A magnetic field in a stationary flow with stretching in Riemannian space

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The problem of the kinematic dynamo at large magnetic Reynolds numbers R_m is considered using as an example artificial flow with exponential particle stretching, simulating the stationary stochastic flow of a conducting fluid. The magnetic fields which have a periodic dependence on only one coordinate grow in time exponentially and without bound. Each Fourier harmonic of the deviation from this growing field increases initially rapidly with a rate independent of R_m during a time interval $t_* \approx t_0 \ln R_m$, and then decays very rapidly.

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1. INTRODUCTION

In spite of substantial progress in the solution of the problem of the behavior of the magnetic field for a prescribed motion of a conducting fluid (the so-called kinematic dynamo, see, e.g., the book of Moffatt[1]), some qualitative aspects of this problem are still unclear. The most difficult and pressing for applications in cosmic physics is the case of large magnetic Reynolds numbers, $R_m = Lv/\nu_m$ (here L and v are characteristic scales of length and velocity and ν_m is the microscopic magnetic diffusion coefficient of the fluid). A decisive role in the generation of the magnetic field is played by the geometric (topological) structure of the velocity field. We restrict our attention to stationary flows of an incompressible fluid ($\operatorname{div} v = 0$). Then, from the point of view of the geometric structure the flows fall into two classes: 1) flows for which the streamlines are on stationary surfaces; 2) stochastic flows, in which individual streamlines fill a spatial region everywhere densely.

In principle, a solution of the problem of the kinematic dynamo is known for the flows of the first class. At $R_m^{-1}=0$, only a nonexponential growth of the initial (intrinsic) magnetic field is possible. Taking into account a small diffusion $R_m^{-1}=\varepsilon \ll 1$ one is led, in certain circumstances, to an exponential instability, however the rate of growth of the field (the argument of the exponential) turns our to be small, proportional to some function of ε, say $\varepsilon^{1/3}$ (Ref. 2, 3). An exception are the degenerate flows along a system of parallel planes or spherical surfaces, when an exponential instability is impossible.

The answer is less clear for flows of the second class, although a number of papers[1,5,6] have been devoted to the solution of the problem of the dynamo for stochastic flows that simulate turbulence. The difficulty of the problem is that, one the one hand, the exponential separation of neighboring trajectories which is characteristic for a stochastic flow leads to an exponential growth of the magnetic field with a rate independent of R_m, and on the other hand, there occurs a sharp fragmentation of the size scales of the field and the role of diffusion increases, and does not become small even in the limit as $R_m \to \infty$.

In the present paper we consider an aritficial example of a flow with exponential stretching of particles, proposed earlier[4] by one of the authors. In this case the problem of the kinematic dynamo is amenable to a detailed investigation. Although the discussion of the example requires passing to a compact manifold with a Riemannian metric, this flow simulates the main peculiarities of a stochastic flow in Euclidean space.

Magnetic diffusion plays an important role. At $R_m^{-1}=0$ the magnetic fields depending on the three spatial coordinates may grow indefinitely. The introduction of an arbitrarily small diffusion ($\varepsilon \ll 1$) changes the result qualitatively. In this case an infinite exponential growth is possible only for fields which have a periodic dependence on one of the spatial coordinates. The Fourier harmonics of the deviation from this solution at first increase exponentially, with an exponent independent of ε, and then decay sharply. The possibility of such a temporal growth of the field was first pointed out in a paper of one of the authors,[7] and the succeeding rapid damping was indicated by Saffman.[8]

2. FORMULATION OF THE PROBLEM OF THE KINEMATIC DYNAMO

The behavior of the magnetic field for a given stationary flow of an incompressible conducting fluid is described by the induction equation, which in terms of dimensionless variables has the form

$$\frac{\partial \mathbf{H}}{\partial t} + (\mathbf{v}\nabla)\mathbf{H} = (\mathbf{H}\nabla)\mathbf{v} + R_m^{-1}\Delta\mathbf{H}, \qquad \operatorname{div}\mathbf{v}=0, \quad \operatorname{div}\mathbf{H}=0. \tag{1}$$

Here $\mathbf{H}(\mathbf{r}, t)$ is the magnetic-field pseudovector, $\mathbf{v}(\mathbf{r})$ is the velocity of the flow, R_m is the magnetic Reynolds number, which is assumed to be large

$$R_m^{-1}=\varepsilon \ll 1.$$

The usual condition imposed on the solutions of the induction equation is the absence of external sources for the field. In infinite space this means that $Hr^3 \to 0$ as $r \to \infty$. However, if the velocity does not decrease at infinity, and is periodic, for example, it suffices to require that H should not increase at infinity (or should be periodic).

