Fluctuation corrections to the hydrodynamics equations for antiferromagnets

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We consider the corrections that must be introduced in the hydrodynamics equations for antiferromagnets to account for spin fluctuations. Expressions for the fluctuation contributions to the energy and momentum densities, as well as to the energy flux density and the stress tensor, are obtained on the basis of a general equation for the correlator matrix. This yields the fluctuation contributions to the equations that describe acoustic oscillations in antiferromagnets. Two frequency ranges of the sound oscillations are considered, low and high; the latter becomes significant near the antiferromagnetic transition point. In both ranges, the nonlocal contributions to the equations and the corrections to the dispersion laws of the sound oscillations are analyzed on the basis of the solution for the correlator matrices of the fluctuating quantities. The calculations are carried through to conclusion within the framework of the isotropic model. It is shown that the fluctuation corrections to the equations increase near the phase-transition point.

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INTRODUCTION

Andreev[1] has shown that nonlocal corrections to the equations of the hydrodynamics of a classical liquid, necessitated by long-wave fluctuations, exceed the Barnett terms. He calculated explicitly the nonlocal fluctuation contributions to the hydrodynamics equations. The same question was considered for the case of superfluid He by Khalatnikov and the authors.[2] Just as in a classical liquid, the fluctuation corrections to the equations of the hydrodynamics of HeII turned out to be small for numerical reasons, with the exception of the temperature region near the λ point, where the fluctuation contribution to the equations increases. The same should be expected in the vicinity of any second-order phase transition.

We consider in this paper fluctuation corrections to the hydrodynamics equations for antiferromagnets. The fluctuation corrections can be revealed in experiment by the corrections to the dispersion laws for the long-wave oscillations. Spin waves have gaps in their spectrum; in addition, there is no direct method of measuring their dispersion law, so that a measurement of the corrections to the spin-wave dispersion law is at present doubtful. We have therefore focused our attention to the calculations of the corrections to the equations for the acoustic variables, paying particular attention to the region near the antiferromagnetic-transition point.

Dzyaloshinskiĭ and Kukharenko[3] have proposed a Lagrangian formalism for the phenomenological description of the long-wave oscillations of antiferromagnets. An alternate and more convenient Lagrangian formalism was formulated by Andreev and Marchenko.[4,5] This formalism, however, is not convenient for a description that includes the magnetic-moment density. We shall use for this purpose a Hamiltonian formalism equivalent to the Lagrangian formalism of Ref. 5. This Hamiltonian formalism yields nondissipative nonlinear spin-dynamics equations, so that particular attention should be paid to the dissipative terms in the equations.

The state of a magnet is macroscopically characterized by the density of the spin S and by an order parameter ψ that has the spin degrees of freedom. A classication of the possible types of order parameter in crystalline magnets was presented by Andreev and Marchenko[5,6] on the basis of symmetry considerations. Not all degrees of freedom of the order parameter, however, can be described hydrodynamically. The hydrodynamic degrees of freedom are rotations of the order parameter inspin state, which are responsible for the Goldstone excitations—the spin waves. Thus, for a weakly inhomogeneous state of a magnet we can write

$$\psi=\exp(i\theta\hat{S})\psi_0. \tag{1}$$

Here ψ_0 is the homogeneous value of the order parameter and $\hat{S}$ is the spin operator. The rotation angles θ together with the density of the spin S are the hydrodynamic variables that describe the spin degrees of freedom of the magnets.

We shall be interested in the corrections to sound-wave dispersion laws, and must therefore describe the acoustic as well as the magnetic degrees of freedom. In place of the traditional elasticity-theory variables we shall use others, more convenient for the description of nonlinear hydrodynamic processes.[7] Besides the specific entropy σ and the momentum density g, we consider the quantities x_μ ($\mu=1,2,3$), whose meaning is that the equation $x_\mu=\text{const}$ (for one index) determines the position in space of some atomic plane of the crystal.[1)] If all three quantities x_μ are fixed, we obtain equations that describe some point of the crystal. i.e., the shear vector u is defined implicitly by the equation $x_\mu(\mathbf{r}+\mathbf{u})=\text{const}$. Under the natural conditions that $x_\mu=r_\mu$ in the undeformed state, we can write down the connection between x_μ and the strain tensor:

$$u_{\nu\mu}=\frac{1}{2}\left(\frac{\partial R_i}{\partial x_\nu}\frac{\partial R_i}{\partial x_\mu}-\delta_{\nu\mu}\right). \tag{2}$$

Here $R_i=r_i+u_i$ is the observation point.

