Emission by electrons and positrons in axial quasichanneling

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Electromagnetic radiation by high-energy particles moving in a crystal at sufficiently small angles to the crystallographic axes is investigated theoretically. The total energy radiation losses and their spectral and angular distributions are calculated as functions of the particle incidence angles relative to the axes. The theoretical calculations are compared with the available experimental results.

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1. INTRODUCTION

The radiation produced by relativistic charged particles as they channel through crystals is intensively investigated at present both theoretically and experimentally. An analysis of the main results obtained in this field, as well as a sufficiently complete list of the original papers, can be found in the review articles by Wedell[1] and Bazylev and Zhevago,[2] and in the proceedings of the First National Conference on Radiation of Charged Particles in Crystals.[3]

In particular, general equations for the calculation of the radiation spectrum at relatively soft frequencies in the case of axial particle channeling were obtained by Bazylev, Glebov, and Zhevago.[4] Actual calculations, however, could be carried out[4–6] only for electrons in the classical approximation and under the assumption that their trajectories in a plane perpendicular to an axis can be regarded as periodic. In addition, Kumakhov and Trikalinos[5] have considered only the dipole and near-dipole cases of emission by electrons, while more general results, when the emission can be essentially non-dipole, was considered in Ref. 4,[1)] as well as by Baĭer, Katkov, and Strakhovenko.[6] No analogous calculations were performed for positrons, owing to the relatively complicated character of their transverse motion, which is quite far from periodic.

Along with axial channeling of particles, there is also axial quasichanneling. In this case, the energy of the transverse motion of the particles exceeds their binding energy in the potential well produced by the axis. The transverse motion of the particles is now infinite, but the field of the axes bends strongly the trajectories of the particles in a plane perpendicular to the axes. As a result, the initially parallel particle beam becomes distributed in the crystal along generators of a cone, producing at the output a characteristic annular transverse-momentum distribution. Such a scattering of particles in a crystal ("doughnut" scattering) was observed by Uggerhoj *et al.*[7,8] in experiments with π^+ and π^- mesons up to particle entry angles, relative to the axis, greatly exceeding the critical angle for their axial channeling.

As shown in our preceding paper (Ref. 3, p. 39), the relative scattering of quasichanneled particles in a transverse plane should lead to intense electromagnetic radiation. For the same reason, in axial quasichanneling the transverse trajectories become aperiodic and therefore the radiation effects connected with the coherence of the action of the different axis becomes inessential. Radiation produced in axial quasichanneling has properties that differ substantially from the analogous properties in axial channeling of electrons.[4–6] The existing theory of coherent bremsstrahlung[9] likewise fails to describe this phenomenon, since no account is taken in this theory of the aforementioned particle scattering.

Investigations of radiation in axial quasichanneling of particles in a crystal seem to us sufficiently important for the understanding of the complete picture of radiation in a crystal and for a correct explanation of the available experimental data.[10–18] The point is that, in contrast to planar channeling of light particles, the fraction of particles in the case of stable axial channeling (hyperchanneling)[2)] is relatively small from the very outset (20–30%).

In the case of electrons the region is that relatively stable channeling sets in only when the particles have a sufficiently large orbital momentum relative to the axis.[20] To this end, the entry angle of the electrons relative to the axis should differ from zero,[3)] and as a result some of the particles acquire transverse energies larger than the depth of the potential well of the channel.

In axial channeling of positrons, the situation is somewhat different. The most favorable conditions for axial channeling of these particles corresponds to a zero angle of incidence of the particles on the axis. The fraction of the channeled (hyperchanneled) particles is then determined by the ratio of the area of the potential well to the total area of the transverse unit cell, and amounts to 10–40%.

Thus, besides the channeled particles there is always present an appreciable fraction of quasichanneled ones, and this must be taken into account in the analysis of the experimental radiation spectra. On the other hand if the incidence angle of the particles on the axis exceeds the critical channeling angle $\theta_L = (2U_w/E)^{1/2}$ (U_w is the depth of the potential well and E is the particle energy), all the beam particles are quasichanneled.

The main theoretical results obtained by us earlier

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for the study in axial quasichanneling are presented in concise form in the proceedings of the conference (see Ref. 3, p. 39). At the same time, Yakamura and Otsuki (see Ref. 3, p. 1), on the basis of the synchrotron formula for the spectrum, calculated by computer simulation the radiation from 56-MeV electrons moving at a small angle to the ⟨110⟩ axis of silicon, with account taken of the contribution of the quasichanneled particles. No account was taken in these calculations, however, of the coherence of the radiation from different sections of the particle trajectory within the confines of the unit cell of the crystal, an important factor at such energies.[4)] Qualitative estimates of the form of the radiation spectrum in the quasichanneling were obtained for some limiting cases by Shul'ga,[22] but he took into account only planar trajectories. A related problem is the subject of a paper by Beloshitskiĭ and Kumakhov,[23] who considered radiation from particles in the region of the transition from axial to planar channeling, when coherence is important in the radiation produced by scattering by different axes.

We present here the results of the theoretical calculation of the spectral and angular characteristics of the radiation produced by electrons and positrons in quasichanneling, and also of the integrated (over the frequencies and angles) losses of energy to radiation, as functions of the angle of incidence of the particles on the crystallographic axes. The developed theory is used to analyze the positron-emission spectra obtained in the experiments of Alguard *et al.*[11]

2. CLASSICAL EQUATIONS OF MOTION OF CHANNELED AND QUASICHANNELED PARTICLES

Let the angle θ_0 of the entry of the particles into the crystal, relative to the crystallographic axes, be close in order of magnitude to the critical Lindhard angle θ_L. It can then be assumed that the particles are acted upon by the continuous potential of the axes, which depends only on the coordinate in the transverse (relative to the axis) plane (see, e.g., Ref. 19).

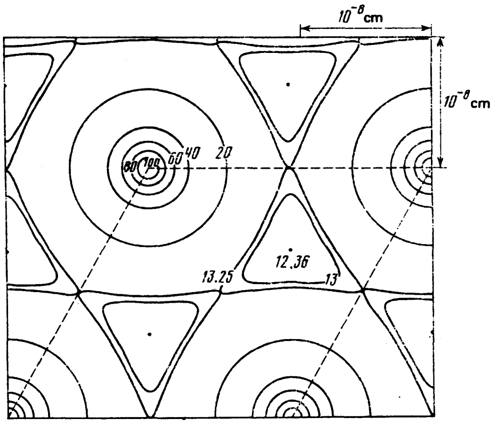

FIG. 1. Potential energy of positron in the field of the ⟨111⟩ axes of a silicon crystal at 297 K. The calculation was based on the Moliere model with allowance for the isotropic thermal oscillations of the crystal atoms [see Eq. (1)]. The unit cell is marked by dashed lines. The numbers on the curves correspond to the values of the potential energy in eV.

Figure 1 shows the equal-potential-energy levels of the particles in the field of the ⟨111⟩ axes of silicon at a temperature $T = 297$ K. The calculation was based on the Moliere approximation for the potential of an individual atom and with allowance for the isotropic thermal oscillations, just as was done by Appleton *et al.*[23] We used, however, a simpler analytic formula[5)]

$$U(\rho)=\mp\frac{e_0^2 Z}{d_s}\sum_{i=1}^{3}\alpha_i\exp(q_i^2)\int_0^1\exp\left(-\frac{\rho^2 t}{u^2}-\frac{q_i^2}{t}\right)\frac{dt}{t}. \qquad (1)$$

In this formula Ze_0 means the charge of the atomic nucleus, d_s is the distance between neighboring atoms, u is the square root of the mean squared amplitude of the thermal vibrations, $\alpha_i = \{0.1; 0.55; 0.35\}$ and $\beta_i = \{6.0; 1.2; 0.3\}$ are the Moliere constants, $q_i = \beta_i u/2a_{TF}$, where $a_{TF} = 0.8853 a_0 Z^{-1/2}$, and $a_0 = 0.529\cdot 10^{-8}$ cm is the Bohr radius. The upper sign in (1) and in all the expressions that follow pertains to the electrons, and the lower to the positrons.

It can be shown by analytic transformations of (1), as well as by numerical computer calculations that expression (1) is fully equivalent to the more cumbersome expression (23) of Ref. 19.

As seen from Fig. 1, in the case considered the electrons move for the greater part of the time in regions in which they are acted upon mainly by the potential of one of the axes, and the influence of the neighboring axis can be neglected. This approximation is accurate enough both for channeled electrons (whose transverse motion around the axis is restricted to approximately the distance between the axes), and for electrons with transverse energy larger than the potential barrier between the neighboring channels (these electrons are scattered in succession by different axes).

