Particle collisions in an expanding universe

G. S. Bisnovatyĭ-Kogan and I. G. Shukhman

Institute of Cosmic Research, USSR Academy of Sciences
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A collision integral is derived for particles interacting only gravitationally in a flat expanding universe described by the nonrelativistic Milne–McCrea solution. The collision integral does not contain divergences at large impact parameters and small deflection angles. This is a consequence of the growth of the expansion velocity of the particles with increasing impact parameter and not Debye screening, as in a plasma. The time required for establishment of a Maxwellian distribution is calculated.

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1. INTRODUCTION

The collisions of stars or other bodies interacting gravitationally occurs in the same way as the collisions of particles in a plasma interacting in accordance with the Coulomb law. However, there is an important difference. In a quasineutral plasma, there are particles of opposite charges, which screen collisions which are not close, so that effectively they take place only within the Debye radius. In a medium of particles which interact only gravitationally, such screening is absent. Because of this, it is impossible to have the existence and correct mathematical description of a homogeneous gravitating liquid or gas at rest, which always serves as the basic model for analyzing collisions in a plasma.

Gravitating systems can exist either in the form of equilibrium objects of finite size, such as galaxies and clusters, or in the form of a homogeneous and infinite but nonstationary gravitating medium which describes a model of an expanding universe. The first studies of gravitational collisions of stars were based on analysis of the motion of a test particle in a homogeneous medium at rest.[1] Thisled to a Fokker-Planck equation, which was identical with the Landau equation derived earlier for a plasma.[2] As was shown on a number of occasions,[3-7] such an approach is incorrect for gravitating particles. In the case of equilibrium systems of finite size, the distribution of the test particles cannot be regarded as homogeneous; it is necessary to take into account the finite size of the system and consider real curved particle trajectories. For a homogeneous but expanding universe, it is necessary to take into account the relative velocity of the separation of the particles when collisions between them are being studied. In these cases, no divergence arises in the collision integral despite the absence of Debye screening.

In Refs. 3–7, collisions in systems of finite size are analyzed. In [8], it is shown that Hubble expansion is equivalent to a negative mean density, which reduces the problem to a plasma problem. In the present paper, we derive a collision integral for gravitating particles in an expanding universe described by the nonrelativistic Milne–McCrea solution. The obtained integral does not contain divergences at small deflection angles and large impact parameters because of allowance for the separation velocity for colliding particles. However, in the equations there is a divergence at small impact parameters, which lead to large deflection angles. The reason for this divergence, which also occurs in a plasma,[2] is the use of the approximation of a small deflection angle in the derivation of the corresponding equations[4,5,7] [see Eqs. (1)–(3) below]. To eliminate this logarithmic divergence, it is assumed that the minimal distance $p_{\min}$ between the particles is equal to the distance at which the corresponding potential energy of the interaction between the particles is equal to the mean kinetic energy: $Gm^2/p_{\min}=\langle mv^2/2\rangle$. The obtained result then agrees with the result of calculations in which close encounters are taken into account correctly in, for example, an integral of Boltzmann type,[9] in which divergences do not arise at small p. The obtained collision integral is used to calculate the relaxation time of the distribution function of gravitating particles in an expanding universe and is applied to nonrelativistic neutrinos of finite mass and to collisions of galaxies.

2. DERIVATION OF THE COLLISION INTEGRAL

We proceed from the system of equations for the single-particle distribution function f_a and the correlation function g_{ab}, so that $f_{ab}=f_af_b+g_{ab}$, derived under a number of simplifying assumptions in Refs. 4, 5, and 7:

$$\frac{\partial f_a}{\partial t}+\mathbf{v}_a\frac{\partial f_a}{\partial \mathbf{r}_a}-\frac{\partial\Phi}{\partial \mathbf{r}_a}\frac{\partial f_a}{\partial \mathbf{v}_a}=\sum_b \mathrm{St}^{a/b}=\sum_b\frac{\partial}{\partial \mathbf{v}_a}\frac{1}{m_a}\int d\mathbf{r}_b\,d\mathbf{v}_b g_{ab}\frac{\partial U_{ab}}{\partial \mathbf{r}_a},\qquad(1)$$

