The inverse scattering formalism in the theory of photon (light) echo

S. M. Zakharov and E. M. Manykin

Moscow Engineering-Physics Institute
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The propagation of several ultrashort light pulses in a dense resonant medium is considered by the inverse scattering formalism with account taken of the response to the transmitted radiation. The scattering coefficients in the photon (light) echo problem are obtained in the case of excitation of the resonant medium by a sequence of two coherent light pulses. The asymptotic part of the solutions, which is connected with formation of solitons, is investigated. The conditions for the onset of solitons and the soliton parameters are determined as functions of the areas of the exciting light pulses, as well as of the time interval between them.

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1. INTRODUCTION

One of the significant achievements in the field of nonlinear equations and soliton theory was the development of the inverse scattering formalism, which made it possible to solve the Cauchy problem by means of nonlinear transformations.[1-3] This method was first performed by Gardner, Greene, Kruskal, and Miura and applied by them to the Korteweg–de Vries equation.[4] Zakarov and Shabat[5] next investigated successfully the self-focusing phenomenon.[5] Ablowitz *et al.* succeeded in establishing certain general limits of the applicability of the method to a large class of equations, including the modified Korteweg–de Vries equation, the simple and generalized sine-Gordon equation, the nonlinear Schrödinger equation, and others.[6] Finally, in 1974 Lamb[7] and Ablowitz and co-workers[8] were able to solve by a similar method the coupled Maxwell-Bloch system of equations that describes coherent propagation of light pulses in a resonant medium with inhomogeneous broadening of the energy levels and, in particular, the McCall and Hahn self-induced transparency phenomenon.[9,10]

A theory that describes coherent interaction of a single light pulse with a resonant medium has by now been well developed.[7-18] Although many theoretical results were initially obtained by simpler methods,[9,10] significant progress in this field became possible by the use of the inverse scattering formalism. First of all, it became possible to investigate analytically the soliton part of the solution of the Maxwell-Bloch equations, which consists of isolated 2π pulses, or of a coupled state of solitons in the form of 0π pulses.[7,14,16] In addition, it became possible to determine exactly how an initial light pulse with arbitrary temporal profile, breaks up at the boundary of a resonant medium into solitons and a certain additional "background effect".[8,13]

Similar problems arise when several light pulses pass through a resonant medium. Under conditions of coherent interaction of electromagnetic radiation with the resonant medium, when the characteristic durations of the incident-radiation pulses are much shorter than the times of irreversible relaxation of the polarization, photon (light) echo can be produced, wherein two light pulses that pass in succession through the medium produce an optical response at the instant of time $2\tau_i$, where τ_i is the interval between the exciting light pulses. The physics of this process has been well investigated in rarefied resonant medium, when the response of the medium to the external action can be neglected. In this case the photon-echo signal remains small compared with the exciting light pulses. Its intensity, however, increases in proportion to the square of a traversed distance and at a certain instant the "given-field" approximation used for rarefied media no longer holds. Strictly speaking, such a medium can no longer be regarded as rarefied but, on the contrary, as dense. In dense resonant media the coherent response in the form of a photon-echo signal becomes comparable in magnitude with the incident pulses themselves, and is capable of producing additional echo-type signals — this is the multiple echo phenomenon. The behavior of these pulses as they penetrate in the interior of the resonant medium is a complicated problem even for powerful numerical methods. This problem was considered by us earlier numerically.[19,20] The inverse scattering formalism provides in this case definite information without resorting to a complex computer calculations. It will be shown below that the asymptotic part of the solution determines the number of pulses (solitons) produced at the "exit" from a dense resonant medium when two pulses with definite amplitudes, durations, and time intervals are applied to its "input." The soliton part of the solution does not describe the dynamics of the transient processes (this is contained in the "background" part of the kernel of the integral equation), but even the number of emerging pulses is an important parameter for the use of this phenomenon.

In the present paper we obtain the asymptotic part of the solution; this part is connected with formation of solitons when two coherent light pulses pass through a resonant medium. We determine the parameters of the solitons and the conditions for their appearance as functions of the areas of the first and second exciting light pulses, as well as of the time interval between them.