As for the positrons, the one-chain approximation is patently violated for them under channeling conditions, when a transverse-motion potential well is produced between axes precisely by the action of several neighboring axes (see Fig. 1). At the same time, for quasichanneled positrons the single-chain approximation is applicable to the same degree as for electrons.

In the approximation employed, the potential $U(\rho)$ acting on the particle depends only on the distance ρ to this axis. The integrals of motion are[6)] the total particle energy E:

$$E=U(\rho)+(1-v_\parallel^2-\dot\rho^2)^{-1/2}, \qquad (2)$$

the longitudinal momentum component $p_\parallel$ of the particle:

$$p_\parallel=v_\parallel(1-v_\parallel^2-\dot\rho^2)^{-1/2}, \qquad (3)$$

as well as the projection M of the angular momentum relative to the axis:

$$M=\rho^2\dot\varphi(1-v_\parallel^2-\dot\rho^2)^{-1/2}. \qquad (4)$$

In (2)–(4), $v_\parallel = v_\parallel(t)$ denotes the longitudinal velocity, $\dot\rho(t)$ is the transverse velocity of the particle, ρ is the radius vector of the particle in the transverse plane, and $\dot\varphi$ is the angular velocity of the transverse motion. All these quantities depend on the time. With the aid of (2)–(4) we obtain for the radial component $\dot\rho$ of the particle trajectory the equation

$$\dot{\rho}^2 = 1 - \frac{E_\parallel^2 + M^2\rho^{-2}}{[E-U(\rho)]^2}, \qquad (5)$$

where $E_\parallel \equiv (1+p_\parallel^2)^{1/2}$ is the longitudinal energy of the particle. The transverse component of the trajectory $\rho(\varphi)$ is given by the equation

$$\left(\frac{d\rho}{d\varphi}\right)^2 = \frac{\rho^4}{M^2}\left\{[E-U(\rho)]^2 - E_\parallel^2 - \frac{M^2}{\rho^2}\right\}, \qquad (6)$$

which follows from relations (2)–(4).

Equations (5) and (6) admit of simplifications connected with the smallness of the potential energy $U(\rho)$ compared with the total particle energy. We designate the transverse energy by $\varepsilon = E - E_\parallel$. Then, neglecting in the right-hand sides of (5) and (6) the terms that are quadratic in $U(\rho)$ and ε, we arrive at the equations of motion in the transverse plane, which coincide in form with Newton's nonrelativistic equations. However, the role of the particle mass in these equations is played by the relativistic mass $E_\parallel$:

$$\dot{\rho}^2 = E^{-2}\{2E[U(\rho)-\varepsilon] - M^2\rho^{-2}\},$$
$$\left(\frac{d\rho}{d\varphi}\right)^2 = \rho^4 M^{-2}\{2E[U(\rho)-\varepsilon] - M^2\rho^{-2}\}. \qquad (7)$$

In the region of distances ρ larger than the radius u of the thermal vibrations of the axis atoms, but smaller than half the distance to the nearest axis $d/2$, the calculated potential can be represented with good accuracy in the form

$$U(\rho) = \mp\alpha/\rho, \qquad (8)$$

where α is a constant that depends on the material of the crystal and on the Miller indices of the axis (see Table I). This form of the potential in the indicated range of distances from the axis is not accidental. It follows also from calculations based on the simpler Nielsen model [see, e.g., Eq. (12) of Ref. 24] for the atom potential $U(r) = Ze_0^2 a_{TF}/1.7706 \cdot r^2$ (which is valid at $r > 2a_{TF}$) and without allowance for the thermal vibrations. The constant α is then expressed in terms of the nuclear charge of the atom Ze_0, the radius of the atom a_{TF}, and the average distance d_s between the atoms of the axis:

$$\alpha = 1.774\frac{a_{TF}Ze_0^2}{d_s}. \qquad (8')$$

More accurate values of α (see Table I) are obtained by fitting a relation similar to (8) to a more exact form of the potential. These values can differ somewhat from those calculated from Eq. (8′). Equation (8′) can be useful for an estimate of the relations between the obtained properties of the radiation in channeling (or quasichanneling) and radiation in an amorphous medium.

TABLE I.

Crystal (axis)	$\alpha \cdot 10^8$, eV · cm	U_0, eV	$d \cdot 10^8$, cm	$\theta_L\gamma^{1/2}$, mrad	I_1, cm^{-1}	$\hbar\omega_1$, eV	L, eV/cm	E_2, GeV
Diamond ⟨110⟩	9.0	100	1.69	19.8	0.23	1735	40.4	60
Si ⟨110⟩	11.5	112	2.58	20.9	0.13	1609	21.8	62
Si ⟨111⟩	12.4	105	2.33	20.3	0.13	1769	23.5	58
Ge ⟨110⟩	20.0	200	2.68	28.0	0.28	2208	64.4	34
Ge ⟨111⟩	15.0	185	2.43	26.9	0.25	2620	66.9	30
W ⟨111⟩	55.0	936	2.71	60.5	1.65	8130	1380	4

In the region $\rho \lesssim u$, the potential of the axis is close to a parabola, but its gradient is relatively small, so that in first approximation at $\rho \lesssim u$ the potential can be regarded as constant $U(\rho) = U_0$, where $U_0 \approx \alpha/u$ is the depth of the potential well (the height of the potential peak in the case of positrons).

Solution of the equations of motion (7) entails no difficulty in this case, in view of the formal similarity of the problem to the Kepler problem. At $\varepsilon < 0$ the electrons move in a transverse plane along elliptic orbits. At $\varepsilon > 0$ the transverse motion of the electron or positron near the axis follows hyperbolic trajectories.

For the model potential (8), the solutions of the more exact equations of motion (5) and (6) can also be represented in analytic form. In particular, the transverse component of the channeling trajectory of an electron ($\varepsilon < 0$), when account is taken of terms quadratic in the potential, is given by

$$\varphi(\rho) = \int_{\rho_{min}}^{\rho} \frac{d\rho}{\rho^2(-A+2B\rho^{-1}-C\rho^{-2})^{1/2}} - \pi C^{-1/2}, \qquad (9)$$
$$A = 2|\varepsilon|EM^{-2}, \quad B = \alpha E M^{-2}, \quad C = 1-\alpha^2 M^{-2},$$

where $\rho_{min} = (B/A) - [(B/A)^2 - (C/A)]^{1/2}$ is the perihelion of the orbit.

Integrating in the right-hand side of (9) we obtain

$$\rho = \frac{P}{1-e\cos(C^{1/2}\varphi)},$$
$$P = \frac{M^2}{\alpha E}\left(1-\frac{\alpha^2}{M^2}\right), \quad e = \left(1 - \frac{2|\varepsilon|M^2}{\alpha^2 E} + \frac{2|\varepsilon|}{E}\right)^{1/2}. \qquad (10)$$

The elliptic trajectories are obtained from (10) by neglecting the small terms $|\varepsilon|/E$ and $\alpha^2/M^2 \lesssim |\varepsilon|/E$. Otherwise the character of the trajectories is somewhat altered. Precession of the ellipse sets in, with a frequency

$$\Omega_{p_e} \approx \frac{\pi\alpha^2}{M^2}\omega_0,$$

where $\omega_0 = (2|\varepsilon|)^{3/2}E^{-1/2}\alpha^{-1}$ is the frequency of the electrons on the elliptic orbit. Corrections of the order of α^2/M^2 arise also in the values of the perihelion and aphelion of the orbit. As expected from the general analysis, the effects connected with allowance for the terms quadratic in the potential are in this case of the order of $|\varepsilon|/E$. This quantity turns out to be in practice so small ($< 10^{-4}$) that the indicated effects do not play a noticeable role. It suffices to state that the calculation of the potential $U(\rho)$, as well as its approximation by the model relation (8), are much less accurate, so that allowance for these effects is meaningless. Similar conclusions follow also from an analysis of the trajectories of the above-barrier electrons and positrons. Thus, in contrast to the usual relativistic Kepler problem (see, e.g., Ref. 25), where the role of the terms quadratic in the potential $U(\rho)$ increases with increasing energy, in the axial-channeling problem the tendency is directly opposite.

3. ENERGY LOST BY PARTICLE TO RADIATION

The simplest to solve is the problem of the energy losses, integrated over the frequencies and over the radiation angles, in the continuous potential of the axis.

