Structure of vortex filament in helium II near the λ point

V. L. Ginzburg and A. A. Sobyanin

P. N. Lebedev Physics Institute, USSR Academy of Sciences
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The structure of a vortex filament near the λ point in helium II is considered on the basis of the Ψ theory of superfluidity ($\Psi = (\rho_s/m)^{1/2}e^{i\phi}$ is a macroscopic wave function that plays the role of the order parameter). The entropy, heat capacity, density, and the thermal coefficient of expansion of helium, which are connected with the presence of the filament, are calculated in addition to the distribution of the density ρ_s of the superfluid part of the helium as a function of the distance from the filament axis and of the parameter M of the theory. The possibilities and the purposes of measuring the corresponding effects are briefly discussed.

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1. The structure of the vortex filament in helium II was discussed for temperatures close to the λ point way back in Ref. 1, and for the temperature region near absolute zero in Refs. 1 and 3 (using the model of a weakly ideal Bose gas as the example). Since the publication of Ref. 1, however, the situation with the description of helium near the λ point has changed substantially (see Ref. 4 and the literature cited therein). Namely, it must be recognized that the Landau theory of phase transitions is in fact not valid for helium near the λ point, and similarity theory must be used. This circumstance can be accounted for within the framework of a phenomenological approach by modifying the temperature dependence of the coefficients of the expansion of the density of the thermodynamic potential $\Phi_{II}(\mu, T, \Psi)$ in powers of the order parameter, the macroscopic wave function $\Psi = (\rho_s/m)^{1/2}e^{i\varphi}$ (ρ_s is the density of the superfluid part of helium II and m is the mass of the helium atom; see Refs. 1, 4, and 5). In this approach it is not only permissible but also generally speaking necessary to take into account in the aforementioned expansion terms of order $|\Psi|^6$, and this is why a new parameter M has been introduced in the theory. The so-generalized Ψ theory of superfluidity was already used to solve a number of problems,[4,6] but the question of the vortex filament structure was not considered on its basis. First, the structure of the filament depends on the parameter, although the dependence in question is indeed quite weak. Second, possibilities of experimentally investigating the filament structure near the λ point are revealed.

2. We express the thermodynamic potential of helium II in the form (see Ref. 4)

$$\tilde{\Phi}_{II}(\mu,T,\Psi)=\int\left[\Phi_{II,0}(\mu,T,|\Psi|^2)+\frac{\hbar^2}{2m}|\nabla\Psi|^2\right]dV$$
$$=\int\left[\Phi_{II,0}(\mu,T,|\Psi|^2)+\frac{\hbar^2}{8m^2}\frac{(\nabla\rho_s)^2}{\rho_s}+\frac{\rho_s v_s^2}{2}\right]dV, \quad (1)$$

$$\Phi_{II,0}(\mu,T,|\Psi|^2)=\Phi_I(\mu,T)+\frac{3T_\lambda\Delta C_\mu}{3+M}\left\{-\tau|\tau|^{1/3}\left|\frac{\Psi}{\Psi_{00}}\right|^2+\frac{1-M}{2}|\tau|^{4/3}\left|\frac{\Psi}{\Psi_{00}}\right|^4+\frac{M}{3}\left|\frac{\Psi}{\Psi_{00}}\right|^6\right\}, \quad (2)$$
$$\tau=[T_\lambda(\mu)-T]/T_\lambda(\mu).$$

Here $T_\lambda(\mu)$ is the temperature of the λ transition as a function of the chemical potential μ, Φ_I is the density of the thermodynamic potential of helium I, ΔC_μ is the discontinuity of the heat capacitance at the λ point ($\Delta C_\mu \approx \Delta C_p = 0.76\cdot10^7$ erg$\cdot$cm^{-3}K^{-1}), M is the aforementioned parameter of the theory, and Ψ_{00} is the coefficient in the expression for the equilibrium value of Ψ:

$$\Psi_e(\tau)=\Psi_{00}\tau^{1/3},\quad \Psi_{00}=(1.43\rho_\lambda/m)^{1/2}T_\lambda^{-1/3}=0.23\cdot10^{12}\ \mathrm{cm}^{-3/2} \quad (3)$$

($\rho_\lambda = 0.14617$ g$\cdot$cm^{-3} is the density of helium at the λ point at the saturated-vapor pressure, and $T_\lambda(\rho_\lambda) = 2.172$ K). The value of Ψ_e is determined from the minimum condition $(\partial\Phi_{II,0}/\partial|\Psi|^2)_{\mu,T}=0$ and is normalized against the experimental data, according to which[7]

$$\rho_{se}=m|\Psi_e|^2=1.43\rho_\lambda T_\lambda^{-2/3}\tau^{2/3}=0.35\tau^{2/3}\ \mathrm{g\cdot cm^{-3}}. \quad (4)$$

The question of the accuracy in the region of the applicability of expressions (1) and (2) is discussed in Refs. 4 and 8 and will not be touched upon here.