2. BASIC EQUATIONS

We shall consider the interaction of coherent light pulses with resonant media in the quasiclassical ap-

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proximation. For slow field and polarization amplitudes we then easily obtain the equations

$$\partial\mathscr{E}/\partial x=\langle 2V_1V_2^*\rangle,$$
$$\frac{\partial V_1}{\partial\tau}+i\frac{\Delta\omega}{2}V_1=\frac{1}{2}\mathscr{E}V_2, \qquad (1)$$
$$\frac{\partial V_2}{\partial\tau}-i\frac{\Delta\omega}{2}V_2=-\frac{1}{2}\mathscr{E}^*V_1.$$

In the system (1), the variables x, τ, and the field amplitude $\mathscr{E}$ are expressed in the following units:

$$x=\frac{2\pi\omega N_0|d_{12}|^2x}{n\hbar c},\quad \tau=t-\frac{xn}{c},$$
$$\mathscr{E}(x,\tau)=d_{21}E(x,\tau)/i\hbar, \qquad (2)$$

where c is the speed of light in vacuum, n is a refractive index of nonresonant type, which takes into account also the dispersion of the matrix of the paramagnetic crystal, ω is the carrier frequency, d_{12} is the matrix element of the dipole-moment operator, and N_0 is the density of the resonant atoms in the medium. The angle brackets in the first equation of (1) denote averaging over the scatter of the energy levels within the limits of the inhomogeneously broadened line:

$$\langle\ldots\rangle=\int_{-\infty}^{+\infty}g(\Delta\omega)(\ldots)d(\Delta\omega)$$

[$g(\Delta\omega)$ is the distribution function over the transition frequencies $\Delta\omega=\omega_{21}-\omega$].

It is known[1-3] that the success of the inverse scattering formalism is due to the ability of finding a pair of operators $\hat{L}$ and $\hat{B}$ that realize the so-called Lax representation. The initial system (1) is described by a single equation equivalent to a certain operator evolution equation

$$i\frac{\partial\hat{L}}{\partial x}=[\hat{B},\hat{L}]. \qquad (3)$$

The determination of the Lax operators $\hat{L}$ and $\hat{B}$ in the general case is a rather complicated problem whose solution, incidentally, is facilitated by the existence of "pseudopotentials."[21] We present only the final form of the Lax pair as applied to the initial Maxwell-Bloch equation (1), first guessed by Ablowitz *et al.*[8]:

$$\hat{L}=i\begin{bmatrix}\dfrac{\partial}{\partial\tau} & -\dfrac{\mathscr{E}(x,\tau)}{2}\\ -\dfrac{\mathscr{E}^*(x,\tau)}{2} & -\dfrac{\partial}{\partial\tau}\end{bmatrix},$$
$$\hat{B}=\begin{bmatrix}A(\zeta,x,\tau) & B(\zeta,x,\tau)\\ B^*(\zeta^*,x,\tau) & -A(\zeta,x,\tau)\end{bmatrix}. \qquad (4)$$

The functions $A(\zeta,x,\tau)$, $B(\zeta,x,\tau)$ of complex argument ζ (the eigenvalue of the operator $\hat{L}$) are determined with the aid of the Hilbert transformation

$$A(\zeta,x,\tau)=-\frac{1}{4}\left\langle\frac{|V_2(\Delta\omega,x,\tau)|^2-|V_1(\Delta\omega,x,\tau)|^2}{(\zeta-\Delta\omega/2)}\right\rangle,$$
$$B(\zeta,x,\tau)=\frac{1}{4}\left\langle\frac{2V_1(\Delta\omega,x,\tau)V_2^*(\Delta\omega,x,\tau)}{(\zeta-\Delta\omega/2)}\right\rangle. \qquad (5)$$

It is possible to associate with the operator $\hat{L}$ a certain scattering problem, in which case the function $\mathscr{E}(x,\tau)$ will play the role of the "potential," and the coordinate x serves as a parameter. Since the spectrum of the eigenvalues ζ of the operator $\hat{L}$ it is independent of x, it is determined only by the limiting form of the function $\mathscr{E}(0,\tau)$. Solving the Cauchy problem in terms of the scattering data, one can find the evolution of the coefficients of the scattering with respect to x. Finally, the inverse problem can be solved, namely reconstruct the potential $\mathscr{E}(x,\tau)$ for arbitrary values of x. In this case, however, it is necessary to solve a Levitan-Gel'fand-Marchenko integral equation whose kernel contains information on the scattering coefficients. The easiest to investigate is the soliton part of the solution, since the kernel of the integral equation turns out to be factorized. At the same time, at large depth of penetration of the light into the resonant medium, it is precisely this asymptotic part of the solution which predominates, since the contribution of the background part of the kernel is exponentially small.[3] This corresponds physically to neglecting in the solution the transient processes that are significant only in a resonant-medium several absorption lengths in size.