$$\left(\frac{\partial}{\partial t}+\mathbf{v}_a\frac{\partial}{\partial \mathbf{r}_a}+\mathbf{v}_b\frac{\partial}{\partial \mathbf{r}_b}-\frac{\partial\Phi}{\partial \mathbf{r}_a}\frac{\partial}{\partial \mathbf{v}_a}-\frac{\partial\Phi}{\partial \mathbf{r}_b}\frac{\partial}{\partial \mathbf{v}_b}\right)g_{ab}=\left(\frac{\partial f_a}{\partial \mathbf{v}_a}\frac{f_b}{m_a}-\frac{\partial f_b}{\partial \mathbf{v}_b}\frac{f_a}{m_b}\right)\frac{\partial U_{ab}}{\partial \mathbf{r}_a}.\qquad(2)$$

Here

$$U_{ab}=-Gm_am_b/|\mathbf{r}_a-\mathbf{r}_b|.\qquad(3)$$

The system of equations (1)–(2) is similar to the system (see, for example, Ref. 10) from which the Landau collision integral[2] is obtained, though it also takes into account the self-consistent potential Φ. In the calculation of the Landau collision integral logarithmic divergences arise at both large and small impact parameters, but these can be eliminated if more accurate allowance is made for the binary correlations associated with screening and Eq. (2) is made somewhat more complicated. For the collisions of gravitating particles in an expanding universe that we consider below, divergences do not arise at large impact parameters.

The averaged gravitational potential Φ can be found from the solution of the Poisson equation, which in a homogeneous expanding Newtonian universe with critical density $\rho=\rho(t)$ has the form[11]

$$\rho(t)=\frac{1}{6\pi Gt^2},\quad \Phi(\mathbf{r})=\frac{2}{3}\pi G\rho(t)r^2,\quad \frac{\partial\Phi}{\partial\mathbf{r}}=\frac{2}{9}\frac{\mathbf{r}}{t^2}. \tag{4}$$

The development of perturbations in such a universe in the absence of collisions was studied in Ref. 12 by means of the collisionless Boltzmann-Vlasov equation. The integrals of the characteristics of the unperturbed motion of this equation have the form

$$\mathbf{C}=3\mathbf{v}t^{1/3}-\mathbf{r}t^{-2/3},\quad \mathbf{u}=\mathbf{v}t^{1/3}-2\mathbf{r}t^{-2/3}/3. \tag{5}$$

An arbitrary function $f(\mathbf{u})$ is a solution of the collisionless equation (1) without right-hand side with the necessary dependence (4) of the density on the time. The collision integral does not vanish for arbitrary $f(\mathbf{u})$, which leads to relaxation of an arbitrary $f(\mathbf{u})$ to the equilibrium $f_0(\mathbf{u})$.

To calculate the collision integral, it is necessary, using (2), to express g_{ab} in terms of f_a and f_b and substitute the result in (1). We seek a solution of (2) by the method of integration along the trajectories, which we write in the form[12]

$$\mathbf{r}=\mathbf{C}t^{2/3}-3\mathbf{u}t^{1/3},\quad \mathbf{v}=-\mathbf{u}t^{-2/3}+2\mathbf{C}t^{-1/3}/3. \tag{6}$$

The solution of (2) has the form

$$g_{ab}=\int_0^t\left(\frac{\partial f_a}{\partial\mathbf{v}_a}\frac{f_b}{m_a}-\frac{\partial f_b}{\partial\mathbf{v}_b}\frac{f_a}{m_b}\right)\frac{\partial U_{ab}}{\partial\mathbf{r}_a}dt'. \tag{7}$$