For all the pairs of variables listed above we can write out expressions for the Poisson brackets in the

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following form (∇ means differentiation with respect to the variable $\mathbf{r}_i$)

$$\{g_k(\mathbf{r}_1), g_i(\mathbf{r}_2)\} = g_i(\mathbf{r}_1)\nabla_k\delta(\mathbf{r}_1-\mathbf{r}_2) + \nabla_i(g_k\delta(r_1-r_2)),$$
$$\{\mathbf{g}(\mathbf{r}_1), S_\alpha(\mathbf{r}_2)\} = S_\alpha(\mathbf{r}_1)\nabla\delta(\mathbf{r}_1-\mathbf{r}_2), \quad \{g(\mathbf{r}_1), \sigma(\mathbf{r}_2)\} = -\nabla\sigma\delta(\mathbf{r}_1-\mathbf{r}_2),$$
$$\{\mathbf{g}(\mathbf{r}_1), x_\mu(\mathbf{r}_2)\} = -\nabla x_\mu\delta(\mathbf{r}_1-\mathbf{r}_2), \quad \{\mathbf{g}(\mathbf{r}_1), \theta_\alpha(\mathbf{r}_2)\} = -\nabla\theta_\alpha\delta(\mathbf{r}_1-\mathbf{r}_2),$$
$$\{S_\alpha(\mathbf{r}_1), S_\beta(\mathbf{r}_2)\} = -\varepsilon_{\alpha\beta\gamma}S_\gamma\delta(\mathbf{r}_1-\mathbf{r}_2), \tag{3}$$
$$\{S_\alpha(\mathbf{r}_1), \theta_\beta(\mathbf{r}_2)\} = \left(\frac{\theta_\alpha\theta_\beta}{\theta^2} + \frac{\theta}{2}\operatorname{ctg}\frac{\theta}{2}\left(\delta_{\alpha\beta} - \frac{\theta_\alpha\theta_\beta}{\theta^2}\right) - \frac{1}{2}\varepsilon_{\alpha\beta\gamma}\theta_\gamma\right)\delta(\mathbf{r}_1-\mathbf{r}_2).$$

Expressions (3) are obtained from the representations of all the variables listed in (3) in terms of normal coordinates.[8]

On the basis of (3) we can now obtain the nondissipative hydrodynamics equations for antiferromagnets. Let

$$E = E(\sigma, \mathbf{g}, S_\alpha, \nabla x_\mu, \theta_\alpha, \nabla\theta_\alpha)$$

be the energy density.[2)] The nondissipative hydrodynamics equations are then

$$\frac{\partial S_\alpha}{\partial t} = -\nabla(\mathbf{v}S_\alpha) + \left[\frac{\partial E}{\partial \mathbf{S}}\times\mathbf{S}\right]_\alpha$$
$$+\left(\nabla\frac{\partial E}{\partial\nabla\theta_\beta} - \frac{\partial E}{\partial\theta_\beta}\right)\left(\frac{\theta_\beta\theta_\alpha}{\theta^2} + \frac{\theta}{2}\operatorname{ctg}\frac{\theta}{2}\left(\delta_{\alpha\beta} - \frac{\theta_\alpha\theta_\beta}{\theta^2}\right) - \frac{1}{2}\varepsilon_{\alpha\beta\gamma}\theta_\gamma\right), \tag{4}$$

$$\frac{\partial\theta_\alpha}{\partial t} = -\mathbf{v}\nabla\theta_\alpha + \frac{\partial E}{\partial S_\beta}\left(\frac{\theta_\beta\theta_\alpha}{\theta^2} + \frac{\theta}{2}\operatorname{ctg}\frac{\theta}{2}\left(\delta_{\alpha\beta} - \frac{\theta_\alpha\theta_\beta}{\theta^2}\right) + \frac{1}{2}\varepsilon_{\alpha\beta\gamma}\theta_\gamma\right), \tag{5}$$

$$\partial\sigma/\partial t = -\mathbf{v}\nabla\sigma, \tag{6}$$
$$\partial x_\mu/\partial t = -\mathbf{v}\nabla x_\mu, \tag{7}$$
$$\partial g_i/\partial t = -\nabla_k\Pi_{ik}. \tag{8}$$

Here $\mathbf{v} = \partial E/\partial\mathbf{g}$ is the velocity, with $\mathbf{v} = \mathbf{g}/\rho$ by virtue of Galilean invariance (ρ is the mass density). The stress tensor is constructed in the following manner:

$$\Pi_{ik} = \left(\mathbf{v}\mathbf{g} + \mathbf{S}\frac{\partial E}{\partial\mathbf{S}} - E\right)\delta_{ik} + g_i v_k + \frac{\partial E}{\partial\nabla_k\theta_\alpha}\nabla_i\theta_\alpha + \frac{\partial E}{\partial\nabla_k x_\mu}\nabla_i x_\mu. \tag{9}$$

From the system (4)–(8) follows the energy conservation law

$$\partial E/\partial t + \nabla\mathbf{Q} = 0. \tag{10}$$

The energy flux density is here

$$\mathbf{Q} = \left(\mathbf{v}\mathbf{g} + \mathbf{S}\frac{\partial E}{\partial\mathbf{S}}\right)\mathbf{v} + \frac{\partial E}{\partial\nabla\theta_\alpha}(\mathbf{v}\nabla)\theta_\alpha + \frac{\partial E}{\partial\nabla x_\mu}(\mathbf{v}\nabla)x_\mu$$
$$-\frac{\partial E}{\partial S_\alpha}\left(\frac{\theta_\alpha\theta_\beta}{\theta^2} + \frac{\theta}{2}\operatorname{ctg}\frac{\theta}{2}\left(\delta_{\alpha\beta} - \frac{\theta_\alpha\theta_\beta}{\theta^2}\right) - \frac{1}{2}\varepsilon_{\alpha\beta\gamma}\theta_\gamma\right).$$

Equations (4) and (5) modify the equations obtained in Ref. 5, with account taken of the elastic motion of the crystal. Equations (6)–(8) are equivalent to the usual system of nonlinear equations of elasticity theory (with the addition, in the right-hand sides, of terms connected with the spin variables). When comparing (6)–(8) with the classical variables of elasticity theory[9] one must bear in mind the relation (2), as well as the fact that, unlike in Ref. 9, the equations (6)–(8) are written in the laboratory frame rather in a frame connected with the crystal.