3. DETERMINATION OF THE SCATTERING PARAMETERS IN THE PHOTON-ECHO PROBLEM

We consider in greater detail the eigenvalue and eigenfunction problem for the operator $\hat{L}$ under conditions when a sequence of two coherent light pulses of rectangular shape, with different amplitudes E_1 and E_2 and durations τ_{p1} and τ_{p2} are incident on the boundary $x=0$ of the resonant medium (Fig. 1).

Using the explicit form of the operator $\hat{L}$, we obtain

$$\frac{\partial V_1}{\partial\tau}+i\zeta V_1=\frac{1}{2}\mathscr{E}V_2,\quad \frac{\partial V_2}{\partial\tau}-i\zeta V_2=-\frac{1}{2}\mathscr{E}^*V_1. \qquad (6)$$

An analogous problem was formulated by Zahkarov and Shabat as applied to the nonlinear Schrödinger equation.[5] They have shown, in particular, that the operator $\hat{L}$ has both a discrete spectrum and a spectrum of continuous eigenvalues ζ. Since $\hat{L}$ is not Hermitian, the discrete eigenvalues correspond, generally speaking, to complex numbers ζ_m. These bound states determine the soliton part of the Marchenko integral equation. The continuous eigenvalue spectrum is obtained for real ζ. We note that at $\zeta=\Delta\omega/2$ Eqs. (6) go over into the last two equations of the system (1).

We stipulate that the function $\mathscr{E}(x,\tau)$ must satisfy for all x the condition

$$\int_{-\infty}^{\infty}|\mathscr{E}(x,\tau)|d\tau<\infty. \qquad (7)$$

Since it is necessary in this case that the field $\mathscr{E}(x,\tau)\to 0$ as $\tau\to\pm\infty$, the system (6) reduces to the equation for oscillators, and for real ζ it has two solutions of the type $\exp(\pm i\zeta\tau)$. We determine the linearly indepen-

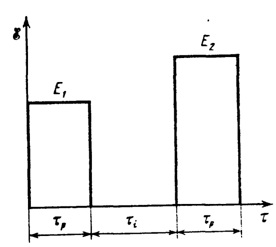

FIG. 1. Profiles of exciting light pulses entering the resonant medium.

dent solutions Φ and $\bar{\Phi}$ of Eqs. (6), whose asymptotic forms are

$$\Phi=\begin{bmatrix}1\\0\end{bmatrix}\exp(-i\zeta\tau),\quad \bar{\Phi}=\begin{bmatrix}0\\-1\end{bmatrix}\exp(i\zeta\tau);$$

$$\tau\to-\infty,\quad \operatorname{Im}\zeta=0. \tag{8}$$

The scattering parameters a and b are then defined for $\tau\to+\infty$ in the following manner:

$$\Phi=\begin{bmatrix}a(\zeta,x)\exp(-i\zeta\tau)\\ b(\zeta,x)\exp(i\zeta\tau)\end{bmatrix},\quad \bar{\Phi}=\begin{bmatrix}\bar{b}(\zeta,x)\exp(-i\zeta\tau)\\ -\bar{a}(\zeta,x)\exp(i\zeta\tau)\end{bmatrix}. \tag{9}$$

It follows from scattering theory that the points of the upper half-plane $\operatorname{Im}\zeta_m>0$, at which $a(\zeta_m)=0$, correspond to discrete eigenvalues of the operator $\hat{L}$.[3] To each such bound state, depending on whether the points ζ_m lie on the imaginary axis or have also a real part, there correspond either 2π pulses or 0π pulses of the Maxwell-Bloch system of equations.

We note that for rectangular exciting light pulses one can arrive at the analog of the one-dimensional Schrödinger equation with potential $U(\tau)=-|\mathscr{E}(\tau)|^2/4$ and energy ζ^2. In fact, differentiating one of the equations of the system (6) with respect to τ and neglecting the δ-function singularities on the fronts, we obtain the equations

$$-\frac{d^2V}{d\tau^2}-\frac{|\mathscr{E}(\tau)|^2}{4}V=\zeta^2V, \tag{10}$$

where $V=V_{1,2}$.