All the functions in the integral (7) are assumed to depend on the time t' in accordance with formula (6). We go over from the variables $(\mathbf{v},\mathbf{r})$ to the variables

$$\mathbf{u},\ \boldsymbol{\xi}=\mathbf{r}/t^{2/3}. \tag{8}$$

Then

$$\frac{\partial}{\partial\mathbf{v}}=t^{1/3}\frac{\partial}{\partial\mathbf{u}},\quad \frac{\partial}{\partial\mathbf{r}}=t^{-2/3}\frac{\partial}{\partial\boldsymbol{\xi}}-\frac{2}{3}t^{-1/3}\frac{\partial}{\partial\mathbf{u}}. \tag{9}$$

For the interaction energy, we have

$$U_{ab}=-\frac{Gm_am_b}{|\mathbf{r}_a-\mathbf{r}_b|}=-t^{-2/3}\frac{Gm_am_b}{|\boldsymbol{\xi}_a-\boldsymbol{\xi}_b|},\quad \frac{\partial U_{ab}}{\partial\mathbf{r}_a}=t^{-2/3}\frac{\partial U_{ab}}{\partial\boldsymbol{\xi}_a}. \tag{10}$$

From (7), we obtain

$$g_{ab}=\int_0^t\left(\frac{\partial f_a}{\partial\mathbf{u}_a}\frac{f_b}{m_a}-\frac{\partial f_b}{\partial\mathbf{u}_b}\frac{f_a}{m_b}\right)\frac{\partial U_{ab}}{\partial\boldsymbol{\xi}_a}dt'. \tag{11}$$

Because of the homogeneity of space, the quantities in the integrand in (11) can be expanded in a Fourier integral:

$$U_{ab}=-\frac{4\pi Gm_am_b}{(2\pi)^3t^{2/3}}\int\frac{d\mathbf{k}}{k^2}\exp[i\mathbf{k}(\boldsymbol{\xi}_a-\boldsymbol{\xi}_b)]. \tag{12}$$

Since $\mathbf{u}$ is an integral of the motion, functions of $\mathbf{u}$ can be taken in front of the integral in (11). Taking into account the time dependence of $\boldsymbol{\xi}$ in accordance with (6) and (8) in the form

$$\boldsymbol{\xi}(t')=\boldsymbol{\xi}(t)+3\mathbf{u}(t^{-1/3}-t'^{-1/3}),$$

we obtain

$$g_{ab}=-\frac{4\pi Gm_am_b}{(2\pi)^3}\left(\frac{\partial f_a}{\partial\mathbf{u}_a}\frac{f_b}{m_a}-\frac{\partial f_b}{\partial\mathbf{u}_b}\frac{f_a}{m_b}\right)\int\frac{i\mathbf{k}\,d\mathbf{k}}{k^2}$$
$$\times\exp[i\mathbf{k}(\boldsymbol{\xi}_a-\boldsymbol{\xi}_b)]\int_0^t\frac{\partial t'}{t'^{4/3}}\exp[3i\mathbf{k}(\mathbf{u}_a-\mathbf{u}_b)(t^{-1/3}-t'^{-1/3})]. \tag{13}$$

Note that in deriving (13) we have not ignored the self-consistent field, as in the derivation of the analogous expression in a plasma.[10] Substituting (13) in the right-hand side of (1), going over there to the variables $(\mathbf{u},\boldsymbol{\xi})$ instead of $(\mathbf{r},\mathbf{v})$, using an expansion of the type (12) for U_{ab} in (1), and the relations

$$\int\exp[i(\mathbf{k}+\mathbf{q})(\boldsymbol{\xi}_a-\boldsymbol{\xi}_b)]d\boldsymbol{\xi}_b=(2\pi)^3\delta(\mathbf{k}+\mathbf{q}),\qquad d\mathbf{v}\,d\mathbf{r}=d\mathbf{u}\,d\boldsymbol{\xi}, \tag{14}$$

we obtain the collision term in the form

$$\mathrm{St}^{a/b}=\frac{(4\pi Gm_am_b)^2}{(2\pi)^3m_a}t^{-1/3}\frac{\partial}{\partial u_{aj}}\int d\mathbf{u}_b\left(\frac{\partial f_a}{\partial u_{ai}}\frac{f_b}{m_a}-\frac{\partial f_b}{\partial u_{bi}}\frac{f_a}{m_b}\right)$$
$$\times\int\frac{d\mathbf{k}(k_ik_j)}{k^4}\int\frac{dt'}{t'^{4/3}}\exp[3i\mathbf{k}(\mathbf{u}_a-\mathbf{u}_b)(t^{-1/3}-t'^{-1/3})]. \tag{15}$$