Generally speaking, in magnets the exchange interaction which determines the spin-wave velocity c_{sp} is stronger than the spin-orbit interaction, which determines the gaps Δ in the spin-wave spectrum. We therefore have the inequality

$$c_{sp} \gg \Delta a. \tag{11}$$

Here a is the atom dimension. For various reasons, however, the exchange interaction may turn out to be weak and (11) may not hold. We shall not consider such antiferromagnets, since the fluctuation corrections will certainly be smaller in this case.

THE CORRELATOR MATRIX

Equations (4)–(8) are the nondissipative part of the equations for the corresponding mean values in the long-wave limit. It turns out that one can write out also the long-wave limit of the equations for the paired correlators of the quantities listed in (4)–(8), and it suffices for this purpose to know the form of the function E and the kinetic coefficients. These equations for the correlators are written out in a way such that they yield the correct hydrodynamic equations for the mean values. For a general case, this program was in fact carried out in the Appendix of Ref. 2, where the fluctuating variables considered were cononically conjugate. We shall use these result below.

Let (p, β) be a complete set of canonically conjugate variables. The equation for the correlator $\langle p_\alpha(\mathbf{r}_1)\beta_\beta(\mathbf{r}_2)\rangle$ is of the form

$$\frac{\partial}{\partial t}\langle p_\alpha(\mathbf{r}_1)\beta_\beta(\mathbf{r}_2)\rangle = -\left\langle\frac{\delta\mathscr{H}}{\delta\beta_\alpha(\mathbf{r}_1)}\beta_\beta(\mathbf{r}_2)\right\rangle + \left\langle p_\alpha(\mathbf{r}_1)\frac{\delta\mathscr{H}}{\delta p_\beta(\mathbf{r}_2)}\right\rangle - I. \tag{12}$$

Here I is the collision integral, and the Hamiltonian

$$\mathscr{H} = \int d^3r E(p, \beta, \nabla\beta). \tag{13}$$

By going to the limit $\mathbf{r}_1 - \mathbf{r}_2 \to \infty$ in (12) we can obtain equations for the mean values $\langle p\rangle$ and $\langle\beta\rangle$. The first two terms in the right-hand side of (12) yield the nondissipative parts of the equations, and the collision integral determines the kinetic terms. However, besides these usual terms, Eq. (12) yields also the fluctuation corrections to the equations for the mean values. We write out these corrections with the pair correlations taken into account.

We introduce the matrix of irreducible pair correlators

$$\begin{pmatrix}\langle\delta p_\alpha(r_1)\delta p_\beta(r_2)\rangle & \langle\delta p_\alpha(r_1)\delta\beta_\beta(r_2)\rangle \\ \langle\delta\beta_\alpha(r_1)\delta p_\beta(r_2)\rangle & \langle\delta\beta_\alpha(r_1)\delta\beta_\beta(r_2)\rangle\end{pmatrix}.$$

We change over to the variables $\mathbf{r} = (\mathbf{r}_1 + \mathbf{r}_2)/2, \mathbf{r}_1 - \mathbf{r}_2$ and take the Fourier transform with respect to the latter. We denote the corresponding matrix by $A(r, q)$. The contributions made by the pair correlators to the equations for the mean value are expressed in terms of the matrix A as follows:

$$\frac{\partial}{\partial t}\langle p\rangle \leftarrow \frac{1}{2}\operatorname{Sp}\int d\tau\left(\nabla\left(\frac{\partial}{\partial\nabla\beta}DA\right) - \frac{\partial D}{\partial\beta}A\right),$$
$$\frac{\partial}{\partial t}\langle\beta\rangle \leftarrow \frac{1}{2}\operatorname{Sp}\int d\tau\left(\frac{\partial D}{\partial p}A\right). \tag{14}$$

Here $d\tau = d^3q/(2\pi)^3$ and the matrix

$$D_{\alpha\beta} = \begin{pmatrix}\dfrac{\partial^2E}{\partial p_\alpha\partial p_\beta} & \dfrac{\partial^2E}{\partial p_\alpha\partial\beta_\beta} + i\mathbf{q}\dfrac{\partial^2E}{\partial p_\alpha\partial\nabla\beta_\beta} \\ \dfrac{\partial^2E}{\partial\beta_\alpha\partial p_\beta} - i\mathbf{q}\dfrac{\partial^2E}{\partial\nabla\beta_\alpha\partial p_\beta} & \dfrac{\partial^2E}{\partial\beta_\alpha\partial\beta_\beta} - i\mathbf{q}\dfrac{\partial^2E}{\partial\nabla\beta_\alpha\partial\beta_\beta} + i\mathbf{q}\dfrac{\partial^2E}{\partial\beta_\alpha\partial\nabla\beta_\beta} + \mathbf{q}\dfrac{\partial^2E}{\partial\nabla\beta_\alpha\partial\nabla\beta_\beta}\mathbf{q}\end{pmatrix}. \tag{15}$$