An equation for Ψ is obtained from the condition that the functional $\tilde{\Phi}_{II}(\mu, T, \Psi)$ be a minimum, and is of the form

$$\frac{\hbar^2}{2m}\Delta_R\Psi-\left(\frac{\partial\Phi_{II,0}}{\partial|\Psi|^2}\right)_{\mu,T}\Psi=0 \quad (5)$$

or

$$\Delta_r\psi-[-\nu+(4\nu-3)|\psi|^2+3(1-\nu)|\psi|^4]\psi=0, \quad (6)$$

where we now use reduced variables (**R** stands for the usual coordinates):

$$\psi=\Psi/\Psi_e(\tau),\quad \mathbf{r}=\mathbf{R}/\xi_0,$$
$$\xi_0=2.74\cdot10^{-8}(T_\lambda-T)^{-2/3}\ \mathrm{cm}=1{,}63\cdot10^{-8}\tau^{-2/3}\ \mathrm{cm},\quad \nu=3/(3+M). \quad (7)$$

3. We consider now a vortex filament in an unbounded liquid (helium II) that is at rest at infinity. The problem has axial symmetry, so that we can use cylindrical coordinates (r, θ, z). For an infinite straight filament we then have

$$\psi(r,\theta)=f(r)e^{in\theta},\quad n=1,2,3,\ldots$$
$$\mathbf{v}_s=\frac{\hbar}{m}\nabla_R\varphi,\quad \oint \mathbf{v}_s d\mathbf{l}=v_{s\theta}\cdot2\pi R=\frac{2\pi n\hbar}{m}, \quad (8)$$
$$v_{s\theta}=n\hbar\xi_0/mr,\quad v_{sr}=v_{sz}=0$$

and

$$\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}+\left(\nu-\frac{n^2}{r^2}\right)f-(4\nu-3)f^3-3(1-\nu)f^5=0. \qquad (9)$$

The solution of interest to us of Eq. (9) takes the following form: at small r ($r<r_1$)

$$f=cr^n\left[1-\frac{\nu r^2}{4(n+1)}+..\right], \qquad (10)$$

and at large r ($r>r_2$)

$$f^2=1-\frac{A}{r^2}-\frac{B}{r^4}-\frac{D}{r^6}-\ldots, \qquad (11)$$

where

$$A=\frac{n^2}{3-2\nu},\quad B=\frac{A}{3-2\nu}[2+3(1-\nu)A]. \qquad (12)$$

$$D=\frac{A}{(3-2\nu)^2}[(1+2\nu)(7-2\nu)+(51-46\nu)A+3(1-\nu)(9-8\nu)A^2].$$

At intermediate r ($r_1<r<r_2$) the solution of (9) and the coefficient c in (10) were obtained numerically.[1)] The results are shown in Fig. 1 for $n=1$ and for a number of values of ν. As seen from this figure, the distribution $f(r)$ depends little on ν. This is not surprising, inasmuch as at small r the terms of (9) that contain the parameter $\nu=3/(3+M)$ are insignificant, and at large r the behavior of the solution is controlled by the length

$$\xi_M{}^-(\tau)=[(3+M)/6(1+M)]^{1/2}\xi_0(\tau),$$

which changes by only a factor $\sqrt{3}$ when M changes from zero to infinity.