We transform the expression (15). From symmetry considerations,

$$I_{ij}=\int\frac{d\mathbf{k}k_ik_j}{k^4}\int\frac{dt'}{t'^{4/3}}\exp[3i\mathbf{k}(\mathbf{u}_a-\mathbf{u}_b)(t^{-1/3}-t'^{-1/3})]=A\frac{w_iw_j}{w^2}+B\delta_{ij}; \tag{16}$$

$$\mathbf{w}=\mathbf{u}_b-\mathbf{u}_a,\quad \tau=t'^{-1/3}-t^{-1/3},\quad k=|k_i|, \tag{17}$$

where A and B are functions of w,k,τ. To find A and B, we calculate I_{ii} and $I_{ij}w_iw_j$:

$$I_{ii}=3\int\frac{d\mathbf{k}}{k^2}\int_0^\infty\frac{d\tau}{(\tau+t^{-1/3})^2}e^{3i\mathbf{k}\mathbf{w}\tau}=A+3B,$$
$$I_{ij}w_iw_j=3\int\frac{d\mathbf{k}(k_iw_i)^2}{k^4}\int_0^\infty\frac{d\tau}{(\tau+t^{-1/3})^2}e^{3i\mathbf{k}\mathbf{w}\tau}=(A+B)w^2. \tag{18}$$

Introducing $x=\cos(\hat{k_i w_i})$, and using $k_iw_i=kwx$, $d\mathbf{k}=-2\pi k^2dkdx$, we obtain from (18)

$$B=-A=2\pi\int_0^\infty\int_0^\infty 3\frac{\sin(3kw\tau)\,d\tau\,dk}{3kw\tau(\tau+t^{-1/3})^2}. \tag{19}$$

In (19), we integrate first over $d\tau$ and then over dk. This choice is explained by the fact that (19) contains the logarithmic divergence mentioned above, which can be eliminated readily from the physical considerations only for this order of integration. We have

$$I=\int_0^\infty\frac{\sin(3kw\tau)\,d\tau}{3kw\tau(\tau+t^{-1/3})^2}=\begin{cases}t^{1/3}, & 3kw\ll t^{1/3}.\\[2ex] \dfrac{\pi t^{2/3}}{6kw}, & 3kw\gg t^{1/3}.\end{cases} \tag{20}$$

The integral in (20) can be interpolated approximately by the formula

$$I=\pi t^{2/3}/(6kw+\pi t^{1/3}). \tag{21}$$

Substituting (21) in (19), we obtain

$$B=-A=6\pi^2t^{2/3}\int_0^{k_{max}}\frac{dk}{6kw+\pi t^{1/3}}=\frac{\pi^2t^{2/3}}{w}\ln\left(1+\frac{6k_{max}w}{\pi t^{1/3}}\right). \tag{22}$$

Finally, for the collisional term in the expanding universe we obtain the expression

$$\mathrm{St}^{a/b}=2\pi\frac{(Gm_am_b)^2}{m_a}\frac{\partial}{\partial u_{aj}}\int du_b\left(\frac{\partial f_a}{\partial u_{ai}}\frac{f_b}{m_a}-\frac{\partial f_b}{\partial u_{bi}}\frac{f_a}{m_b}\right)\left(\frac{\delta_{ij}}{w}-\frac{w_iw_j}{w^3}\right)\Lambda_{ab},$$

$$\Lambda_{ab}(w)=\ln\left(1+\frac{6k_{max}w}{\pi t^{4/3}}\right). \quad (23)$$

The maximal value $k_{\max}$ corresponds to the minimal distance $p_{\min}$ between the particles, which is determined from the condition

$$m_{ab}\langle v^2\rangle\sim Gm_am_b/p_{min},$$
$$m_{ab}=m_am_b/(m_a+m_b), \quad \langle v^2\rangle=\langle(\mathbf{v}_a-\mathbf{v}_b)^2\rangle.$$