If no account is taken of the correlators, the expressions for the energy and momentum densities in terms of the mean canonically conjugate quantities are of the form

$$E=E(p,\beta,\nabla\beta),\quad \mathbf{g}=-p\nabla\beta.$$

When the pair correlations are taken into account the following are added to these equations

$$E_{\mathrm{fl}}=\frac{1}{2}\mathrm{Sp}\int d\tau DA,\quad \mathbf{g}_{\mathrm{fl}}=\frac{1}{2}\mathrm{Sp}\int d\tau \mathbf{q}JA. \tag{16}$$

here

$$J=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}. \tag{17}$$

Now, taking into account (14) as well as the equation obtained for the matrix A in the Appendix of Ref. 2, we can find the fluctuation contributions to the energy flux density and to the stress tensor[3]:

$$P_{\mathrm{fl}}=\frac{1}{2}\mathrm{Sp}\int d\tau DJ_zA+\frac{1}{2}\mathrm{Sp}\int d\tau p\frac{\partial D}{\partial p}A, \tag{18}$$

$$\Pi_{ik}^{\mathrm{fl}}=P_{\mathrm{fl}}\delta_{ik}+\frac{1}{2}\mathrm{Sp}\int d\tau\frac{\partial D}{\partial q_k}q_iA+\frac{1}{2}\mathrm{Sp}\int d\tau\nabla_i\beta\frac{\partial D}{\partial\nabla_k\beta}A, \tag{19}$$

$$\mathbf{Q}_{\mathrm{fl}}=\frac{1}{2}\mathrm{Sp}\int d\tau(DJ\mathbf{D}_q+\mathbf{D}_q^{+}JD)A$$
$$-\frac{1}{2}\mathrm{Sp}\int d\tau\left(\frac{\partial E}{\partial\nabla\beta}\frac{\partial D}{\partial p}A+\frac{\partial E}{\partial p}\frac{\partial D}{\partial\nabla\beta}A\right). \tag{20}$$

Here

$$J_z=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$
$$\mathbf{D}_q^{\alpha\beta}=\begin{pmatrix}0 & 0\\ -i\dfrac{\partial^2E}{\partial\nabla\beta_\alpha\partial p_\beta} & -i\dfrac{\partial^2E}{\partial\nabla\beta_\alpha\partial\beta_\beta}+\dfrac{\partial^2E}{\partial\nabla\beta_\alpha\partial\nabla_i\beta_\beta}q_i\end{pmatrix}. \tag{21}$$

In the case $\Omega\gg\omega$, where Ω is the characteristic frequency of the fluctuations (i.e., in the integrals that contain the matrix A), and ω is the frequency of the hydrodynamic motion, the matrix A is a sum of terms that pertain to each oscillation mode and are characterized by a distribution functions n_a (Ref. 2). In this case expressions (16) and (18)–(20) assume the much simpler form

$$E_{\mathrm{fl}}=\sum_a\int d\tau\Omega_a n_a, \tag{22}$$

$$\mathbf{g}_{\mathrm{fl}}=\sum_a\int d\tau\mathbf{q}n_a, \tag{23}$$

$$P_{\mathrm{fl}}=\sum_a\int d\tau p\frac{\partial\Omega_a}{\partial p}n_a+\sum_a\int d\tau\Omega_a n_a\langle JJ_z\rangle_a, \tag{24}$$

$$\Pi_{ik}^{\mathrm{fl}}=P_{\mathrm{fl}}\delta_{ik}+\sum_a\int d\tau\frac{\partial\Omega_a}{\partial q_k}n_aq_i+\sum_a\int d\tau\nabla_i\beta\frac{\partial\Omega_a}{\partial\nabla_k\beta}n_a, \tag{25}$$

$$\mathbf{Q}_{\mathrm{fl}}=\sum_a\int d\tau\Omega_a\frac{\partial\Omega_a}{\partial\mathbf{q}}n_a. \tag{26}$$

The index a numbers here the different oscillations modes, Ω_a are the eigenvalues of the matrix JD and have the meaning of the dispersion laws of the long-wave oscillations, $\langle\ \rangle_a$ denotes averaging over the eigenvector of the matrix JD having an eigenvalue Ω_a and normalized in accordance with the Appendix of Ref. 2.

We shall be interested in the spin-fluctuation corrections to the acoustic equations. The variables describing the acoustic oscillations are chosen to be the mean values x_μ and the densities of the energy E and of the momentum $\mathbf{g}$. Accordingly, the right-hand sides of the equations for these quantities should be expressed in terms of the same variables. As for expressions (16) and (18)–(26), they make it possible to express both the right- and left-hand sides of the equations for x_μ, E, and $\mathbf{g}$ in terms of the mean values of the canonically conjugate variables. Allowance for the contribution (16) and for the absence of a fluctuation contribution to the mean values x_μ (which are themselves canonically conjugate) allows us to express the stress tensor and the energy flux density in terms of the mean x_μ, E, and $\mathbf{g}$:

$$\Pi=\Pi_0+\Pi_{\mathrm{fl}}-\frac{\partial\Pi_0}{\partial\sigma}\frac{E_{\mathrm{fl}}-\mathbf{v}\mathbf{g}_{\mathrm{fl}}}{\rho T}-\frac{\partial\Pi_0}{\partial\mathbf{g}}\mathbf{g}_{\mathrm{fl}},$$
$$\mathbf{Q}=\mathbf{Q}_0+\mathbf{Q}_{\mathrm{fl}}-\frac{\partial\mathbf{Q}_0}{\partial\sigma}\frac{E_{\mathrm{fl}}-\mathbf{v}\mathbf{g}_{\mathrm{fl}}}{\rho T}-\frac{\partial\mathbf{Q}_0}{\partial\mathbf{g}}\mathbf{g}_{\mathrm{fl}}. \tag{27}$$