The integrated losses over the entire time of interaction of the particle with the axis can be represented in the form (see, e.g., Ref. 26, § 73) in the form

$$\Delta E=\frac{2e_0^2}{3}\int_{-\infty}^{\infty}|\nabla U(\rho)|^2\frac{1-\dot\rho^2}{1-\dot\rho^2-v_\parallel^2}dt, \tag{11}$$

where $\nabla U(\rho)/e_0$ is the intensity of the electric field of the axes. With the aid of the equations of motion (2) and (3) we obtain

$$1-\dot\rho^2-v_\parallel^2=[E-U(\rho)]^{-2},\quad 1-\dot\rho^2=E_\parallel^2[E-U(\rho)]^{-2}. \tag{12}$$

Thus, expression (11) for the energy losses takes the form

$$\Delta E=\frac{2e_0^2}{3}E_\parallel^2\int_{-\infty}^{\infty}|\nabla U(\rho(t))|^2\,dt. \tag{13}$$

The energy loss per unit time (the integrated radiation intensity) by quasichanneled particles is obtained by dividing (13) by the time of flight of the particle through the region of its interaction with the crystal axis. For channeled particles that execute finite transverse motion in the range from $\rho_{\min}$ to $\rho_{\max}$, it is reasonable to introduce the energy lost per unit time, averaged over the period T_ρ of the radial vibrations:

$$\frac{dE}{dt}=\frac{2e_0^2}{3}\frac{E_\parallel^2}{T_\rho}\int_0^{T_\rho}|\nabla U(\rho(t))|^2\,dt. \tag{14}$$

Accurate to corrections of the order of $|\varepsilon|/E$, the longitudinal energy in relations (13) and (14) can be replaced by the total energy, and the particle trajectory $\rho(t)$ can be calculated using Eq. (7).

Thus, the energy lost by an electron or positron to radiation in the continuous potential of the crystal axis turns out to be proportional to the square of the particle energy and to the square of the gradient of the potential that acts in the effective region of particle motion. These conclusions are valid provided that classical electrodynamics is applicable. It must therefore be assumed that the energy loss is due mainly to emission of sufficiently soft photons with energies $\hbar\omega\ll E$.

To avoid from the very beginning the difficulties connected with the singularity of the potential (8) at $\rho = 0$, we use in the calculation of the radiation losses a more correct form of the potential:

$$U(\rho)=\begin{cases}\mp\alpha/\rho & u\leqslant\rho\leqslant d/2\\ \mp U_0 & \rho\geqslant u\end{cases}. \tag{15}$$

The channeled-electron trajectories in a plane perpendicular to the axis are in a field of the form (8) the ellipses (see Ref. 27, § 15):

$$x(t)=a(\cos\xi-e),\quad y(t)=a(1-e^2)^{1/2}\sin\xi,\quad t=a(E/2|\varepsilon|)^{1/2}(\xi-e\sin\xi), \tag{16}$$

where ξ is a parameter that runs through values from $-\infty$ to ∞, $\varepsilon<0$ is the transverse energy of the electron, E is its total energy, $a=\alpha/2|\varepsilon|$ is the minor semiaxis of the ellipse, $e=(1-2|\varepsilon|M^2/E\alpha^2)^{1/2}\leqslant 1$ is the eccentricity of the ellipse, and M is the orbital momentum of the particle about the axis. This form of the trajectory is retained also in a field of the type (15) provided that the perihelion of the orbit $\rho_{\min}=a(1-e)$ exceeds the radius of the thermal vibrations. In the opposite case ($\rho_{\min}<u$) part of the particle orbit inside the radius of the thermal vibrations is transformed into a segment tangent to the ellipse at the point where the ellipse crosses the circle $\rho=u$. It is convenient to calculate the energy radiation loss (15) by changing from the variable t to the particle rotation angle about the axis, $dt=E\rho^2M^{-1}d\varphi$, and using at the same time the trajectory equations in the form (10) with allowance for the inequality $|\varepsilon|/E\ll 1$. The results of the calculations for the channeled electrons of the form

$$\frac{dE_{ch}^-}{dl}=\frac{8L\gamma^2}{\pi}D^2\frac{x^4}{(1-e^2)^{5/2}}\left[\left(1+\frac{e^2}{2}\right)\varphi_1+\left(\frac{e^2\cos\varphi_1}{2}-2e\right)\sin\varphi_1\right]$$
$$\times\left[\frac{e(1-e^2)^{1/2}\sin\varphi_1}{1-2e\cos\varphi_1+e^2}+\operatorname{arctg}\left(\left(\frac{1+e}{1-e}\right)^{1/2}\operatorname{tg}\frac{\varphi_1}{2}\right)\right]^{-1}\eta(e^2-2|x|+1),$$
$$\gamma=E/mc^2,\quad D=dU_0/\alpha,\quad x=\varepsilon/U_0, \tag{17}$$
$$L=\frac{2\pi e_0^2U_0^2}{3d^2(mc^2)^2},\quad \varphi_1=\arccos\left(\max\left\{\frac{2|x|-1+e^2}{2e|x|};-1\right\}\right),$$

where e_0 is the electron charge, $\eta(x)$ is the Heaviside step function: $\eta(x)=0$ at $x<0$ and $\eta(x)=1$ at $x\geqslant 0$.

The trajectories of the quasichanneled electrons and positrons in the region $\rho\geqslant u$ are hyperbolas:

$$x(t)=a(e\mp\operatorname{ch}\xi),\quad t=a(E/2\varepsilon)^{1/2}(e\operatorname{sh}\xi\mp\xi),$$
$$y(t)=a(e^2-1)^{1/2}\operatorname{sh}\xi,\quad a=\alpha/2\varepsilon \tag{18}$$

with eccentricity $e=(1+2\varepsilon M^2/\alpha^2E)^{1/2}$. Inside the region of the thermal vibrations ($\rho\leqslant u$) the trajectories are straight lines. We consider only particles with transverse energy ε greatly exceeding the potential energy on the boundary of the region $2\alpha/d$ of the interaction of the particle with the axis. This corresponds to the one-chain approximation used above. On the other hand, the contribution of particles having a transverse energy that does not satisfy this condition should be relatively small because of the small gradient of the potential on the periphery of the region where the particle interacts with the axis. If $\varepsilon\gg 2\alpha/d$, the time of flight of the particle through the region of interaction with the axis is determined by the formula $\tau\approx d/(2\varepsilon/E)^{1/2}$. Then the energy lost by quasichanneled particles per unit path in the crystal takes the form

$$dE_q^\mp/dl=L\gamma^2[\Phi^\mp(x,e,\varphi_0^\mp)-\Phi^\mp(x,e,\varphi_2^\mp)],$$
$$\Phi^\mp(x,e,\varphi)=\frac{16x^3d}{\pi u(e^2-1)^{5/2}}\left[\left(1+\frac{e^2}{2}\right)\varphi+\left(\frac{e^2}{2}\cos\varphi\pm 2e\right)\sin\varphi\right], \tag{19}$$
$$\varphi_0^\mp=\arccos(\mp 1/e),\quad \cos\varphi_2^\mp=\min\{1,(e^2-1\mp 2x)/(2ex)\}.$$

In the limit of large transverse energies compared with the depth of the well (the height of the peak when dealing with positrons), the energy loss (19) tends to the value $L\gamma^2$. This result is obtained also in the following manner. Since $\varepsilon\gg U_0$, the trajectories can be regarded in this limit as straight line (see Problem 1, § 73 in Ref. 26), so that the distance from the particle to the axis is determined by the equation $\rho^2=b^2+(2\varepsilon/E)t^2$, where $(2\varepsilon/E)^{1/2}$ is the transverse velocity of the particle and b is the impact parameter. The asymptotic value of the energy loss averaged over all the impact parameters from u to $d/2$ takes in this case the form

$$\frac{dE_\infty}{dt}=\frac{2e_0^2E^2\alpha^2}{3}\frac{1}{\tau d/2}\iint_S\frac{dt\,db}{(b^2+t^2\,2\varepsilon/E)^2},$$

where the integration is carried out in the plane of the variables t and b over an area S bounded by two concentric circles with radii u and $d/2$. The integral can be easily calculated in cylindrical coordinates, and the result takes the form $dE_\infty/dt = L\gamma^2$.

