^\sigma(\omega,\Omega)=[(\omega^2-\omega_m^2)/\Omega_m^2]\,[1+\sigma\,\mathrm{th}\,(\Omega^2/\Omega_m^2)], \qquad (21)$$

where the constant ω_m is the lower edge of the turbulence spectrum, while Ω_m determines the noise-variation scale along the Ω axis. The prescription of the turbulence spectrum $l^\sigma(\omega_d,\Omega)$ at the boundary of the amplification region fully determines the turbulence spectrum in the region (ω_d,ω_m), as well as the energy flux transported by the Langmuir waves from the parametric-absorption region into the Cherenkov-damping region:

$$q_l=(1/2\pi)\int_{\omega_m}^{\omega_d}\omega d\omega\,(N_l^+-N_l^-).$$

The obtained solution (21) is realized in the case in which the increment of the primary instability (or, more exactly, in which the spectrum at $\omega=\omega_d$) is specially chosen, and is, from this standpoint, of limited interest. At the same time, the characteristics of this solution are typical of other increments ξ_0 that are closer to reality. In the first place, the quasistationary spectrum of the Langmuir turbulence preserves the energy flux. In the second plane, the characteristic value of the turbulence level $l^+(\omega,\Omega)$ depends weakly on the coordinate Ω, and has the same order of magnitude both in the excitation region and in the neighborhood of the reversal line for the Langmuir waves and near the Cherenkov-absorption region for these waves.

These circumstances allow us to estimate the spectral width $\delta\omega=\omega_d-\omega_m$ of the Langmuir turbulence even in the case in which an explicit solution to the equations (20) cannot be constructed. Indeed, on account of the above characteristics of the turbulence, the energy flux $q_l\approx(1/2\pi)\omega_0\delta\omega N_l$ transported by the Langmuir waves into the Cherenkov-absorption region is equal to the energy flux absorbed in the excitation region:

$$q_{abs}=\int dxQ=L_1\Delta\omega_0\omega_0\xi_0 N_l/2\pi.$$

Then, using the fact that N_l depends weakly on the coordinate Ω, we obtain

$$\delta\omega\approx\Delta\omega_0\xi_0 L_1\ll\omega_0. \qquad (22)$$

Consequently, the spectral width of the Langmuir turbulence is proportional to the spectral width $\Delta\omega_0$ of the amplification band, the coefficient of proportionality (which is much greater than unity) being the ratio of the width $L_1=3(k_m r_{De})^2L$ of the entire parametric resonance region [see (9)] to the parametric amplification length ξ_0^{-1}.

In the case, considered here, of strong acoustic-wave attentuation [i.e., when $\xi_0(\partial\Delta k_x/\partial\omega)^{-1}<\gamma_s$] the amplification band width is of the order of the sound-damping constant, i.e., $\Delta\omega_0\approx\gamma_s$, and therefore $\delta\omega$ is proportional to the increment of the instability. In the case of a more intense pumping, $\Delta\omega_0\propto\xi_0$, and therefore $\delta\omega$ is proportional to the square of the increment.

The region of applicability of the formula (22) is limited by the requirement that the parametric turbulence region be free from the secondary aperiodic instability, i.e., the requirement that the Langmuir-noise level be low in the long-wave region, where such aperiodic instability is possible. Recognizing that it follows from the particular solution (21) that the turbulence level near the reversal line for the Langmuir waves is of the same order as in the amplification region, we can rewrite with the aid of (16) the condition (8) for the absence of aperiodic instability in the form of a limitation on the growth rate of the primary instability:

$$\xi_0<\left(\frac{\omega_{Li}}{\omega_{Le}}\right)^2\frac{L}{r_{De}}\frac{\Delta\omega_0}{v_{Te}}. \tag{23}$$

The presence of the inhomogeneity scale in (23) implies that an increase in L broadens the region of applicability of the weak-turbulence theory. The limitation on L from above is given by the inequality (22), and is due to the requirement that the turbulence spectrum be narrow. In the limit of the homogeneous plasma (i.e., for $L\to\infty$), the applicability of the theory is due to the allowance for the collision-induced attenuation of the Langmuir waves. The spectral width of the turbulence in this case is given by the relation[7] $\delta\omega=k_m v_s\gamma_0/\nu_{ei}$, where ν_{ei} is the electron–ion collision rate. A comparison of this formula with (22) with allowance for the substitution $\gamma_0\leftarrow\xi_0 v_i$ allows us to give the condition for the negligibility of the collisions in an inhomogeneous plasma: $v_{Te}/\nu_{ei}>L_1\Delta\omega_0/\omega_{Li}$. It corresponds to the requirement that the dimension of the region of parametric amplification of the waves be small compared to the electron mean free path.

7. The transport of the energy of the Langmuir oscillations into rarefied layers of the plasma is accompanied by a decrease in the phase velocity of the oscillations, and thereby leads to an increase in the Cherenkov damping of them. The energy of the Langmuir oscillations is then transferred to the electrons. This raises the question of the electron distribution function that results from the Cherenkov absorption of the waves.