Here $T=(\partial E/\partial\sigma)/\rho$ is the temperature, Π_0 and $\mathbf{Q}_0$ are the nondissipative expressions (9) and (11), and the dependence of (27) on σ is equivalent to the dependence on the energy density. To obtain an explicit dependence on E it is necessary to invert the function $E(\sigma)$.

SPIN DYNAMICS

The spin part of the energy density takes in the quadratic approximation the form

$$E_{sp}={}^1/_2\chi_{ik}^{-1}S_iS_k+{}^1/_2\eta_{iklm}\nabla_i\theta_l\nabla_k\theta_m+{}^1/_2\xi_{ik}\theta_i\theta_k. \tag{28}$$

Here χ is the susceptibility tensor, η determines the spin-wave velocities, and ξ determines the gaps in the spin-wave spectrum. In the isotropic model

$$\chi_{ik}=\chi\delta_{ik},\quad \xi_{ik}=\chi\Delta^2\delta_{ik},$$
$$\eta_{iklm}=\chi c_\parallel^2\delta_{il}\delta_{km}+\chi c_\perp^2(\delta_{ik}\delta_{lm}-\delta_{il}\delta_{km}). \tag{29}$$

Here Δ is the gap in the spin-wave spectrum, and $c_\parallel$ and $c_\perp$ are respectively the longitudinal and transverse velocities of the spin waves.

To write down the equation for the matrix A, we must know the form of the kinetic terms in the hydrodynamic equations. It is known that these terms can be obtained with the aid of the dissipative function R. In our case we are interested in the dependence on the spin variables

$$R=R\left(\frac{\partial E}{\partial S_\alpha},\nabla\frac{\partial E}{\partial S_\alpha},\frac{\delta\mathcal{H}}{\delta\theta_\alpha}\right).$$

The dissipative contributions to the right-hand sides of the equations for the spin variables are of the form

$$\frac{\partial S_\alpha}{\partial t}\leftarrow-\frac{1}{2}\frac{\partial R}{\partial(\partial E/\partial S_\alpha)}+\frac{1}{2}\nabla\frac{\partial R}{\partial(\nabla\partial E/\partial S_\alpha)},$$
$$\frac{\partial\theta_\alpha}{\partial t}\leftarrow-\frac{1}{2}\frac{\partial R}{\partial(\delta\mathcal{H}/\delta\theta_\alpha)}. \tag{30}$$

In the isotropic model

$$R=\zeta_1\left(\frac{\delta\mathcal{H}}{\delta\theta}\right)^2+\zeta_2\left(\nabla\frac{\partial E}{\partial S_\alpha}\right)^2+\zeta_3\left(\nabla\frac{\partial E}{\partial\mathbf{S}}\right)^2+\zeta_4\frac{\partial E}{\partial S_\alpha}\frac{\delta\mathcal{H}}{\delta\theta_\alpha}. \tag{31}$$

We are interested in the matrix A_{sp} of the paired irreducible correlators of the spin variables. We consider the equation for A_{sp} in the presence of long-wave acoustic motion, which leads to the appearance of disequilibrium in the matrix A_{sp}. It turns out that to obtain this equation for A_{sp}, it suffices to know the equations for the spin variables in the approximation linear in these variables, and consequently it suffices to use relation (3) for $\{S_\alpha, \theta_\beta\}$ in the zeroth approximation in θ. In this approximation, S_α and θ_α turn out to be canonically conjugate variables, with S_α the generalized momenta and θ_α the generalized coordinates. Expression (28) for the energy density can now be regarded as its dependence on the canonically conjugate variables (for the spin degrees of freedom).

The sound-wave dispersion laws can be represented in the form

$$\omega_s = c_s q - i\Gamma_s q^2. \tag{32}$$

Here c_s is the speed of sound, and the second term in (32) determines the damping decrement of the sound wave. In the limit $c_{sp}q \gg \Delta$ [which exists by virtue of the inequality (11)], a similar expression holds for the spin-wave dispersion law, and the corresponding coefficient of q^2 will be designated Γ_{sp}. In the isotropic model

$$\Gamma_\perp = 0.5(\zeta_1\chi c_\perp^2 + \zeta_2/\chi), \quad \Gamma_\parallel = 0.5(\zeta_1\chi c_\parallel^2 + (\zeta_2+\zeta_3)/\chi). \tag{33}$$