Expressions (17) and (19) obtained for the energy lost to radiation must be averaged over all possible particle trajectories in the crystal, i.e., over the eccentricities of the orbits and the transverse energies. We shall assume for simplicity that the crystal is thin enough to be able to neglect incoherent scattering processes that lead to nonconservation of the transverse energy. Then, as is well known, the transverse energy of the particles is determined by the relation $\varepsilon = E\theta_0^2/2 + U(\rho_0)$, where ρ_0 is the coordinate of the point of entry of the particle into the crystal and θ_0 is the angle of entry relative to the axis. Assuming ρ_0^2 to be equally probable in the range from 0 to $d/2$, and using a model potential of the form (15), we obtain the following probability distribution for the transverse energies referred to the depth of the well:

$$\frac{dP^{\mp}}{dx} = \frac{8}{D^2-4}\frac{1}{|(\theta_0/\theta_L)^2 - x|^3},$$

$$x^-_{max} = (\theta_0/\theta_L)^2 - 2D^{-1}, \quad x^-_{min} = (\theta_0/\theta_L)^2 - 1, \tag{20}$$

$$x^+_{max} = (\theta_0/\theta_L)^2 + 1, \quad x^+_{min} = (\theta_0/\theta_L)^2 + 2D^{-1},$$

where $\theta_L = (2U_0/E)^{1/2}$ is the critical angle, whose numerical values are listed in Table I. Since the impact parameters of the particles in the transverse plane are equally probable, the averaging over the orbital momenta reduces in this case to integration over the eccentricities of the orbit in the range from $e^-_{min} = 1 - 2\theta_0^2(\theta_0^2 + |x|\theta_L^2)^{-1}$ to $e^-_{max} = \min\{1 - 2|x|; D|x| - 1\}$ (for channeled electrons), or from unity to $e^{\mp}_{max} = Dx \pm 1$ (for quasichanneled particles).

The energy lost to radiation, averaged in this manner, is shown in Fig. 2 as a function of the ratio of the entry angle to the critical angle. The quantity L, which characterizes the energy loss per unit path, is listed in Table I for different axes and crystals.

The energy lost by the electrons increases with decreasing entry angle from θ_L to 0, this being due to the increase of the fraction of electrons that become axially channeled. For positrons the picture is reversed. The reason is that when the positron entry angle θ_0 decreases from θ_L to zero the peak in the positron transverse-energy distribution shifts towards smaller ε [see (20)].

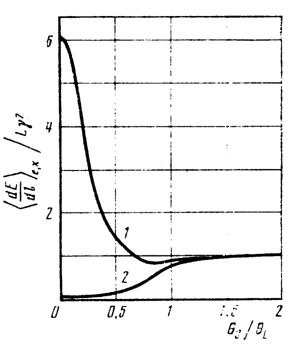

FIG. 2. Average radiation energy $\langle dE^+/dl\rangle_{e,x}$ lost by an electron (curve 1) and positron (curve 2) per unit path in a silicon crystal, as a function of the ratio θ_0/θ_L of the incidence angle of the particles on the axis to the Lindhard angle.

Positrons with such values of ε are prevented by the repulsion from the axis from penetrating into the region of a relatively large field gradient, and this leads to a decrease of the radiation energy losses.

4. SPECTRAL DISTRIBUTION OF THE RADIATION FROM QUASICHANNELED PARTICLES

In the classical-electrodynamics approximation, the spectral-angular distribution of the radiation energy during the entire time of particle interaction with the field is given by (Ref. 26, § 66)

$$\frac{d^2W}{d\omega\, do} = \left(\frac{e_0\omega}{2\pi}\right)^2 |[\mathbf{n}\times\mathbf{j}(\mathbf{k},\omega)]|^2, \tag{21}$$

$$\mathbf{j}(\mathbf{k},\omega) = \int_{-\infty}^{\infty} \mathbf{v}(t)\exp[i\mathbf{k}\mathbf{r}(t) - i\omega t]dt, \tag{21'}$$

where $\mathbf{r}(t)$ and $\mathbf{v}(t)$ are respectively the coordinate and velocity of the particle at the instant of time t; ω and $\mathbf{k}$ are the frequency and wave vector of the radiated wave, do is the differential of the solid angle and e_0 is the particle charge.

In the case of motion of quasichanneled electrons and positrons in an axis field of the type (8), the dependence of the transverse coordinates on the time is given by relations (18).

The longitudinal velocity of the particles can be expressed in terms of the square of the transverse velocity and the total particle energy E, using the approximate equation

$$\dot z(t) \approx 1 - \frac{1}{2}[E^{-2} + \dot\rho^2];$$
$$\dot\rho^2 = \frac{2\varepsilon}{E}(e\,\mathrm{ch}\,\xi \pm 1). \tag{22}$$

This equation is obtained from the condition (3) for the conservation of the longitudinal momentum in the limit of ultrarelativistic energies $E \gg 1$ and with account taken of the weaker condition $E \gg \varepsilon$.

Integration of (22) yields the following dependence of the longitudinal coordinate on the time:

$$z(t) = a\left(\frac{E}{2\varepsilon}\right)^{1/2}\left[\left(1 - \frac{1}{2E^2}\right)(e\,\mathrm{sh}\,\xi \mp \xi) - \frac{\varepsilon}{E}(e\,\mathrm{sh}\,\xi \pm \xi)\right]. \tag{23}$$

When expressions (18), (22), and (23) are substituted in the general equations (21) for the spectral-angular distribution of the radiation energy, one can neglect in the pre-exponential factor the difference between the longitudinal particle velocity and the speed of light. At the same time, when calculating the phase $ik_z z(t)$ account must be taken of the small term $1/2E^2$ in the expression for the longitudinal coordinate, a term resulting from the difference between $v_\parallel$ and the velocity of light, as well as the term proportional to ε/E, which is the consequence of the influence of the axis field on the longitudinal velocity of the particle [see (22)]. Allowance for these small terms becomes essential because of the cancellation of the principal terms in the phase of the exponential, with allowance for the second term, proportional to ε/E, needed only at sufficiently high particle energies, when the inequality $\varepsilon/E \gtrsim 1/$

$(2E^2)$ is satisfied. The resultant effects are considered in detail in Ref. 3 in the analysis of the emission spectra at ultrahigh particle energies.

Calculations of the Fourier components of the particle current (21′) lead to the result

$$j_x=2a\exp\left(\pm\frac{\pi\nu}{2}\right)\left[-\frac{\nu}{\zeta}K_{i\nu}(\zeta)\operatorname{sh}\xi_0+iK_{i\nu}'(\zeta)\operatorname{ch}\xi_0\right],$$
$$j_y=2a(e^2-1)^{1/2}\exp\left(\pm\frac{\pi\nu}{2}\right)\left[\pm\frac{\nu}{\zeta}K_{i\nu}(\zeta)\operatorname{ch}\xi_0+iK_{i\nu}'(\zeta)\operatorname{sh}\xi_0\right], \quad (24)$$
$$j_z=\frac{2a}{\theta}\exp\left(\pm\frac{\pi\nu}{2}\right)\left[\pm\left(\frac{\nu e}{\zeta}\operatorname{ch}\xi_0-1\right)K_{i\nu}(\zeta)+ieK_{i\nu}'(\zeta)\operatorname{sh}\xi_0\right].$$

Here

$$\theta_1=(2\varepsilon/E)^{1/2},\quad \nu=(\omega a/2\theta_1)(\theta^2-\theta_1^2+E^{-2}),$$
$$\zeta=(\delta^2-\eta^2)^{1/2},\quad \delta=\omega a\left[-\frac{e}{2\theta_1}(\theta^2+\theta_1^2+E^{-2})+\theta(e^2-1)^{1/2}\sin\varphi\right],$$
$$\eta=\mp\omega a\theta\cos\varphi,\quad \operatorname{sh}\xi_0=\eta/\zeta,$$

$K_{i\nu}(\zeta)$ and $K'_{i\nu}(\zeta)$ are the Macdonald function and its derivative. It was assumed in the calculation that the particles are ultrarelativistic and that the radiation angles θ with the crystallographic axis are small: $\theta\ll 1$. The azimuthal radiation angle φ is measured from the symmetry axis of the transverse trajectory (hyperbola). Inessential phase factors that are common to all three current components have also been left out of (24).

After substituting the Fourier components in (21) and carrying out additional algebraic transformations, the results can be represented in the form

$$\frac{d^2W_q}{d\omega\, do}=\left(\frac{e_0\omega a}{\pi}\right)^2\exp(\pm\nu\pi)\left\{k_1K_{i\nu}^2(\zeta)+k_2\left[\frac{d}{d\zeta}K_{i\nu}(\zeta)\right]^2\right\},$$
$$k_1=1-\frac{\nu^2}{\zeta^2}-\frac{1}{2\varepsilon E}\left(1-\frac{e\nu\delta}{\zeta^2}\right)^2,\quad k_2=1-\frac{1}{2\varepsilon E}\left(\frac{e\eta}{\zeta}\right)^2. \quad (25)$$

The general expression obtained from the spectral-angular distribution of the radiation energy in scattering of a particle by the continuous potential of the axis admits of substantial simplification in a number of limiting cases.