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Time does not enter explicitly into Eq. (1). Therefore one can set in (1)

$$H\sim e^{\gamma t},\quad \gamma\mathbf{H}=\hat{P}\mathbf{H},$$

where, in general, γ is complex since the differential operator $\hat{P}$ is not self-adjoint.

3. STATIONARY FLOW WITH EXPONENTIAL PARTICLE STRETCHING

The problem (1) can be solved to the end for the case of an artifical example proposed by one of the authors,[4] which is a model of the fundamental property of a stochastic flow, namely the exponential stretching of the fluid particles. The domain of the flow is a three-dimensional compact manifold, which in Cartesian coordinates can be constructed as the product of the two-dimensional torus $\{(x, y) \bmod 1\}$ with the segment $0\leq z\leq 1$, for which the end-tori are identified according to the law

$$(x, y, 0)\equiv(2x+y, x+y, 1).$$

On this manifold one can introduce a Riemannian metric as the metric in R^3 which is invariant under the transformations

$$\begin{gathered}(x, y, z)\to(x+1, y, z),\\ (x, y, z)\to(x, y+1, z),\\ (x, y, z)\to(2x+y, x+y, z+1).\end{gathered}\tag{2}$$

The last transformation is implemented by the area-preserving matrix

$$A=\begin{pmatrix}2&1\\1&1\end{pmatrix},$$

which has the eigenvalues

$$\begin{gathered}\lambda_{1,2}=(3\pm\sqrt{5})/2,\quad \lambda_1\lambda_2=1,\\ \lambda_1\approx 2.11>1,\quad \lambda_2\approx 0.34<1.\end{gathered}\tag{3}$$

Changing from the coordinates x, y, z to the coordinates p, q, z, where p has the direction of the eigenvector with $\lambda_2<1$, and q is along the eigenvector with the eigenvalue $\lambda_1>1$, then the metric given by the line element

$$ds^2=e^{-2\mu z}dp^2+e^{2\mu z}dq^2+dz^2,\qquad \mu=\ln\lambda_1\approx 0.75\tag{4}$$

is invariant with respect to the transformations (2) and therefore determines an analytic Riemannian structure on the compact three-dimensional manifold.

On this Riemannian manifold we consider a flow with the stationary velocity field

$$\mathbf{v}=(0, 0, v),\tag{5}$$

where v=const, so that $\operatorname{div}\mathbf{v}=0$ and $\operatorname{curl}\mathbf{v}=0$. When moving in this velocity field, each fluid particle is exponentially stretched in the q-direction and exponentially contracted along the p-axis (cf. Ref. 4).

In the space with the metric (4) and coordinates p, q, z the differential operations have the form

$$\nabla=(e^{\mu z}\nabla_p,\quad e^{-\mu z}\nabla_q,\ \nabla_z),$$

$$\operatorname{div}\mathbf{H}=e^{\mu z}\frac{\partial H_p}{\partial p}+e^{-\mu z}\frac{\partial H_q}{\partial q}+\frac{\partial H_z}{\partial z},$$

$$\operatorname{rot}_p\mathbf{H}=e^{-\mu z}\left(\frac{\partial H_z}{\partial q}-\frac{\partial e^{\mu z}H_q}{\partial z}\right),$$

$$\operatorname{rot}_q\mathbf{H}=e^{\mu z}\left(\frac{\partial e^{-\mu z}H_p}{\partial z}-\frac{\partial H_z}{\partial p}\right),$$

$$\operatorname{rot}_z\mathbf{H}=e^{\mu z}\frac{\partial H_q}{\partial p}-e^{-\mu z}\frac{\partial H_p}{\partial q},$$

$$\Delta=e^{2\mu z}\frac{\partial^2}{\partial p^2}+e^{-2\mu z}\frac{\partial^2}{\partial q^2}+\frac{\partial^2}{\partial z^2}.$$

The Laplacian $\Delta\mathbf{H}$ is identified with $-\operatorname{curl}\operatorname{curl}\mathbf{H}$.