We consider first the case of low frequencies $\omega \ll \Omega$, where ω is the frequency of the hydrodynamic motion and Ω is the characteristic frequency of the spin fluctuations. As already noted, the matrix A_{sp} reduces in this case to a number of distribution functions n_a. Let q be the characteristic wave vector of the spin fluctuations. We consider the region $c_{sp}q \gg \Delta$ (if the inequality is reversed the fluctuation corrections are small). In this case we can disregard the gaps in the spin-wave spectrum. The condition imposed on q, the inequality $\Omega \gg \omega$, and the necessary condition $q \gg k$ (k is the wave vector of the hydrodynamic motion) yield the following inequalities for the frequency of the hydrodynamic motion:

$$\frac{c_{sp}^2}{\Gamma_{sp}}, \quad \frac{c_s^2}{\Gamma_{sp}} \gg \omega \gg \frac{\Delta_{sp}^2\Gamma_{sp}}{c_{sp}^2}. \tag{34}$$

The region of the existence of the integral (34) is ensured by the inequality $c_{sp}^2 \gg \Delta_{sp}\Gamma_{sp}$, which means weak damping of the spin waves at frequencies not greatly exceeding the gap, and by the experimental condition $c_{sp} \sim c_s$.

In the presence of long-wave hydrodynamic motion, the distribution function n_{sp} for each spin-wave mode acquires a nonequilibrium increment

$$n_{sp} = n_0 + \delta n. \tag{35}$$

With account taken of the dispersion law $n_0 = T/c_{sp}q$, an equation for the function n was obtained in the Appendix of Ref. 2 and takes the form of the usual kinetic equation. The solution of this equation for long-wave motion with frequency ω and wave vector k is

$$\delta n(\mathbf{q}) = \frac{i}{-i\omega + ik_i(\partial c_{sp}/\partial n_i + c_{sp}n_i) + 2\Gamma_{sp}(\mathbf{n})q^2}\frac{T}{c_{sp}q} \tag{36}$$
$$\times\left(-\omega\delta\ln T + \omega\delta\ln c_{sp} + k_i\left(\frac{\partial c_{sp}}{\partial n_i} + c_{sp}n_i\right)\delta\ln T + k_i\left(\frac{\partial \ln c_{sp}}{\partial n_i} + n_i\right)n_k v_k\right).$$

Allowance is made here for the dependence of the spin-wave velocity on the direction of the vector $\mathbf{q}$, and n_i is a unit vector in the direction of $\mathbf{q}$.

We must now substitute n in the integrals (22)–(26). The integrals of n_0 are independent of ω and k, and result an inessential redefinition of the thermodynamic quantities. The same applies to the integral of δ_n with respect to large q, so that we take into account in the integrals (22)–(26) only the pole part of (36). Thus, the integral with respect to the modulus of q reduces to taking the residue, after which we are left with an integral with respect to the angles, which depends generally speaking on the form of the functions $c_{sp}(\mathbf{n})$ and $\Gamma_{sp}(n)$. To estimate the numerical factors that arise in the integration we carry out the integration for the case when c_{sp} and Γ_{sp} do not depend on the direction of $\mathbf{q}$, as is the case, in particular, in the isotropic model. It is necessary here to take into account the equation $\langle JJ_z\rangle_a = 0$, which holds because the energy density (28) contains no cross terms in θ and S. As a result, each spin-wave mode makes the following contribution:

$$E_{\mathrm{fl}} = \frac{i+1}{16\pi}\left(\frac{\omega}{\Gamma^3}\right)^{1/2}[\omega(\delta\ln c - \delta\ln T)I_1 + ck\,\delta\ln T I_2 + \mathbf{kv}I_3],$$
$$Q_{\mathrm{fl}} = \frac{i+1}{16\pi}\left(\frac{\omega}{\Gamma^3}\right)^{1/2} Tc\frac{\mathbf{k}}{k}[\omega(\delta\ln c - \delta\ln T)I_2 + ck\,\delta\ln T I_3 + \mathbf{kv}I_4],$$
$$g_{\mathrm{fl}} = Q_{\mathrm{fl}}/c^2, \tag{37}$$
$$\Pi_{ik}^{\mathrm{fl}} = \frac{i+1}{16\pi}\left(\frac{\omega}{\Gamma^3}\right)^{1/2} T\Big\{[\omega(\delta\ln c - \delta\ln T)I_3 + ck\,\delta\ln T I_4 + \mathbf{kv}I_5]\delta_{ik}$$
$$+\frac{1}{2}(I_3 - I_5)(k_i v_{\perp k} + v_{\perp i}k_k)\Big\} - \frac{\partial\ln c}{\partial u_{ik}}E_{\mathrm{fl}}.$$

Equations (37) are written in an approximation linear in the acoustic variables,

$$\delta\ln c = \frac{\partial\ln c}{\partial u_{ik}}u_{ik} + \frac{\partial\ln c}{\partial\sigma}\delta\sigma$$

and so forth. In these equations $I_i = I_i(ck/\omega)$, and the expressions for I_i are given in the Appendix of Ref. 2.

FLUCTUATIONS NEAR A TRANSITION POINT

We consider the situation near the antiferromagnetic-transition point. In this case both the gaps and the velocities in the spin-wave spectrum are small. An important role is therefore assumed by the high-frequency region, in which the inequality

$$\omega \gg \Omega$$

holds. It turns out that the inequality $c_{sp}q \gg \Delta$ is then automatically satisfied, since the characteristic wave vector q is determined by the expression $q \sim (\omega/\Gamma_{sp})^{1/2}$. Taking into account the need for satisfying the inequality $q \gg k$, and requiring also weak damping of the sound wave, we obtain the following inequalities for the frequency:

$$c_s^2/\Gamma_{sp}, \; c_s^2/\Gamma_s \gg \omega \gg c_{sp}^2/\Gamma_{sp}. \tag{38}$$

The existence of this interval is ensured by the smallness of c_{sp} near the transition point.