5. DIPOLE RADIATION

When the particle travels at the impact distance b from the axis, it acquires a transverse momentum $\Delta p_\perp \sim(\alpha/b^2)\tau$, where α/b^2 is the average force acting on the particle, $\tau\sim b/v_\perp$ is the effective time of action of the potential, and $v_\perp\sim[2(\varepsilon-U(b))/E]^{1/2}$ is the transverse velocity of the particle. The particle deflection angle $\theta_{\rm def}$ is proportional to the ratio of $\Delta p_\perp$ to the longitudinal momentum of the particle $p_\parallel\approx E$. We thus obtain the relation

$$\theta_{def}\sim\frac{\alpha}{b}\frac{1}{[2E(\varepsilon-U(b))]^{1/2}}.$$

The maximum deflection angle of the electron corresponds to impact parameters $b\approx u$, therefore, recognizing that $U_0=\alpha/u$, we obtain

$$\theta_{def}^{(max)}\sim U_0/[2E(\varepsilon+U_0)]^{1/2}.$$

Let the maximum angle of particle deflection by the field be much less than the effective radiation angle $\theta_{\rm rad}\sim 1/E$, i.e., assume the following inequality (in the case of electrons)

$$\frac{U_0E}{2|1+\varepsilon/U_0|^{1/2}}\ll 1. \quad (26)$$

Let also the transverse energies of the particle not be too large: $\varepsilon\sim U_0$. Then the average angle at which the particle moves relative to the crystal axis is also small compared with the effective radiation angle.

In this case the following approximations are valid:

$$k_1\approx(1-e^{-2})\left[1-(\theta E)^2\left(\frac{2\sin\varphi}{(\theta E)^2+1}\right)^2\right],$$
$$k_2\approx 1-(\theta E)^2\left(\frac{2\cos\varphi}{(\theta E)^2+1}\right)^2, \quad (27)$$
$$\nu\approx\frac{\omega a}{2\theta_1}(\theta^2+E^{-2}),\quad \zeta\approx e\nu.$$

As a result, the general expression (25) coincides with the analogous expression (18) of our preceding paper (Ref. 3, p. 39), obtained in the dipole approximation, for all velocities and angles θ, if we assume in the latter equation that the radiation angles are small, $\theta\ll 1$, and the particle velocity is relativistic, $v_z\approx 1-1/(2E^2)$.

We integrate the spectral-angular distribution (25) over the radiation angle. Taking the approximations (27) into account, the integration with respect to the azimuthal angle φ is elementary. Integration with respect to the polar angle can be carried out only numerically. We divide (25) by the time of interaction of the particle with an individual axis and average next the result over all the impact parameters of the collision (over the eccentricities of the orbits). After integrating with respect to the aximuthal angle, the spectral-angular distribution of the energy radiated by the particle per unit path takes the form

$$\left\langle\frac{d^3W_q^{\mp}}{dl\,d(\hbar\omega)\,d(\theta\gamma)^2}\right\rangle_e=I_1\gamma^{1/2}x^{1/2}G^{\mp}(\Omega,\theta\gamma;x),$$

where

$$I_1=\frac{2^{7/2}e_0^2}{\pi\hbar c}\frac{U_0^{1/2}\alpha}{d^2(mc^2)^{3/2}},\quad \Omega=\frac{\omega}{\omega_1(\gamma x)^{3/2}},$$
$$\omega_1=\frac{2^{5/2}U_0^{3/2}}{m^{1/2}\alpha},\quad x=\varepsilon/U_0,\quad \nu=\Omega[(\theta\gamma)^2+1], \quad (28)$$
$$G^{\mp}(\Omega,\theta\gamma,x)=\Omega^2\left[1-\frac{2}{(\theta\gamma)^2+1}\left(1-\frac{1}{(\theta\gamma)^2+1}\right)\right]$$
$$\times\exp(\pm\nu\pi)\int_{e^{\mp}_{min}}^{e^{\mp}_{max}}[(1-e^{-2})K_{i\nu}^2(\nu e)+(K'_{i\nu}(\nu e))^2]\,de.$$

The quantities $I_1\gamma^{1/2}$ and $\omega_1\gamma^{3/2}$, as will be made clear below, determine in this case the characteristic values of the intensity and frequency of the radiation of a particle with transverse energy $\varepsilon\approx U_0$.

At relatively high frequencies, when $\Omega\gg 1$, the largest contribution to the intensity is made by trajectories with eccentricities $e\approx 1$, so that the Macdonald function and its derivative can be replaced by the asymptotic values[27]

$$K_{i\nu}(\zeta)\approx e^{-\pi\nu/2}\frac{w}{3^{1/2}}K_{1/3}\left(\frac{\nu w^3}{3}\right),$$
$$K_{i\nu}'(\zeta)\approx e^{-\frac{\pi\nu}{2}}\frac{\zeta w^3}{3^{1/2}(\zeta^2-\nu^2)^{1/2}}K_{2/3}\left(\frac{\nu w^3}{3}\right);$$
$$w=[(\zeta/\nu)^2-1]^{1/2}\approx(e^2-1)^{1/2}.$$

Taking into account the character of the behavior of the functions $K_{1/3}$ and $K_{2/3}$ at large values of the argument, we obtain the following result.

The spectral-angular density of the positron-radiation intensity decreases exponentially at frequencies and angles determined by the condition

$$\Omega[1+(\theta E)^2]\gg 1.$$

The analogous condition for the electrons is

$$\Omega(e^2-1)^{1/2}[1+(\theta E)^2]\gg 1.$$

In the region of relatively soft frequencies, when $\Omega \ll 1$, we can use the approximations[24]

$$K_{i\nu}(\nu e)\approx -\ln[\nu e/2C_E],\quad K_{i\nu}'(\nu e)\approx 1/\nu e,$$

where $C_E \approx 1.782$ is the Euler constant. Thus, in the region of relatively low frequencies, the spectral-angular density of the intensity of radiation by both electrons and positrons is practically independent of frequency.

The results did not take into account the effects of polarization of the medium, i.e., of the fact that the crystal has a nonzero average dielectric susceptibility $\chi'(\omega) = -\omega_p^2/\omega^2$, where ω_p is the plasma frequency of the medium.

As shown by us in Ref. 3 (p. 39), the polarization effect can be taken into account by replacing Ω in (28) by

$$\Omega'=\Omega[1+(\omega_p\gamma/\omega)^2]. \tag{29}$$

This suppresses the radiation intensity in the frequency region $\omega \lesssim \omega_p E$. At harder frequencies, $\omega \gg \omega_p E$, the polarization has a negligible effect.

We now average the spectral-angular distribution (28) over the transverse energies of the particles, using for this purpose the distribution function (20). The numerical calculations yield the spectral-angular intensity density of the radiation as a function of the particle incidence angle on the axis. Figure 3 shows the results for the incidence angle $\theta_0 = \theta_L$ and two values of the radiation observation angle, $\theta = 0$ and $\theta = \gamma^{-1}$. We took into account in the calculations the polarization of the medium, so that the behavior of the radiation spectra at relatively soft frequencies, other conditions being equal, depends on the ratio $\omega_p\gamma/\omega$ [see (29)]. To illustrate the polarization effect, the curves of Fig. 3 are plotted for different $\gamma = mc^2/E$, namely 10^2, 10^3, and 10^4, and at the plasma frequency $\omega_p = 30.8$ eV corresponding to silicon. The spectra integrated over the radiation angles, as functions of the ratio of the particle incidence angle on the axis to the Lindhard angle, are similar in form.

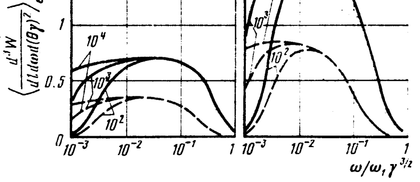

FIG. 3. Spectral-angular density of the radiation intensity (28) in silicon, averaged over the transverse energies of the particles, as a function of the ratio of the radiation frequency to $\omega_1\gamma^{3/2}$ and of the observation angle for the electrons (a) and for the positrons (b). The solid curves correspond to an observation angle $\theta = \gamma^{-1}$, and the dashed curves to $\theta = 0$. The numbers at the curves indicate the values of the Lorentz factor of the particles γ. The values $\langle d^3W/dld\omega d(\theta\gamma)^2\rangle_{e,x}$ are measured in units of $I_1\gamma^{1/2}$, and the angle of incidence θ_0 of the particles on the axis is taken equal to θ_L.