We consider below the Cherenkov heating of the electrons within the framework of the quasilinear approximation.[1)] Under the conditions of the one-dimensional inhomogeneity, when only the x component of the wave vector varies as a result of the wave departure, it is sufficient to use the one-dimensional quasilinear equation, which we write in the following form (cf., for example, Ref. 17):

$$\sigma\frac{\partial}{\partial x}f^\sigma(\varepsilon,x)=\frac{\partial}{\partial\varepsilon}D^\sigma(\varepsilon,x)\frac{\partial}{\partial\varepsilon}f^\sigma(\varepsilon,x), \tag{24}$$

where f^σ is the distribution function of the electrons, ε is their energy, $\sigma\ (=\pm1)$ is the sign of the electron-velocity component along the inhomogeneity direction, $D^\sigma=2\pi e^2\omega_{Le}N_l^\sigma(\omega_R(\varepsilon,x),x)$ is the diffusion coefficient, $\omega_R(\varepsilon,x)=\omega_{Le}(x)(1+3T_e/4\varepsilon)$ is the frequency of the Langmuir wave resonantly interacting with the electron with energy ε at the point x.

If the spectrum of the Langmuir waves lies in the interval $(\omega_4,\omega_4-\delta\omega)$, then the quasilinear-interaction region in the (ε,x) plane is a band bounded by the lines $\varepsilon_1(x)=\frac{3}{2}T_eL(x-x_1)^{-1}$ and $\varepsilon_2(x)=\frac{3}{2}T_eL(x-x_2)^{-1}$

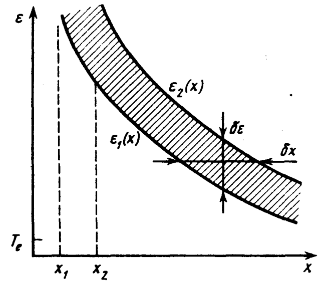

FIG. 2. Region of quasilinear diffusion.

(see Fig. 2), where the points x_1 and x_2 are the reversal points for the Langmuir waves of frequencies ω_4 and $\omega_4-\delta\omega$ respectively.

Equation (24) has the character of the diffusion equation, and consequently leads to the formation of a plateau in the distribution function in the case of a sufficiently large dimension of the diffusion region. Therefore, under the conditions of strong diffusion, i.e., for $D\delta x\gg(\delta\varepsilon)^2$, where δx and $\delta\varepsilon$ are the dimensions of the diffusion region in the neighborhood of the given point x (see Fig. 2), we can approximate the function f^σ inside the interval $(\varepsilon_1,\varepsilon_2)$ by a constant. Then it is not difficult to verify that the distribution of the electrons flying out of the plasma should have the form depicted in Fig. 3. The dashed curve is the plot of the electron distribution function $f_0^+(\varepsilon)$ for $x<x_1$. We shall, for definiteness, assume this to be a Maxwellian distribution with temperature T_e. Beyond the diffusion region, i.e., in the region $\varepsilon>\varepsilon_2(x)$, the electron distribution function $f_H^+(\varepsilon)$ differs from the Maxwellian distribution, since it describes particles that have already interacted with the waves.

To find the distribution function f_H^+, it is sufficient to use the electron-flux conservation law [which follows from (24)]:

$$j^\sigma=m_e^{-1}\int_0^\infty d\varepsilon f^\sigma(\varepsilon,x)=\text{const}.$$

Differentiating this relation with respect to x, we obtain the following equation for the determination of f_H^+:

$$\delta\varepsilon\frac{df_H^+(\varepsilon)}{d\varepsilon}=[f_H^+(\varepsilon)-f_0^+(\varepsilon-\delta\varepsilon)]\frac{d\varepsilon_1/dx}{d\varepsilon_2/dx}, \tag{25}$$

where $\delta\varepsilon=\varepsilon_2(x)-\varepsilon_1(x)\approx\frac{4}{3}(\delta\omega/\omega_0)\varepsilon^2/T_e$. Let us note that this equation has been obtained in the strong-dif-

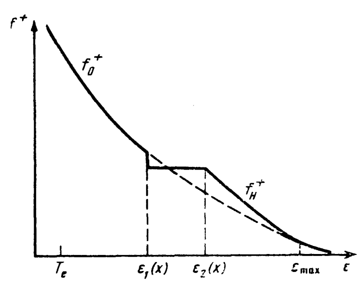

FIG. 3. Shape of the distribution function $f^+(\varepsilon, x)$ under conditions of intense diffusion.

fusion limit, i.e., for $D\delta x \gg (\delta\varepsilon)^2$, which, according to (16) and (22), imposes a definite limitation on the energy of the accelerated electrons:

$$\varepsilon < \varepsilon_{max} = (\omega_{Le}/\xi_0 v_s)^{1/3} T_e. \quad (26)$$