We have

$$k_{max}=t^{2/3}/p_{min}=t^{2/3}\langle v^2\rangle/G(m_a+m_b). \quad (24)$$

Taking into account approximately $w=\langle v^2\rangle^{1/2}t^{2/3}$ under the logarithm in (23), we obtain

$$\Lambda_{ab}=\ln\left(1+6t\langle v^2\rangle^{3/2}/\pi G(m_a+m_b)\right). \quad (25)$$

The mass of weakly interacting objects in the universe varies in a wide range from 5×10^{-32} g for neutrinos if thier rest energy is 30 eV (Ref. 13) to the mass $10^{14}M_\odot=2\times10^{47}$ g of clusters of galaxies. Accordingly, Λ varies from

$$\Lambda_{\nu\nu}=\ln\left(1+\frac{6\cdot6\cdot10^{17}\cdot216\cdot10^{15}}{\pi\cdot6.7\cdot10^{-8}\cdot5\cdot10^{-32}}\right)\approx170 \quad (26)$$

for neutrinos to

$$\Lambda_{GG}=\ln\left(1+\frac{6\cdot6\cdot10^{17}\cdot10^{24}}{\pi\cdot6.7\cdot10^{-8}\cdot2\cdot10^{47}}\right)\approx4.5$$

for clusters of galaxies. Here, we have assumed $t=6\times10^{17}$ sec, corresponding to $z=0$, $\langle v_\nu^2\rangle^{1/2}=6\times10^5$ cm/sec (see, for example, Ref. 14), and $\langle v_G^2\rangle^{1/2}=10^8$ cm/sec.

3. EQUALIZATION OF THE TEMPERATURES OF PARTICLES IN AN EXPANDING UNIVERSE

We consider a universe filled with two species of collisionless objects with masses m_a and m_b. We assume that both species have Maxwellian distributions but with different "invariant temperatures" Θ_a and Θ_b, i.e., (see Ref. 12)

$$f_a=\frac{\alpha}{6\pi Gm_a}\left(\frac{m_a}{2\pi\Theta_a}\right)^{3/2}\exp\left(-\frac{m_au_a^2}{2\Theta_a}\right),$$
$$f_b=\frac{1-\alpha}{6\pi Gm_b}\left(\frac{m_b}{2\pi\Theta_b}\right)^{3/2}\exp\left(-\frac{m_bu_b^2}{2\Theta_b}\right), \quad (27)$$
$$m_a\int f_a\,d\mathbf{v}_a+m_b\int f_b\,d\mathbf{v}_b=1/6\pi Gt^2.$$

If $\Theta_a=\Theta_b$, then the functions (27) satisfy the kinetic equation

$$\frac{\partial f_a}{\partial t}+\mathbf{v}_a\frac{\partial f_a}{\partial \mathbf{r}}-\frac{\partial\Phi}{\partial\mathbf{r}}\frac{\partial f_a}{\partial\mathbf{v}_a}=\mathrm{St}^{a/a}+\mathrm{St}^{a/b}=0, \quad (28)$$

where $St^{a/b}$ is defined in (23); the equation for f_b has the same form.