The radiation spectra of quasichanneled particles, shown in Fig. 3, differ substantially in character from the corresponding radiation spectra of channeled particles. This difference should manifest itself most noticeably when observed at fixed angles θ ($\Delta\theta \ll \theta$). The spectra of the channeled particles should consist in this case of individual lines or bands corresponding to different harmonics.[4-6] At the same time, the spectra of the radiation of the quasichanneled particles (Fig. 3) are continuous without any characteristic maxima.[7)] This is the result of the random transverse motion of the quasichanneled particles.

6. COMPARISON OF THE RESULTS OF THE THEORY WITH THE EXPERIMENTAL DATA

The results enable us to draw a number of general conclusions concerning the observed increase of the radiation intensity when the crystallographic axis are oriented along the beam of the incident particles. Distinct maxima in the electron radiation spectra, observed in the experiments[13-16] at different electron energies and for different single crystals, can be attributed only to the fact that a substantial fraction of the electrons passed axially channeled through the targets. On the other hand, the absence of maxima of this type in the spectra obtained by Swent *et al.*[12] indicates that the predominant contribution to these spectra was made by quasichanneled electrons.

Since the form of the radiation spectrum in axial channeling and quasichanneling turns out to be quite sensitive to the distribution of the particles with respect to the transverse energies, an important role in the detailed comparision of the experimental results with the calculation of the particles in the beam, and also of the subsequent evolution of the particle distribution.

In most experiments[12-15] the initial angle scatter of the particle was comparable with the critical angle. The angular distribution of the particles was not measured in this case. On the other hand, the kinetic theory of axial channeling of particles is much more complicated than in the case of planar channeling, and the calculation of the particle distribution as they move in a relatively thick crystal is a rather complicated problem in itself.

We confine ourselves here therefore to an analysis of the spectra of the radiation produced by positrons of energy 56 MeV, when a beam of particles having an angular divergence $\Delta\theta \approx 3 \times 10^{-3}$ in the vertical direction and 9×10^{-3} in the horizontal direction passes along the $\langle 110\rangle$ axis of a silicon crystal 18 μm thick. The regis-

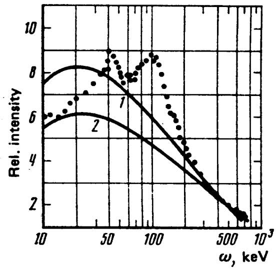

FIG. 4. Spectral intensity $\langle d^2W/dld\rangle_{e,x}$ of positron emission in a silicon crystal in the interval of the angle from zero to $8\cdot10^{-4}$, in relative units. These units are chosen such that the theoretical value of $\langle d^2W/dld\omega\rangle_{e,x}$ is equal to two at $\hbar\omega=500$ keV. In the calculations, the spectra were also averaged over the angles θ_0 of the positron incidence on the ⟨110⟩ axis in the range from zero to 1.5 θ_L (curve 1) or $2\theta_L$ (curve 2) under the assumption that the angles θ_0 have an equiprobable distribution. The points show the experimental results of Aulgard *et al.*[11]

tered radiation was contained in an angle interval $\Delta\theta \leq 1.6\cdot10^{-3}$ around the direction of the crystal axis.[11]

Under these conditions, the overwhelming majority of the positrons pass through the crystal in the quasi-channeling regime, with the beam divergence in both directions exceeding the critical angle $\theta_L = 2.0\cdot10^{-3}$ (see Table I). The results of the theoretical calculation are given in Fig. 4.

The spectra were calculated by us in the dipole approximation (28), which in this case was accurate enough. The redistribution of the transverse energies of the positrons propagating in the crystal was neglected, since the crystal was thin enough. Since the true form of the initial angular distribution of the positrons was unknown, the sensitivity of the spectrum to the initial angular distribution of the positrons was illustrated by choosing two angle widths, within which the distribution was assumed equally probable.

The theoretical curves explain the principal characteristic feature of the measured spectra, namely the presence of a broad frequency range ≲100 keV, in which the spectra density of the radiation energy exceeds unity noticeably. The small characteristic peaks in the observed spectra were attributed[11] to the contribution from the positrons that become planarly channeled. We did not take this contribution into account.

It should be noted that the assumption made in Ref. 11 that at sufficiently high frequencies the radiation spectrum in an oriented crystal should coincide with the spectrum in an amorphous medium is unfounded. As shown by the calculation above, in a sufficiently hard region of the spectrum the radiation in the crystal can be suppressed compared with an amorphous target. Therefore the ratio of the spectral radiation densities shown in Fig. 4 cannot be regarded as the true ratio of these quantities for oriented and amorphous targets.

7. QUANTUM THEORY OF RADIATION IN QUASICHANNELING

The classical theory of radiation in axial quasichanneling shows that the integrated energy lost by the particle to radiation is proportional to the square of the particle energy. At sufficiently low particle energies ($U_0E \lesssim 1$), when the radiation is dipole, the characteristic radiation frequencies are proportional to $E^{3/2}$, and the spectral density of the radiation power is proportional to $E^{1/2}$. In the opposite case, when $U_0E \gg 1$, the characteristic frequency is proportional to E^2, and the spectral power density does not depend on the particle energy E. These results are in fact independent of the model of the axis potential.

The restrictions on the applicability of the theory developed above are due to quantum effects. A distinction can be made between two types of such effects. The first can be significant at relatively low particle energies (~1 MeV),[17,18] when the de Broglie wavelength of the transverse motion $\lambda_D \approx (2E\varepsilon)^{-1/2}$ is of the same order as the dimensions of the unit cell, and a quantum description of the particle scattering is essential. The second should manifest itself when the photon energy becomes comparable with the particle energy and the recoil accompanying the photon emission is significant, as well as the interaction of the particle spin with the radiation field. A corresponding theory that takes both these effects into account is considered below. We note that according to classical estimates (Ref. 3, p. 39) the particle energy at which the characteristic radiation frequency in the average potential reaches the region $\omega\sim E$ is determined by the quantity $E_2 = \alpha m^3c^5/(\hbar U_0^2)$ (see Table I). However, channeled and quasi-channeled particles with transverse energy ε can in principle emit photons with energy $\omega\sim E$ also at energies $E_1 \sim 1/\varepsilon$ which are lower than E_2. To this end, the change of the transverse energy upon radiation should be of the order of the initial energy ε, and the radiation probability turns out to be relatively low. At still lower energies, $E \lesssim E_1$, the energy and longitudinal-momentum-component conservation laws forbid the emission of photons with energy $\omega\sim E$ [see (32) below] in the average potential of the axes. Such photons can be emitted in this case only when account is taken of the deviations of the potential of the atomic chain from the mean value, and also as a result of the scattering of the particles by the crystal electrons. These possibilities were not considered in detail in this study.

The Dirac relativistic equation that describes the particle motion in an average potential $U(\rho)$ can be reduced to a Schrödinger equation for the wave function of the transverse motion. This procedure is described in detail, e.g., in Sec. 1 of Ref. 29 for the case of planar channeling of the particles. In perfect analogy, for axial channeling the wave function of the electron in the initial state can be written in the form

$$\Psi_i(\rho,z,t)=\left(\frac{E_i+1}{2E_i}\right)^{1/2}\begin{pmatrix}\varphi_i\\ \sigma\mathbf{p}\varphi_i/(E_i+1)\end{pmatrix}\exp(ip_i^{(z)}z)\psi_i(\rho)\exp(-iE_it), \tag{30}$$

where E_i is the total initial energy of the particle, $p_i^{(z)}$ is the initial particle momentum component parallel to the axis, σ are Pauli matrices, $\mathbf{p} = -i\nabla$ is the particle momentum operator, and φ_i is the spinor corresponding to the initial spin state. The wave function of the transverse motion satisfies in this case the equation

$$\left[-\frac{1}{2E_i}\Delta_\rho+U(\rho)\right]\psi_i(\rho;E_i)=\varepsilon_i(E_i)\psi_i(\rho;E_i). \quad (31)$$

Most important is the circumstance that the wave functions and the corresponding transverse energies depend parametrically on the total energy of the particles and consequently make up two different bases. This difference turns out to be significant in calculations of emission spectra, even if the photon energy is low compared with the particle energy, so that E_f differs little from E_i. A classical analog of this effect is the influence of the field of the axis on the longitudinal velocity of the particle [see Eq. (3)] in the emission process.[2]