The quantity ε_{max} is the maximum energy of the hot electrons, and does not, for the typical parameters realized in a laser plasma, exceed $(20\text{-}30)T_e$. It is natural to require that $f_H^+(\varepsilon_{max}) \approx f_0^+(\varepsilon_{max})$. Then the solution to Eq. (25) can be represented in the form

$$f_H^+(\varepsilon) = \int_\varepsilon^{\varepsilon_{max}} d\varepsilon \frac{1}{\delta\varepsilon(\varepsilon')} f_0^+(\varepsilon' - \delta\varepsilon(\varepsilon'))$$

$$\times \exp\left(-\int_\varepsilon^{\varepsilon'} \frac{d\varepsilon''}{\delta\varepsilon(\varepsilon'')}\right) + f_0^+(\varepsilon_{max}).$$

From this it can be seen that a significant difference between f_H^+ and f_0^+ arises in the case in which the width of the diffusion region $\delta\varepsilon > T_e$. Such a situation is realized at electron energies $\varepsilon \gtrsim T_e(\omega_0/\delta\omega)^{1/2}$.

The relations obtained for the distribution function allow us to find the relation for the variation of the energy flux of the Langmuir waves as a result of the Cherenkov absorption of them. The total energy flux of the Langmuir waves and the electrons is conserved in the course of the quasilinear interaction, i.e.,

$$q_l^+(x) + m_e^{-1} \int_0^\infty d\varepsilon \varepsilon f^+(\varepsilon) = \text{const}.$$

Differentiating this relation with respect to the coordinate x, and using the explicit form of the function f_H^+, we find the law governing the decrease of the energy of the Langmuir oscillations as a result of the Cherenkov absorption:

$$\frac{dq_l^+}{dx} = -\frac{(\delta\varepsilon)^2}{2m_e} \left|\frac{d\varepsilon_1}{dx}\right| \frac{f_0^+(\varepsilon_1(x))}{\delta\varepsilon + T_e}. \quad (27)$$

In contrast to the linear Landau damping, the rate of decrease of the energy flux does not, according to (26), depend on the magnitude of the flux itself, and is determined only by the number of resonant particles. The relation obtained provides an estimate for the minimum energy of the accelerated electrons for a given Langmuir-wave energy flux q_l:

$$\varepsilon_{min} \approx T_e \ln(\delta\omega n_e T_e v_{Te}/\omega_{Le} q_l) \approx T_e \ln(\omega_{Li}/\xi_0 v_{Te}). \quad (28)$$

Here for q_l we have used the expression

$$q_l \approx L_1 Q \approx (144/\pi)(k_m r_{De})^3 n_e T_e v_{Te} L(\Delta\omega_0/\omega_{Le})(\xi_0 v_{Te}/\omega_{Li}),$$

which is derived in Sec. 5, and which relates the increment of the primary instability with the energy of the pumping wave in the instability excitation region.

Within the limits of applicability of the quasilinear theory, the quantity under the logarithm sign in (28) is always greater than unity. This means that the entire energy absorbed in the plasma is transferred to the epithermal electrons.

8. The foregoing analysis allowed us to estimate the fraction of the heating-radiation energy that can be absorbed in an inhomogeneous plasma as a result of the excitation of parametric instabilities, and to follow how this energy is transferred to the plasma particles. The redistribution of the absorbed energy in the plasma occurs in several stages.

First, the nonlinear interaction of the parametrically excited waves leads to the formation in the plasma of a spectrum of Langmuir turbulence that ensures the transfer of the absorbed energy across the spectrum and in space, so that short-wave Langmuir energy equal to the absorbed energy flows toward the region of lower densities. Secondly, the Cherenkov interaction of the short-wave Langmuir waves with the electrons leads to their attenuation and the formation of a "tail" of accelerated electrons.

The estimates obtained in the homogeneous-plasma model for the pumping-wave energy parametrically absorbed in the plasma retain their values in the case of convective instabilities in an inhomogeneous plasma if the instability growth rate γ_0 is replaced by the product of the spatial growth rate ξ_0 and the group velocity of the Langmuir wave.

It is also important to note the broader region of applicability of the theory of weak turbulence in a spatially inhomogeneous plasma. Thus, if the theory of weak turbulence in a collisionless homogeneous plasma does not, strictly speaking, have a region of applicability at all (the nonlinear evolution of any perturbation leads to the formation of a Langmuir condensate), in an inhomogeneous plasma the convective transport of the energy of the Langmuir waves into the Cherenkov-absorption region guarantees the absence of aperiodic instability even under conditions of fairly intense pumping. Thus, the formulas obtained above for the ion-acoustic parametric instability are valid if the condition $L/r_{De} > (m_i/m_e)^{3/2}$, which follows from the relation (23), is fulfilled. A similar condition obtains for the two-stream parametric instability: $L/r_{De} > m_i/m_e$.

The significant broadening of the conditions of applicability of the theory of weak turbulence in an inhomogeneous plasma leads to a qualitative broadening of the range of real plasma-physics problems to which the results of such a theory can be applied.

[1)] A similar problem in the case of the convective saturation of the two-stream instability is considered in Ref. 18.

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