If $\omega \gg \Omega$, the solution of the equation for the matrix A_{sp} becomes more complicated, since A_{sp} cannot be represented in terms of n_a. The kinetic equation for the matrix A_{sp} is of the form[2]

$$\frac{\partial A_{sp}}{\partial t} = -\frac{1}{2}\{JD + \mathbf{vq}, A_{sp}\} + \frac{1}{2}\{A_{sp}, JD + \mathbf{vq}\} - G\delta A_{sp} - \delta A_{sp}G^T. \tag{39}$$

Here $\{,\}$ are Poisson brackets in $\mathbf{q}$ and $\mathbf{r}$; G is the kinetic-coefficient matrix; D is a matrix constructed in accordance with (15) from the spin part of the energy density (28); δA_{sp} is the deviation from the equilibrium value

$$A_0 = TD^{-1}.$$

Linearizing (39), we obtain, taking (38) into account, the following equation for δA_{sp}:

$$\frac{\partial \delta A_{sp}}{\partial t}+G\delta A_{sp}+\delta A_{sp}G^{T}=-\frac{\partial}{\partial t}(TD^{-1})-\frac{1}{2}\{JD+\mathbf{vq},TD^{-1}\}+\frac{1}{2}\{D^{-1},TDJ+\mathbf{vq}\}. \tag{40}$$

By virtue of $c_{sp}^2\gg\Delta_{sp}\Gamma_{sp}$, the inequality (38) ensures the condition $cq\gg\Delta$. It is accordingly necessary to take into account in the expressions for D and G only the terms of principal order in q. In this approximation we have in the isotropic model

$$D=\begin{pmatrix}\delta^{\alpha\beta}/\chi & 0\\ 0 & \chi c_\parallel^2q^2\delta_\parallel^{\alpha\beta}+\chi c_\perp^2q^2\delta_\perp^{\alpha\beta}\end{pmatrix}, \tag{41}$$

$$G=\begin{pmatrix}\Gamma_p^\parallel\delta_\parallel^{\alpha\beta}q^2+\Gamma_p^\perp\delta_\perp^{\alpha\beta}q^2 & 0\\ 0 & \Gamma_\theta^\parallel\delta_\parallel^{\alpha\beta}q^2+\Gamma_\theta^\perp\delta_\perp^{\alpha\beta}q^2\end{pmatrix}. \tag{42}$$

Here

$$\Gamma_p^\parallel=\frac{1}{\chi}(\zeta_2+\zeta_3),\quad \Gamma_p^\perp=\frac{\zeta_2}{\chi},\quad \Gamma_\theta^\parallel=\chi\zeta_1c_\parallel^2,\quad \Gamma_\theta^\perp=\chi\zeta_1c_\perp^2.$$

The solution of (4) is a sum of terms determined by the eigenvectors of the matrix G. If the acoustic motion takes place with frequency ω, each such term has a denominator in the form

$$1/(\omega+2i\Gamma q^2).$$

It is the residues determined by these denominators which contribute to the fluctuation corrections. In the approximation linear in the acoustic motion we obtain

$$E_\Pi=\frac{1+i}{16\pi}\omega^{1/2}T\Big[-\Big(\frac{1}{2\Gamma_p^{\parallel 3/2}}+\frac{1}{\Gamma_p^{\perp 3/2}}\Big)\delta\ln(T\chi)+\frac{1}{2\Gamma_\theta^{\parallel 3/2}}\Big(\frac{2}{3}\frac{\mathbf{k}}{\omega}\mathbf{v}-\delta\ln\frac{T}{\chi c_\parallel^2}\Big)+\frac{1}{\Gamma_\theta^{\perp 3/2}}\Big(\frac{2}{3}\frac{\mathbf{k}}{\omega}\mathbf{v}-\delta\ln\frac{T}{\chi c_\perp^2}\Big)\Big],$$

$$\mathbf{g}_\Pi=\sqrt{2}\frac{i+1}{8\pi}\omega^{3/2}T\Big[\frac{1}{(\Gamma_p^\parallel+\Gamma_\theta^\parallel)^{3/2}}+\frac{2}{(\Gamma_p^\perp+\Gamma_\theta^\perp)^{3/2}}\Big]\frac{\mathbf{k}}{k}\delta\ln T,$$

$$\mathbf{Q}_\Pi=\frac{i+1}{8\pi}\sqrt{2}\,\omega^{1/2}T\Big[c_\parallel^2\delta\ln T\frac{1}{(\Gamma_p^\parallel+\Gamma_\theta^\parallel)^{1/2}}+2c_\perp^2\delta\ln T\frac{1}{(\Gamma_p^\perp+\Gamma_\theta^\perp)^{1/2}}+\frac{(c_\parallel^2-c_\perp^2)^2}{c_\parallel^2}\frac{1}{(\Gamma_p^\perp+\Gamma_\theta^\parallel)^{1/2}}\Big(\delta\ln T+\delta\ln\frac{c_\perp^2}{c_\parallel^2}\Big)\Big]\frac{\mathbf{k}}{k},$$