The spectral-angular distribution of the probability of the emission of a photon of energy ω and polarization e per unit time is determined in the quantum case by the Fourier component of the correlation function [see, e.g., Eq. (4) of Ref. 29]. By calculating it with the aid of the wave functions (30), we obtain the spectral-angular distribution of the probability, per unit time, of emission by an unpolarized electron (positron):

$$\frac{d^2w}{d\omega do}=\frac{e^2\omega}{2\pi}\sum_f\left\{\left(1+u+\frac{u^2}{2}\right)[|[\mathbf{n}_\rho\mathbf{I}_{if}^{(2)}]|^2\right.$$
$$\left.+|\theta I_{if}^{(1)}-\mathbf{n}_\rho\mathbf{I}_{if}^{(2)}|^2]+\frac{u^2}{2E^2}|I_{if}^{(1)}|^2\right\}\delta\left[\frac{\omega}{2}\left(\theta^2+\frac{1}{E(E-\omega)}\right)-\omega_{if}\right], \quad (32)$$

where $u=\omega/(E-\omega)$, E is the initial energy of the particle, $\mathbf{n}_\rho$ is a unit vector in the direction of the projection of the photon momentum on a plane perpendicular to the crystallographic axis $\mathbf{k}_\perp=\omega\theta\mathbf{n}_\rho$, $\omega_{if}=\varepsilon_i(E)-\varepsilon_f(E-\omega)$ is the difference between the transverse energies in the initial and final states,

$$I_{if}^{(1)}=\int\psi_i(\rho;E)\psi_f^*(\rho;E-\omega)\exp(-i\mathbf{k}_\perp\rho)\,d^2\rho,$$
$$I_{if}^{(2)}=\frac{-(-1)^{1/2}}{E}\int\frac{d\psi_i(\rho;E)}{d\rho}\psi_f^*(\rho;E-\omega)\exp(-i\mathbf{k}_\perp\rho)\,d^2\rho \quad (33)$$

are the matrix elements of the longitudinal and transverse components of the particle current. The summation in (32) is over all the quantum numbers f of the final states of the transverse motion.

If it is assumed formally that the transverse transition current has only one spatial component $(\mathbf{I}_{if}^{(2)})$, then the expression in the curly brackets of (32) coincides with the analogous expression (10) of Zhevago's paper[29] for the probability of emission of unpolarized photons in planar channeling. The difference between the arguments of the δ-functions in Eq. (12) of Ref. 29 and in our equation (32) is due to the somewhat different character of the influence of the quantum recoil on the longitudinal motion in the field of the axis and of the planes. Compared with the results of Baĭer *et al.* [see Eq. (3.5) of Ref. 6], our general expression (32) takes into account a number of possible effects. First, the quantum character of the particle motion, second the influence of the quantum recoil on the transverse motion, which turns out to be essential for hard frequencies $\omega\sim E$ (for more details see Sec. 2 of Ref. 30), and third, Eq. (32) remains valid when the transverse particle trajectories become nonperiodic, so that they can be used to investigate also the radiation in the case of axial quasichanneling.

For soft frequencies ($\omega\ll E$) and relatively low energies ($U_0E\ll 1$), when the dipole approximation is valid, expression (32) takes on a much simpler form

$$\frac{d^2w}{d\omega do}=\frac{e^2\omega}{2\pi}\sum_f\left\{(\theta^2+E^{-2})^2|[\mathbf{n}_\rho\times\rho_{if}]|^2\right.$$
$$\left.+(\theta^2-E^{-2})^2|\mathbf{n}_\rho\rho_{if}|^2\right\}\delta\left[\frac{\omega}{2}(\theta^2+E^{-2})-\varepsilon_i(E)+\varepsilon_f(E)\right];$$
$$\rho_{if}=\int\psi_i(\rho;E)\rho\psi_f^*(\rho;E)\,d^2\rho. \quad (34)$$

At low particle energies, $E\sim 1$ MeV, a definite role can be played by effects of diffraction of a transverse wave function by a periodic potential $U(\rho)$. To this end we transform Eq. (34) for the radiation probability, taking into account the Bloch form of the wave functions

$$\psi_i(\rho;E)=e^{i\varkappa\rho}\phi_{n,\varkappa}(\rho),\quad \psi_f(\rho;E)=e^{i\varkappa'\rho}\phi_{n'\varkappa'}(\rho), \quad (35)$$

where $i=\{\varkappa,n\}$; $f=\{\varkappa',n'\}$ are the quantum numbers (the quasimomentum and the number of the band) of the initial and final states respectively; $\phi_{n,}(\rho)$, $\phi_{n,}(\rho)$ are periodic functions having the period R of the two-dimensional lattice, and the quasimomenta $\varkappa$ and $\varkappa'$ are assumed to be referred to the first Brillouin zone.

We substitute expression (35) in the general formula (34) for the matrix elements of the dipole moment of the transition, and transform the integral over the entire crystal cross section into an integral over the area S of the unit cell. Using the periodicity of the function ϕ, we obtain

$$\rho_{if}=\sum_{\mathbf{R}}e^{i(\varkappa-\varkappa')\mathbf{R}}\int_S\phi_{n,\varkappa}(\rho)\rho\phi_{n',\varkappa'}^*(\rho)\,d^2\rho. \quad (36)$$

We substitute next (36) in (34) and sum over the quasimomenta of the final states. We take account here of the relation

$$\sum_{\mathbf{R}}e^{i(\varkappa-\varkappa')\mathbf{R}}=\delta_{\varkappa,\varkappa'},$$

where δ is the Kronecker symbol. The spectral-angular probability density of the dipole radiation can be rewritten in the form

$$\frac{d^2w}{d\omega do}=\frac{e^2\omega^3}{8\pi}\sum_{n'}\{(\theta^2+E^{-2})^2|[\mathbf{n}_\rho\times\rho_{nn'}(\varkappa)]|^2$$
$$+(\theta^2-E^{-2})^2|\mathbf{n}_\rho\rho_{nn'}(\varkappa)|^2\}\,\delta\left[\frac{\omega}{2}(\theta^2+E^{-2})-(\varepsilon_{n,\varkappa}-\varepsilon_{n',\varkappa})\right];$$
$$\rho_{nn'}(\varkappa)=\int_S\phi_{n,\varkappa}(\rho)\,\rho\phi_{n',\varkappa}^*(\rho)\,d^2\rho.$$

The functions $\phi(\rho)$ are normalized by the condition

$$\int_S\phi_{n,\varkappa}(\rho)\phi_{n',\varkappa'}^*(\rho)\,d^2\rho=\delta_{nn'}.$$

The quasimomentum is thus conserved in dipole radiative transitions. The radiation takes place only in interband transitions and there are no intraband transitions ($n=n'$) in the dipole approximation. The more general matrix elements of the current (33), with the wave functions (35) of the Block type, can similarly be written in one of the following forms:

$$I_{if}^{(1)}=\int_S\psi_{n,\varkappa}(\rho)\psi_{n',\varkappa-\mathbf{k}_\perp-\mathbf{K}}^*(\rho)\,e^{-i\mathbf{k}_\perp\rho}d^2\rho=\int_S e^{i\mathbf{K}\rho}\phi_{n,\varkappa}(\rho)\phi_{n',\varkappa-\mathbf{k}_\perp-\mathbf{K}}^*(\rho)\,d^2\rho,$$

$$\mathbf{I}_{if}^{(2)}=\frac{-(-1)^{1/2}}{E}\int_s \frac{d\psi_{n,\varkappa}(\rho)}{d\rho}\psi^{*}_{n',\varkappa-\mathbf{k}_\perp-\mathbf{K}}(\rho)\,e^{-i\mathbf{k}_\perp\rho}d^2\rho$$

$$=-\frac{\varkappa}{E}I_{if}^{(1)}-\frac{(-1)^{1/2}}{E}\int_s e^{i\mathbf{k}\rho}\frac{d\phi_{n,\varkappa}(\rho)}{d\rho}\phi^{*}_{n',\varkappa-\mathbf{k}_\perp-\mathbf{K}}(\rho)\,d^2\rho,$$

where K is the reciprocal-lattice vector that refers the quasimomentum $\varkappa'=\varkappa-\mathbf{k}_\perp-\mathbf{K}$ to the first Brillouin zone. However, allowance for the particle diffraction under non-dipole radiation conditions ($E\gg 100$ MeV) is meaningless, for when the particle energy is increased the diffraction effects vanish and the problem of radiation produced in axial quasichanneling (channeling) can be solved in the approximation of the isolated axis (cell). In this case the quantum number for the quasichanneled particles is the two-dimensional vector $\mathbf{p}_\perp$, which has the meaning of the transverse momentum of the particle in a region where the action of the potential of the axis can be neglected. The summation in (32) over the final numbers f must be understood as integration with respect to $d^2p_\perp/(2\pi)^2$.