With the aid of the obtained solutions it is easy to calculate the contribution of the vortex filament to the thermodynamic potential of helium II [i.e., to obtain the difference between the values of the thermodynamic potential $\Delta\tilde{\Phi}_{II}(\mu, T)$ in the presence and absence of the vortex filament, referred to a unit filament length]:

$$\Delta\tilde{\Phi}_{II}(\mu,T)=\frac{\pi\hbar^2}{m^2}\rho_{se}N,\quad N=N_1+N_2,$$

$$N_1=\int\limits_0^\infty\left[\frac{1}{2}-\nu f^2+\frac{1}{2}(4\nu-3)f^4+(1-\nu)f^6+\left(\frac{df}{dr}\right)^2\right]rdr. \qquad (13)$$

$$N_2=n^2\int\limits_0^{r_{max}}f^2\frac{dr}{r}.$$

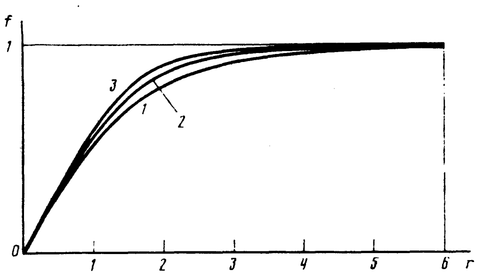

FIG. 1. Distribution of the order parameter near the axis of a vortex filament with a single circulation quantum ($n=1$) in helium II at $\nu=1$ (curve 1), $\nu=0.5$ (curve 2) and $\nu=0$ (curve 3).

The integral N_2 represents here the contribution made to $\Delta\tilde{\Phi}_{II}$ by the kinetic energy of the vortex [i.e., by the term $\rho_s v_s^2/2$ in (1)]. We note that N_2 diverges logarithmically at the upper limit. We have therefore restricted the integration region to a certain $r_{max}=R_{max}/\xi_0$, where the role of R_{max} can be assumed by the radius of the vessel or by the mean distance between the vortex filaments.

Differentiating (13) with respect to the temperature T and the chemical potential μ [the latter dependence enters in (13) via the dependence of the λ-point temperature T_λ on μ], we obtain the contribution of the vortex filament to the entropy $\tilde{S}$, the specific heat $\tilde{C}_\mu \approx \tilde{C}_p$, the helium density $\tilde{\rho}$, and the thermal expansion coefficient $\tilde{\beta}_\mu$ (per unit length of one filament):

$$\Delta\tilde{S}=-\left(\frac{\partial\Delta\tilde{\Phi}_{II}}{\partial T}\right)_\mu=-\frac{\pi\hbar^2}{m^2}\left(\frac{\partial\rho_{se}}{\partial T}\right)_\mu(N+1)$$
$$=0.851\cdot 10^{-8}(N+1)\tau^{-1/3}\ \mathrm{erg\cdot cm^{-1}\cdot K^{-1}}, \qquad (14)$$

$$\Delta\tilde{C}_\mu=-T\left(\frac{\partial^2\Delta\tilde{\Phi}_{II}}{\partial T^2}\right)_\mu=0.284\cdot 10^{-8}(N-1)\tau^{-4/3}\ \mathrm{erg\cdot cm^{-1}\cdot K^{-1}}, \qquad (15)$$

$$\Delta\tilde{\rho}=-\left(\frac{\partial\Delta\tilde{\Phi}_{II}}{\partial\mu}\right)_T=-\Delta\tilde{S}\frac{dT_\lambda}{d\mu}=0.107\cdot 10^{-16}(N+1)\tau^{-}\ \mathrm{g\cdot cm^{-1}}. \qquad (16)$$

$$\Delta\tilde{\beta}_\mu=\frac{1}{\rho_\lambda}\left(\frac{\partial\Delta\tilde{\rho}}{\partial T}\right)_\mu=-\frac{\Delta\tilde{C}_\mu}{\rho_\lambda T_\lambda}\frac{dT_\lambda}{d\mu}=0.114\cdot 10^{-16}(N-1)\tau^{-1}\ \mathrm{cm^2\cdot K^{-1}}. \qquad (17)$$

Similarly, in the case of solutions of ^3He in superfluid ^4He, differentiating $\Delta\tilde{\Phi}_{II}$ with respect to the chemical potential μ_3 of the ^3He admixture, we can find the excess content of the ^3He in the vortex filament:

$$\Delta\tilde{X}_3=-\left(\frac{\partial\Delta\tilde{\Phi}_{II}}{\partial\mu_3}\right)_{\mu_4,T}=-\frac{\pi\hbar^2}{m^2}\left(\frac{\partial\rho_{se}}{\partial\mu_3}\right)_{T,\mu}(N+1). \qquad (18)$$

Finally, substantial interest attaches also to the "contribution" of the vortex filament to the decrease of the content of the superfluid component (just as the contributions indicated above, this contribution was not calculated in Refs. 1–4)

$$\Delta M_s=\int(\rho_s-\rho_{se})dV=-2\pi\rho_{se}\xi_0^2N_3=-5{,}84\cdot 10^{-16}N_3\tau^{-}\ \mathrm{g\cdot cm^{-1}}.$$
$$N_3=\int\limits_0^{r_{max}}(1-f^2)rdr. \qquad (19)$$