Projecting Eq. (1) with the velocity field (5) onto the directions $e^{-\mu x}\nabla_p$, $e^{\mu x}\nabla_q$, ∇_z we obtain

$$\begin{gathered}\frac{\partial H_p}{\partial t}+v\frac{\partial H_p}{\partial z}=-\mu vH_p+\varepsilon\left[(\Delta-\mu^2)H_p-2\mu e^{\mu z}\frac{\partial H_z}{\partial p}\right],\\ \frac{\partial H_q}{\partial t}+v\frac{\partial H_q}{\partial z}=\mu vH_q+\varepsilon\left[(\Delta-\mu^2)H_q+2\mu e^{-\mu z}\frac{\partial H_z}{\partial q}\right],\\ \frac{\partial H_z}{\partial t}+v\frac{\partial H_z}{\partial z}=\varepsilon\left(\Delta-2\mu\frac{\partial}{\partial z}\right)H_z.\end{gathered}\tag{6}$$

The equation for the z-component of the field separated, therefore asymptotically, for $t\to\infty$ the component H_z decays. Indeed, let us multiply the last equation by H_z and integrate over the volume contained between the planes $z=0$ and $z=1$. Recognizing that the integrals $\oint H_z^2dpdq$ on these planes coincide, we obtain

$$\frac{d}{dt}\int H_z^2dpdqdz=-2\varepsilon\int(\nabla H)^2dpdqdz.$$

The negative character of the right-hand side of this equation proves the assertion. On the basis of this result we can for simplicity assume in the sequel that the component H_z of the field vanishes.

The equations for the p and q components differ only by the substitution $\mu\to-\mu$, it is therefore sufficient to consider only the q-component, which we denote by $H_q\equiv H$ with $\mu>0$:

$$\frac{\partial H}{\partial t}+v\frac{\partial H}{\partial z}=\mu vH+\varepsilon(\Delta-\mu^2)H.\tag{7}$$

Let us formulate the boundary conditions. This is simplest in the original coordinate system x, y, z. The symmetry (2) means that the function H is periodic in x, y:

$$\begin{aligned}H(x, y, z, t)&=\sum_{n,m}H_{nm}(z, t)\exp[2\pi i(nx+my)]\\ &=\sum_{\alpha,\beta}h_{\alpha\beta}(z, t)\exp[i(\alpha p+\beta q)],\end{aligned}$$

where n, m are integers and α, β are related to $2\pi n$, $2\pi m$ by a linear transformation corresponding to the transition from the coordinates x, y to p, q [a rotation of the cartesian axes x, y by an angle $\arctan(2-\lambda_1)\simeq 72°$]. The symmetry with respect to a shift along the z axis:

$$H(x, y, z, t)=H(2x+y, x+y, z+1, t)$$

allows us to impose restrictions on the Fourier amplitudes. Substituting the last relation into Eq. (8), we obtain

$$H_{\mathbf{n}}(z+1)=H_{A'\mathbf{n}}(z),$$

where $\mathbf{n}=(n, m)$, and A' is the transpose of the matrix A;

in the case under discussion $A'=A$.

Thus, a shift along the z axis is equivalent to a transition from the Fourier amplitudes with indices $(n, m)=\mathbf{n}$ to the Fourier amplitudes with indices $A\mathbf{n}$. An exception is the case $n=m=0$, when the magnetic field does not depend on x, y, or p, q. In that case $A0=0$. In the general case, applying the matrix A to the vector $\mathbf{n}$ shifts the point n, m along a hyperbola in the (n, m) plane, see the figure.