It is now easy to write down the solution of the system (6) during the time of action of the exciting light pulses and in the intervals between them:

$$V_i=V_i(\tau_0)\cos\lambda(\tau-\tau_0)+\frac{V_i'(\tau_0)}{\lambda}\sin\lambda(\tau-\tau_0),$$

$$\lambda^2=\zeta^2+\frac{|E|^2}{4}, \tag{11}$$

$$V_i=V_i(\tau_0)\exp[\mp i\zeta(\tau-\tau_0)],$$

where the minus and plus signs are chosen for V_1 and V_2, respectively.

It is clear that the scattering coefficients a and b can be written, after the termination of the action of the last optical pulse, in the form

$$a(\zeta)=V_1(\tau_0)\exp(i\zeta\tau_0),\quad b(\zeta)=V_2(\tau_0)\exp(-i\zeta\tau_0), \tag{12}$$

where $\tau_0=\tau_{p1}+\tau_i+\tau_{p2}$.

Solving the system (6) with account taken of relations (11) and (12), we obtain the scattering coefficient following the action of the first light pulse

$$a(\zeta)=\left(\cos\lambda\tau_p-i\frac{\zeta}{\lambda}\sin\lambda\tau_p\right)\exp(i\zeta\tau_p),$$

$$b(\zeta)=-\frac{E^*\sin\lambda\tau_p}{2\lambda}\exp(-i\zeta\tau_p). \tag{13}$$

We note that the solution (13) agree with the results of Kaup,[13] who considered the propagation of a single light pulse of "rectangular shape." Similarly, after the action of two optical pulses, we obtain the expressions

$$a(\zeta)=\exp[i\zeta(\tau_{p1}+\tau_{p2})]\left\{\prod_{k=1,2}H_k^-(\zeta)-\frac{E_1^*E_2}{4\lambda_1\lambda_2}\sin\lambda_1\tau_{p1}\sin\lambda_2\tau_{p2}\times\exp(2i\zeta\tau_i)\right\},$$

$$b(\zeta)=-\frac{1}{2}\exp[-i\zeta(\tau_{p1}+\tau_{p2})]\left\{\exp(-2i\zeta\tau_i)\frac{E_2^*}{\lambda_2}H_1^-(\zeta)\times\sin\lambda_2\tau_{p2}+\frac{E_1^*}{\lambda_1}H_2^+(\zeta)\sin\lambda_1\tau_{p1}\right\},\quad H_k^\pm(\zeta)=\cos\lambda_k\tau_{pk}\pm i\frac{\zeta}{\lambda_k}\sin\lambda_k\tau_{pk}. \tag{14}$$

Finally, putting $a(\zeta)=0$ in (14), we obtain an equation for the determination of the bound states in the direct scattering problem. We consider solutions lying on the imaginary axis of the half-plane $\operatorname{Im}\zeta>0$, i.e., $\zeta=i\eta$. These solutions correspond to solitons in the form of 2π pulses:

$$\mathscr{E}(x;\tau)=4\eta\,\mathrm{sech}\left[2\eta\left(\tau-\tau_0-x\int_{-\infty}^{\infty}\frac{g(\Delta\omega)d(\Delta\omega)}{4\eta^2+\Delta\omega^2}\right)\right]. \tag{15}$$

We obtain for them

$$\prod_{m=1,2}(F_m\cos F_m+\eta\tau_{pm}\sin F_m)=\exp(-2\eta\tau_i)\prod_{k=1,2}\frac{\Theta_k}{2}\sin F_k. \tag{16}$$

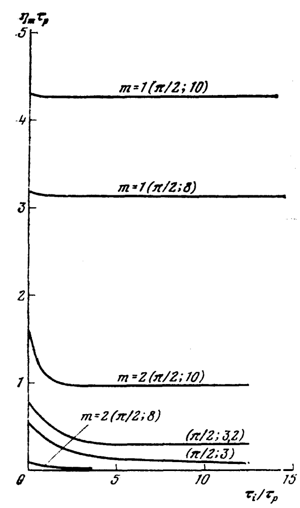

FIG. 2. Dependence of the coefficient η_m, which determines the energy of the bound state in the scattering problems and the soliton parameters, as a function of the interval between the exciting light pulses τ_i at $\Theta_1=\pi/2$ and at different values of Θ_2 [the number m labels the different branches of the solutions of Eq. (16), while the values of the areas of the exciting light pulses are indicated in the parentheses].