If $\Theta_a\neq\Theta_b$, then the collisions result in energy transfer and equalization of the "temperatures." Assuming Θ_a and Θ_b to be functions of the time, and substituting (27) in (28), we obtain as in Ref. 2

$$\left(\frac{m_au_a^2}{2\Theta_a}-\frac{3}{2}\right)\frac{f_a}{\Theta_a}\frac{d\Theta_a}{dt}=-\frac{2\pi(Gm_am_b)^2}{m_a}\frac{\partial}{\partial u_{aj}}$$
$$\times\int d\mathbf{u}_bf_af_bu_{ai}\left(\frac{1}{\Theta_a}-\frac{1}{\Theta_b}\right)\left(\frac{\delta_{ij}}{w}-\frac{w_iw_j}{w^3}\right)\Lambda_{ab}. \quad (29)$$

The integral on the right-hand side of (29) with the functions (27) was calculated in Ref. 2 for the case $m_a/m_b\ll1$ and in the general case in Ref. 15. Multiplying (29) by $m_au_a^2/2$, and integrating over $d\mathbf{u}_a$ in accordance with Ref. 15, we obtain

$$\frac{d\Theta_a}{dt}=\frac{4(1-\alpha)}{9\pi}Gm_a^2m_b(2\pi m_am_b)^{1/2}\Lambda_{ab}\frac{\Theta_b-\Theta_a}{(m_a\Theta_b+m_b\Theta_a)^{3/2}}. \quad (30)$$

The characteristic relaxation time τ of the "temperatures" in a flat universe with nonrelativistic free particles is

$$\tau=\Delta_{ab}\Big/\left(\frac{\alpha}{m_a}+\frac{1-\alpha}{m_b}\right),\quad \Delta_{ab}=\tau_a\frac{1-\alpha}{m_b}=\tau_b\frac{\alpha}{m_a},$$
$$\tau_a=\frac{9\pi}{4(1-\alpha)}\frac{(m_a\Theta_b+m_b\Theta_a)^{3/2}}{Gm_a^2m_b(2\pi m_am_b)^{1/2}}\frac{1}{\Lambda_{ab}}. \quad (31)$$

Although (31) has been obtained as the time of establishment of equilibrium between two species of particle, it can be used in order of magnitude to estimate the relaxation time in a medium of particles of one species. For this, it is necessary to set $m_a=m_b=m$, $\Theta_a=\Theta_b=\Theta$, bearing in mind that $\Theta=\langle\frac{1}{2}mu^2\rangle$.

We make estimates for two different situations. If the neutrinos have mass, then on the transition from the relativistic to the nonrelativistic state the distribution function becomes a nonequilibrium function[16]:

$$f\sim\exp(-muc/\Theta), \quad (32)$$

and it relaxes to the nonrelativistic Maxwell distribution (27) because of collisions. We consider the idealized situation when perturbations of the density are absent and homogeneity is preserved.[1)] At the time $t=6\times10^{17}$ sec for $m_\nu=5\times10^{-32}$ g,[13] the mean velocity is $\langle v^2\rangle\approx6$ km/sec (Ref. 14) and $\Theta\approx m_\nu\langle v^2\rangle t^{4/3}\approx9\times10^3$. Then for the relaxation time we obtain the huge value $\tau_\nu\approx10^{90}$ sec, i.e., there is virtually no relaxation.

Obviously, with increasing mass of the objects their relaxation time decreases. Using the approximate relation $\Theta=m\langle v^2\rangle t^{4/3}$, we rewrite (31) in the form

$$\tau=9\pi^{1/2}\langle v^2\rangle^{3/2}t^2/2Gm\Lambda. \quad (33)$$

For clusters of galaxies with $m_G=10^{14}M_\odot\approx2\times10^{47}$ g,

$$\tau=3\cdot10^{17}\left(\frac{\langle v^2\rangle}{4\cdot10^{14}}\right)^{3/2}\left(\frac{t}{6\cdot10^{17}}\right)^2\left(\frac{5}{\Lambda_{GG}}\right)\left(\frac{2\cdot10^{47}}{m_G}\right)\text{ sec.} \quad (34)$$

Note that (34) is valid for objects corresponding to the maximal inhomogeneity scale in the universe under the condition that the diameter of the objects is much less than the distances between them. By the present epoch, the random velocities of clusters of galaxies in the expanding universe could have achieved a Maxwellian distribution.

[1)]In the real universe, there are inhomogeneities in the density and velocity distributions. This leads to the development of gravitational instability, and the real situation with massive neturinos differs from the idealized case considered here.

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