If the function $H(x, y, z, t)$ is analytic its Fourier harmonics $h_{\alpha\beta}(z, t)$ must decay exponentially in α and β. This means, according to Eq. (2) that the function $h_{\alpha\beta}(z, t)$ must decrease not slower than a double exponential. If the function is k times differentiable, then the decay will be according to a power law, as is well known.

Thus, the solutions to Eq. (7) must be periodic in (x, y). If the solution does not depend on p, q then $H(z, t)$ is periodic in z. If there is a dependence on p, q, then the Fourier harmonics $h_{\alpha\beta}(z, t)$ of this solution must decrease rapidly with the increase of $|z|$.

We first consider the case $\varepsilon=0$. Then, going over into a Lagrangian reference frame, it is easy to obtain a solution to the Cauchy problem. Returning to Eulerian coordinates we have

$$H(t, p, q, z)$$
$$=e^{\mu v t}H(0, p, q, z-vt). \tag{9}$$

When the initial field does not depend on p and q this solution can be represented as a superposition of the eigenfunctions $\exp(2\pi imz)\exp(\gamma_m t)$ belonging to the eigenvalues

$$\gamma_m=\mu v-2\pi imv,\ m=0, \pm1, \pm2, \ldots.$$

This is easily seen by expanding Eq. (9) into a Fourier series with respect to z. If the initial field depends on p and q (i.e., is a periodic function of x and y), then, as shown above, the Fourier harmonics of the expansion of (9) in terms of p and q, must decrease with the increase of $|z|$. Therefore the indicated set of functions does not describe solutions satisfying the boundary conditions. In fact this is related to the circumstances that the translation operator along the z axis has a continuous spectrum.

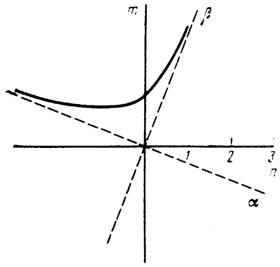

FIG. 1. The dashed axes indicate the directions of the eigenvectors of the matrix A corresponding to the eigenvalues $\lambda_1>1>\lambda_2$. Since $\lambda_1\lambda_2=1$, the product nm is conserved under the action of A on a vector with components (n, m), i.e., there occurs a shift along a hyperbola. One such hyperbola is shown in the figure.

We now go over to the general case $\varepsilon\neq0$. As before, Eq. (7) has solutions periodic in z which are independent of p and q, with the set of eigenvalues

$$\gamma_m=\mu v+\varepsilon(4\pi^2m^2-\mu^2)-2\pi imv. \tag{10}$$

However, when the initial field depends on p and q, the character of the solution is completely different from (9). The shift $z-vt$ along the z axis is equivalent to a translation (along the hyperbola) of the labels of the harmonics $h_{\alpha\beta}(z, t)$ for fixed z. Therefore any given harmonic will shift with time into the region of large wave numbers, where dissipation becomes important, and asymptotically, for $t\to\infty$, will decay independently of the magnitude of ε. Let us describe this process.

We look for solutions of the form (8)

$$h_{\alpha\beta}(z, t)e^{i(\alpha p+\beta q)}.$$

Then the equation takes on the form

$$\frac{\partial h_{\alpha\beta}}{\partial t}+v\frac{\partial h_{\alpha\beta}}{\partial z}+(\varepsilon\mu^2-\mu v+\varepsilon\alpha^2e^{2\mu z}+\varepsilon\beta^2e^{-2\mu z})h_{\alpha\beta}=\varepsilon\frac{\partial^2h_{\alpha\beta}}{\partial z^2}. \tag{11}$$

It is natural to assume that for $\varepsilon\to0$ the leading role is played by terms containing the exponentials. Therefore we consider the reduced equation

$$\frac{\partial h_{\alpha\beta}}{\partial t}+v\frac{\partial h_{\alpha\beta}}{\partial z}+[\varepsilon(\alpha^2e^{2\mu z}+\beta^2e^{-2\mu z})-\mu v]h_{\alpha\beta}=0.$$