In analogy with the three-dimensional scattering case,[31] we can introduce a system of wave functions of the continuous spectrum of the transverse energies in the axis field, $\{\psi^{(+)}_{\mathbf{p}\perp}(\rho)\}$, and also the system $\{\psi^{(-)}_{\mathbf{p}\perp}(\rho)\}$. The initial wave functions in the matrix elements of the transition current are chosen from the system $\psi^{(+)}$, and the final from the system $\psi^{(-)}$. This corresponds to radiation by a quasichanneled particle scattered by the potential of the axis with change of the transverse momentum by an amount $\mathbf{p}_\perp'-\mathbf{p}_\perp$.

For a model potential of the type $(-\alpha/\rho)$, the explicit form of the wave functions is obtained by solving Eq. (31) in planar parabolic coordinates. The solutions are expressed in terms of confluent hypergeometric functions $\Phi(a,c;x)$ in the following manner[8)]:

$$\psi^{(+)}_{\mathbf{p}_\perp}(\rho)=C^{(+)}e^{i\mathbf{p}_\perp\rho}\Phi(i\mu,{}^1/_2;i(p_\perp\rho+\mathbf{p}_\perp\rho));$$

$$\psi^{(-)}_{\mathbf{p}'_\perp}(\rho)=C^{(-)}e^{i\mathbf{p}_\perp'\rho}\Phi(-i\mu',{}^1/_2;-i(p_\perp\rho+\mathbf{p}_\perp\rho));$$

$$C^{(+)}=\pi^{-1/2}\Gamma({}^1/_2-i\mu)e^{\pi\mu/2};\quad C^{(-)}=\pi^{-1/2}\Gamma({}^1/_2+i\mu')e^{\pi\mu'/2};$$

$$p_\perp=(2\varepsilon E)^{1/2},\quad p_\perp'=[2(E-\omega)\varepsilon']^{1/2},\quad \mu=\pm\alpha E/p_\perp,$$

$$\mu'=\pm\alpha(E-\omega)/p_\perp',$$

where ε and ε' are the initial and final transverse energies, the upper signs in front of the parameter of the potential α correspond to electrons, and the lower to positrons. The functions $\psi^{(\pm)}$ are normalized to unity amplitude in the incident wave.

The wave functions $\psi^{(+)}_{\mathbf{p}\perp}$ describe the scattering of a planar flux of particles by a two-dimensional potential. At sufficiently large ρ, the function $\psi^{(+)}(\rho)$ has asymptotically the form of a sum of a plane wave and a diverging cylindrical wave, which in this case are somewhat distorted because of the long-range character of the model potential

$$\psi^{(+)}(\rho)\approx\exp[i\mathbf{p}_\perp\rho-i\mu\ln(p_\perp\rho-\mathbf{p}_\perp\rho)]+\frac{f(\chi)}{\rho^{1/2}}$$

$$\times\exp\left[ip_\perp\rho+i\ln(p_\perp\rho-\mathbf{p}_\perp\rho)-\frac{i\pi}{4}\right].$$

The amplitude of the scattered wave is

$$f(\chi)=\frac{\Gamma({}^1/_2-i\mu)}{\Gamma(i\mu)}[p_\perp(1-\cos\chi)]^{-1/2},\qquad (37)$$

where χ is the scattering angle in the transverse plane. The poles of the scattering amplitude in the complex plane of the variable $\rho_\perp$, as is well known, determine the energy levels of the channeled particles. From the condition that the argument of the δ-function in the numerator (37) be a negative integer $(-N)$ we obtain the condition for the quantization of the transverse energy for the channeled electron in the potential $(-\alpha/\rho)$. According to (37), the cross section for scattering of quasichanneled particles in a transverse plane is of the form

$$\sigma(\chi)=\frac{\alpha}{4\varepsilon\sin^2(\chi/2)}\,\mathrm{th}\frac{\pi\alpha E}{p_\perp}.$$

The cross section has the dimension of length, since in this case it constitutes the probability of scattering through an angle χ in a unit time, referred to a unit planar particle flux. It is interesting to note that in contrast to the three-dimensional case of scattering in a Coulomb field, $\sigma(\chi)$ does not coincide now, generally speaking, with its classical limit $\sigma_c(\chi)=\alpha/4\varepsilon\sin^2(\chi/2)$. The matrix elements of the current (33) can be expressed in terms of the unit integral

$$J(\lambda,\mathbf{q},\mathbf{p}_\perp,\mathbf{p}_\perp')=C^{(+)}C^{(-)*}\int\rho^{-1}\exp(-\lambda\rho+i\mathbf{q}\rho)$$

$$\times\Phi(i\mu,{}^1/_2;i(p_\perp\rho-\mathbf{p}_\perp\rho))\Phi(i\mu',{}^1/_2;i(p_\perp'\rho+\mathbf{p}_\perp'\rho))d^2\rho\qquad (38)$$

by means of the relations

$$I_{if}^{(1)}=\lim_{\lambda\to0}\frac{\partial J}{\partial\lambda};\qquad \mathbf{I}_{if}^{(2)}=\frac{\mathbf{p}_\perp}{E}I_{if}^{(1)}+\frac{ip_\perp}{E}\lim_{\lambda\to0}\frac{\partial J}{\partial\mathbf{p}_\perp},\qquad (39)$$

after which it is necessary to put $\mathbf{q}=\mathbf{p}_\perp-\mathbf{p}_\perp'-\mathbf{k}_\perp$ in (39). The integral J is calculated by an analytic method similar to that used by Nordsieck[32] and Sommerfeld[25] to calculate the bremsstrahlung in a Coulomb field. The results of the calculations takes in this case the form

$$J=\exp[{}^1/_2\pi(\mu+\mu')]\Gamma({}^1/_2-i\mu)\Gamma({}^1/_2-i\mu')(2/\tau)^{1/2}$$

$$\times(1+\beta/\tau)^{-i\mu}(1+\gamma/\tau)^{-i\mu'}F(i\mu,i\mu',{}^1/_2;z).$$

The following notation was introduced:

$$\tau=(\lambda^2+q^2)/2,\qquad \beta=-(\mathbf{q}\mathbf{p}_\perp+i\lambda p_\perp),$$

$$\gamma=\mathbf{q}\mathbf{p}_\perp'-i\lambda p_\perp',\quad \delta=-(p_\perp p_\perp'+\mathbf{p}_\perp\mathbf{p}_\perp'),\quad z=\frac{\beta\gamma-\tau\delta}{(\tau+\beta)(\tau+\gamma)},$$

and $F(a,b,c;z)$ is a hypergeometric function expressed at $c=1/2$ in terms of Legendre functions.[33]

For soft frequencies $\omega\ll E$ and large parameters $\mu\gg1$, and under the condition that the change of the transverse energy in the course of the radiation is small, $|\mu'-\mu|\ll\mu$, the matrix elements (39) go over into the previously calculated Fourier components (24). A detailed analysis of the dipole radiation of quasichanneled electrons and positrons, with quantum effects taken into account, is published in Ref. 35.

We note in conclusion that the existing theory of formation of electron-positron pairs by a photon in a crystal, based on the Born approximation,[9,34] should be also reviewed to take into account the effects of quasichanneling of the particles in the final state. The criterion for the influence of the quasichanneling is in this case that the incidence angle of the photon on the axis and the pair-separation angle, $\theta_{eff}\sim1/\omega$, be close in order of magnitude to the critical angle $\theta_L\sim(U_0/\omega)^{1/2}$. The corresponding theory can be developed on the basis of the results obtained in Sec. 7 and of the cross-symmetry properties of the amplitudes of the processes considered.

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1) Equation (35) of this reference contains an error that can be corrected by introducing the factor (−2) in front of the product $\omega\omega_0 a$ in the definitions of the parameters $\chi, \bar{\chi}$, and μ.

2) Quasichanneled positrons are sometimes called also simply channeled. The truly channeled positrons, which execute finite transverse motion, are then called hyperchanneled (Ref. 19, p. 193).

3) In the opposite case, relatively rapid dechanneling takes place.

4) More correct calculation with allowance for coherence were subsequently performed by these authors in another paper.[21]

5) This representation of $U(\rho)$ was obtained by V. I. Glebov and one of us (N. Zh.)

6) We use a system of units in which $m = c = 1$.

7) A weakly pronounced maximum appears only as a result of the polarization of the medium, when the particle energies are not too high.

8) At $c = 1/2$ the hypergeometric functions $\Phi(a, c, x)$ reduce to the parabolic cylinder functions[28] $D_{-a}(\pm x^{1/2})$. The results presented below were obtained also in a recent paper by Bazylev and Demura.[35]

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