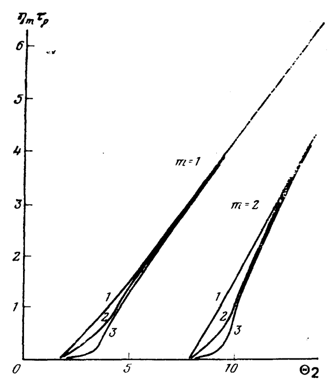

FIG. 3. Dependence of the coefficient η on the area of the second exciting pulse at $\Theta_1=\pi/2$. Curves 1, 2, and 3 pertain to different time intervals τ_i, given respectively by τ_i/τ_p = 0, 1, and 10.

In (16), Θ_k stands for the area of the k-th exciting pulse of light: $\Theta_k = E_k\tau_{pk}$ and $F_m^2 = \Theta_m^2/4 - \eta^2\tau_{pm}^2$.

Equation (15) describes the simplest single-soliton solution of the coupled system of equations (1), when Eq. (16) admits of the existence of one unique solution. In our case we construct N-soliton solutions at which the number of solitons is determined by the number of branches or roots of Eq. (16).[11,18]

4. DISCUSSION OF RESULTS

Using the results of scattering theory, as well as the analogy between the system (6) and the one-dimensional Schrödinger equation, we can assume that each bound state in the scattering problem corresponds to a certain discrete "energy level" $\varepsilon_m = -\eta_m^2$, which determines the parameters of the soliton (15).

In the approximation $\eta\tau_i \gg 1$ it is seen from (16) that the asymptotic form of the solution does not depend on the "interaction" of the incident pulses, and there are determined separately by the initial areas Θ_1 and Θ_2 of the initial light pulses. Equation (16) breaks up in this case into two, each of which takes the same form as in the case of scattering by a spherically symmetrical potential:

$$F_i \cos F_i + \eta\tau_p \sin F_i = 0, \quad i = 1, 2. \tag{17}$$

At $\tau_i = 0$ and $E_1 = E_2$, Eq. (16) reduces to the form

$$\cos\lambda(\tau_{p1}+\tau_{p2}) + \frac{\eta}{\lambda}\sin\lambda(\tau_{p1}+\tau_{p2}) = 0, \tag{18}$$

i.e., the scattering is determined by a potential well with effective width $\tau_{p1}+\tau_{p2}$. In the general case the right-hand side of (16) is responsible for the interaction of two rectangular potential wells, due to the overlap of the wave functions.

A numerical analysis of the solutions of the transcendental equation (16) confirms the noted features of the behavior of the roots. Figure 2 shows the dependence of the coefficient η_m, which determines the "energy" of the bound state in the scattering problem and the soliton

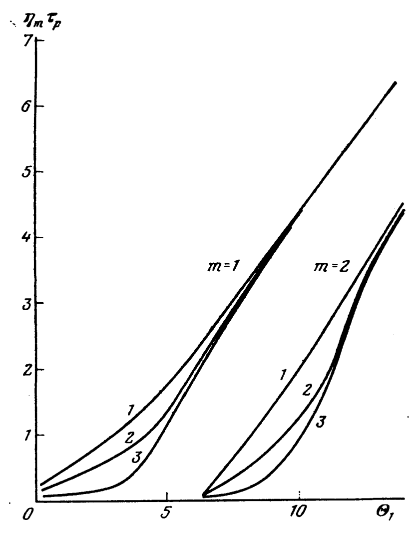

FIG. 4. The same as Fig. 3, as a function of the area of the first exciting light pulse at $\Theta_2 = \pi$.

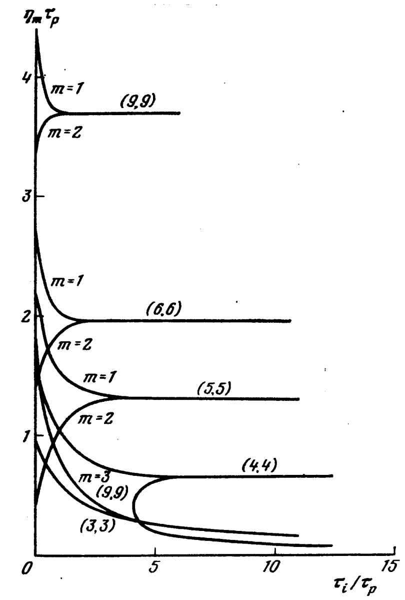

FIG. 5. The same as Fig. 2, as a function of the interval between the exciting pulses in the case of equal areas $\Theta_1 = \Theta_2$.

parameters, as a function of the quantity τ_i, i.e., of the interval between the exciting light pulses, for $\Theta_1 = \pi/2$ and for different values of Θ_2. In the region $\pi < \Theta_1 + \Theta_2 < 3\pi$ there is only one bound state. However, the $\eta(\tau_i)$ dependence turns out to be substantially different for $\Theta_2 \gtrless \pi$. Thus, at $\Theta_2 = 3$ the value of η tends asymptotically to zero as $\tau_i \to \infty$. At $\Theta_2 = 3.2$ the asymptotic value of η as $\tau_i \to \infty$ is already different from zero.