It has two first integrals

$$I_1=z-vt,$$
$$I_2=h_{\alpha\beta}(z, t)\exp\left[-\mu z+\frac{\varepsilon}{2\mu v}(\alpha^2e^{2\mu z}-\beta^2e^{-2\mu z})\right],$$

with the help of which it is easy to construct a solution of the Cauchy problem with the initial field $h_{\alpha\beta}(z, 0)$:

$$h_{\alpha\beta}(z, t)=h_{\alpha\beta}(z-vt, 0)\exp\left\{\mu vt-\frac{\varepsilon}{2\mu v}\right\}\alpha^2e^{2\mu z}(1-e^{-2\mu vt})$$
$$-\beta^2e^{-2\mu z}(1-e^{2\mu vt})]\Big\}\approx h_{\alpha\beta}(z-vt, 0)\exp\left\{\mu vt-\frac{\varepsilon}{2\mu v}\beta^2e^{-2\mu z+2\mu vt}\right\}. \tag{12}$$

We see that for a prescribed initial function which is bounded in z, $h_{\alpha\beta}(z, 0)$, each $\alpha\beta$-harmonic will first increase exponentially in proportion to $e^{\mu vt}$, being at the same time translated to the right along the z axis with velocity v, and then the growth is replaced by a sharp decay[1)]

$$\exp\left\{-\frac{\varepsilon\beta^2}{2\mu v}e^{-2\mu z+2\mu vt}\right\}$$

after the lapse of a characteristic time

$$t_*\approx\frac{z}{v}+\frac{1}{2\mu v}\ln\frac{1}{\varepsilon\beta^2}. \tag{13}$$

However, during this process the scale of the field in the z direction starts changing rapidly [approximately over the same time interval, only with $\ln(\varepsilon\beta^2)^{-1}$ replaced by $\ln(\varepsilon|\beta|)^{-1}$], i.e., the transition to the compact equation is no longer justified.

In order to find the asymptotic solution for $t\to\infty$ we go over from the equation (11) to an equation of the Schrödinger type:

$$\Delta\psi+(k-U)\psi=0;$$

$$\psi=H\exp\left(-\frac{v}{2\varepsilon}z-\gamma t\right),\quad k=-\gamma-\frac{v^2}{4\varepsilon}+\mu v-\mu^2\varepsilon,$$

$$U(z)=\alpha^2e^{2\mu z}+\beta^2e^{-2\mu z}=2|\alpha\beta|^{1/2}\operatorname{ch}\mu\tilde{z},\quad \tilde{z}=z+\frac{1}{2\mu}\ln\left|\frac{\alpha}{\beta}\right|.$$

The potential U has a minumum at the point $\tilde{z}=0$:

$$U_{min}=2|\alpha\beta|^{1/2}$$

and increases exponentially rapidly on both sides of the minumum. One can roughly estimate that the lowest "energy" level is of the order of $U_{\min}$. This leads to

$$\gamma\approx-\frac{v^2}{4\varepsilon}+\mu v-\mu^2\varepsilon-U_{min}\underset{\varepsilon\to0}{\longrightarrow}-\frac{v^2}{4\varepsilon}.$$

It can be shown that for $z\to\infty$ the corresponding eigenfunction has the form

$$h_{\alpha\beta}(z,t)\approx\exp\left[\left(-\frac{v^2}{4\varepsilon}+\mu v\right)t+\frac{v}{2\varepsilon}\tilde{z}-2|\alpha\beta|^{1/2}\operatorname{ch}\tilde{z}\right].$$

We thus come to the conclusion that asymptotically, for $t\to\infty$, only the solution which is independent of p and q survives.

We want to stress the fundamentally three-dimensional character of the problem. The expansion occurs in the p, q plane, and the velocity of the flow is along the z axis. It was the shift along the z axis which was responsible for the increasing factor $e^{\mu v t}$, which does not depend on p and q, in the field. On the other hand, the same shift along the z axis gives rise to the continuous spectrum, leading to the sharp decay of the harmonics in the expansion with respect to p and q.

[1)]Roughly speaking we have $k_{eff}=k_0e^{\mu v t}$, $h_{\alpha\beta}=h_{\alpha\beta}(0)e^{-ck^2}$, which leads to the exponential in the argument of the exponential.

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