Since the integrals N and N_2 diverge at the upper limit, it is convenient to represent them in the form

$$N=n^2\ln(x_N r_{max})=n^2\ln\left(x_N\frac{R_{max}}{\xi_0(\tau)}\right),$$
$$N_3=\frac{n^2}{3-2\nu}\ln(x_3 r_{max})=\frac{n^2}{3-2\nu}\ln\left(x_3\frac{R_{max}}{\xi_0(\tau)}\right). \qquad (20)$$

The values of x_N, x_3, N_1 and of the coefficient c in the asymptotic expression (10), as well as the distance r_2 up to which the numerical integration of (9) was carried out, are listed in Table I for n equal to 1 and 2 and for a number of values of the parameter $\nu=3/(3+M)$.

By measuring the corresponding quantities one can hope to determine the parameter M and, most importantly, verify the Ψ theory of superconductivity itself.

In experiment, of course, one can count primarily on measuring the contribution of not one but of a large number of vortex filaments. If the average distance

TABLE I.

ν	M	$n=1$					$n=2$				
		x_N	N_1	x_3	c	r_2	x_N	N_1	x_3	c	r_2
1	0	1,46	0,78	2.40	0.58319	5,7	0.59	2.41	0.94	0,15338	6.2
0,9375	0.2	1.50	0.78	2.85	0.58459	5,4	0.61	2.42	1.10	0,15069	6.1
0.857	0,5	1.56	0.79	3.47	0.58650	6.9	0.62	2,44	1.32	0,14737	6,1
0,75	1	1.60	0,80	4.34	0.58916	6,1	0.63	2.45	1.66	0,14322	5.9
0.5	3	1.72	0.81	6.72	0.59583	4,9	0,67	2.50	2.61	0,13448	5,7
0,2	12	1.86	0.82	10,21	0.60437	4.4	0,70	2,54	4,13	0.12553	5.5
0	∞	1.94	0,83	12,82	0.61026	4.1	0.73	2,57	5,40	0,12033	5,1

between filament exceeds ξ_0 by at least several times, then all the expressions above should be simply multiplied by the density of the vortex filaments (i.e., by their total length per cm^3). The experiments can be performed both in rotating helium II and with helium II flowing in channels at superfluid-flow velocities exceeding the critical vortex-formation velocity $v_{sc\,1}$. In the case of rotation, as follows from the equations given, observation of an increase in C_μ or of a decrease in $\bar{\rho}_s$ by 10% at $\tau \sim 10^{-5}$ calls for rotation velocities $\Omega \sim 5\times 10^4$ rad/sec, since the average density of the vortex filaments in a rotating vessel is

$$n_0=4\pi\Omega m/\hbar=2\cdot 10^3\Omega \text{ cm}^{-2}.$$

It appears that sufficient vortex densities are easier to obtain in experiments with translational motion of the superfluid helium (for example, by heat flow).

A more detailed discussion of the experimental possibilities is not our aim, and we confine ourselves to recalling that it is most convenient to measure the decrease of $\bar{\rho}_s$ by determining the change of the second-sound velocity, and the increase of $\bar{\rho}$ by determining the change of the dielectric constant (of the refractive index). Observation of light scattering (intensity and spectral composition) may also be quite effective. Thus, in rotating helium the ratio of the intensity of light scattered by the vortex filaments to the intensity of the thermal (molecular) scattering by density fluctuations is[9]

$$\frac{I_{\text{vort}}}{I_{\text{mol}}}=\frac{(\Delta\bar{\rho})^2 h n_0}{k_B T\rho^2\chi_T}=0{,}12\cdot 10^{-8}(N+1)^2 n_0\tau^{-4/3}h, \tag{21}$$

where h is the height of the scattering volume and coincides with the length of each filament in the light beam (it is assumed that the scattered light beam is perpendicular to the rotation axis), and $\chi_T=\rho^{-1}(\partial\rho/\partial p)_T$ is the compressibility of helium. Substitution of numerical values shows that, e.g., at $\tau=10^{-3}$ and $h=1$ cm the intensity I_{vort} becomes comparable with I_{mol} even at a vortex filament density $n_0\sim 10^6$ cm^{-2}, i.e., at angular velocities $\Omega\sim 5\times 10^2$ rad/sec.

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