At large values of Θ_2, such that $\Theta_1 + \Theta_2 > 3\pi$, two bound states η_1 and η_2 are already produced. But the limiting values of η_1 and η_2 as $\tau_i \to \infty$ again depend on Θ_2. Thus, $\eta_2 \to 0$ as $\tau_i \to \infty$ and at $\Theta_2 = 8$ ($\Theta_2 < 3\pi$), but $\eta_2 \to \mathrm{const} \neq 0$ as $\tau_i \to \infty$ and $\Theta_2 = 10$ ($\Theta_2 > 3\pi$). Generally speaking, the presence of two closely lying potential wells facilitates the appearance of "shallow" energy levels in the wells (the second branch of η_2 for $\Theta_2 = 3$, 3.2, and 8, see Fig. 2). The asymptotic values of η_m with increasing distance between the potential wells, and the fact that the shallow energy levels coalesce with the continuous spectrum, are determined in this case by the value of Θ_2 and are in splendid agreement with the solution of Eq. (17) for a single light pulse.[13]

The threshold character of the onset of bound states with increasing Θ_2 at $\Theta_1 = \pi/2$ is illustrated in Fig. 3. It shows two branches of the solution of Eq. (16), with curves 1, 2, and 3 corresponding to different delay times $\tau_i/\tau_p = 0$, 1, and 10.

Similar features of the behavior of the energy levels for different Θ_1 and for $\Theta_2 = \pi$ are shown in Fig. 4. Figure 5 shows a plot of $\eta_m(\tau_i)$ for equal areas of the exciting pulses $\Theta_1 = \Theta_2$. It is seen from the figure that the function $\eta_m(\tau_i)$ does not decrease in all the branches. At $\tau_i = 0$ the number of roots and their values are determined by the solution of Eq. (16) at the duration $2\tau_p$.

At sufficiently large distances between the potential wells, the energy levels in them turn out to be degen-

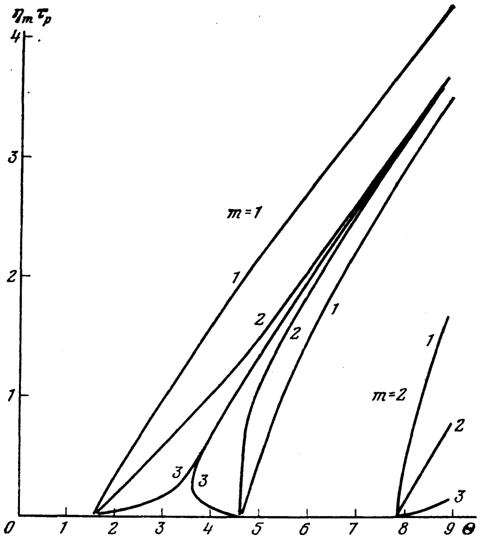

FIG. 6. The same Figs. 3 and 4 in the case of equal areas $\Theta_1 = \Theta_2$.

erate. The degeneracy can be easily attributed to differences in the symmetry of the wave function. When the wells come close together (the interval between the exciting light pulses decreases) the degeneracy is lifted in the region where the wave functions overlap, and the parameters of the solitons turn out to be different.

We note an interesting feature of the behavior of the roots. With increasing interval between the exciting light pulses, a new branch of the solutions appears at certain values $\Theta_1 = \Theta_2$ and τ_i (Figs. 5 and 6). This means that the number of solitons leaving a dense resonant medium is determined not only by the areas of the exciting light pulses Θ_1 and Θ_2, but also by the time interval τ_i between the pulses.

Thus, the number of branches of the equation that determines the bound states in the scattering problem, and with it also the number of solitons leaving the dense resonant medium, are determined by the total area $\Theta_1 + \Theta_2$ of the exciting light pulses, by the time delay τ_i between them, while the asymptotic form of the solutions as $\tau_i \to \infty$ depends strongly on each of the values Θ_1 and Θ_2 separately.

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