$$P_\Pi=\frac{i+1}{16\pi}\omega^{1/2}T\Big[-\Big(\frac{1}{2\Gamma_p^{\parallel 3/2}}+\frac{1}{\Gamma_p^{\perp 3/2}}\Big)\delta\ln(T\chi)-\frac{1}{2\Gamma_\theta^{\parallel 3/2}}\Big(\frac{2}{3}\frac{\mathbf{k}}{\omega}\mathbf{v}-\delta\ln\frac{T}{\chi c_\parallel^2}\Big)-\frac{1}{\Gamma_\theta^{\perp 3/2}}\Big(\frac{2}{3}\frac{\mathbf{k}}{\omega}\mathbf{v}-\delta\ln\frac{T}{\chi c_\perp^2}\Big)\Big], \tag{43}$$

$$\Pi_{ik}^\Pi=P_\Pi\delta_{ik}+\frac{i+1}{16\pi}\omega^{1/2}T\Big\{\frac{1}{3\Gamma_\theta^{\parallel 3/2}}\Big[\frac{2}{5\omega}(\delta_{ik}\mathbf{kv}+k_iv_k+k_kv_i)-\delta_{ik}\delta\ln\frac{T}{\chi c_\parallel^2}\Big]+\frac{2}{3\Gamma_\theta^{\perp 3/2}}\Big[\frac{2}{5\omega}(\delta_{ik}\mathbf{kv}+k_iv_k+k_kv_i)-\delta_{ik}\delta\ln\frac{T}{\chi c_\perp^2}\Big]+\frac{\sqrt{2}}{(\Gamma_\theta^\parallel+\Gamma_\theta^\perp)^{3/2}}\frac{(c_\parallel^2-c_\perp^2)^2}{c_\parallel^2c_\perp^2}$$
$$\times\frac{2}{3\omega}\Big(-\frac{6}{5}\mathbf{kv}\delta_{ik}+\frac{4}{5}k_iv_k+\frac{4}{5}k_kv_i\Big)-\frac{1}{2\Gamma_p^{\parallel 3/2}}\frac{\partial\ln\chi}{\partial u_{ik}}\delta\ln(\chi T)-\frac{1}{\Gamma_p^{\perp 3/2}}\frac{\partial\ln\chi}{\partial u_{ik}}\delta\ln\chi-\frac{1}{2\Gamma_\theta^{\parallel 3/2}}\frac{\partial}{\partial u_{ik}}\ln(\chi c_\parallel^2)\Big(\frac{2}{3}\frac{\mathbf{k}}{\omega}\mathbf{v}-\delta\ln\frac{T}{\chi c_\parallel^2}\Big)-\frac{1}{\Gamma_\theta^{\parallel 3/2}}\frac{\partial}{\partial u_{ik}}\ln(\chi c_\perp^2)\Big(\frac{2}{3}\frac{\mathbf{k}}{\omega}\mathbf{v}-\delta\ln\frac{T}{\chi c_\perp^2}\Big)\Big\}.$$

CORRECTIONS TO THE SOUND VELOCITIES

It is impossible in the general case to correct the sound-wave velocities for the fluctuations. We confine ourselves therefore from the very outset to the isotropic model. In this case the acoustic part of the energy density is

$$E_*=\frac{1}{2\rho}g^2+E_0(\sigma)-(P+E_0)u_{ii}+(\rho c_t^2-P-E_0)u_{ik}^2+\frac{1}{2}(\rho c_l^2-2\rho c_t^2-P-E_0)u_{ii}^2. \tag{44}$$

Here P is the pressure at $u=0$, c_l and c_t are respectively the longitudinal and transverse sound velocities. This form of energy density corresponds to the following energy flux density and stress tensor (in the linear approximation):

$$\mathbf{Q}_0=\frac{P+E_0}{\rho}\mathbf{g},\quad \Pi_{ik}^0=P\delta_{ik}+2\rho c_t^2u_{ik}+\rho(c_l^2-2c_t^2)u_{ii}. \tag{45}$$

The last expression takes into account the nonlinear dependence of the strain tensor u_{ik} on ∇x_μ. Knowing expressions (16)–(26), we can now write down the fluctuation corrections for the equations for the energy and momentum densities in accord with (27). Recognizing that the spin fluctuations do not alter the form Eq. (7) for x_μ, we know thus the fluctuation contributions to all the equations for the acoustic variables. These contributions allow us to find the corrections to the dispersion laws of the sound waves; these corrections affect both the real parts of the spectrum and the damping decrements.

In the frequency range (34), the fluctuation corrections to the equations are given by expressions (37). Substituting (27) in them and linearizing with respect to the variations of the variables in the sound wave, we can obtain the corrections to the dispersion laws of the longitudinal and transverse sound waves. They are of the form

$$\delta c_t=\frac{T}{64\pi\rho c_t}\Big(\frac{\omega}{\Gamma}\Big)^{3/2}(I_3'-I_5'), \tag{46}$$

$$\delta c_l=\frac{T}{32\pi\rho c_l}\Big(\frac{\omega}{\Gamma}\Big)^{3/2}\Big[(\gamma+\varphi_1)^2I_1'-2\varphi_1\frac{c}{c_l}(\gamma+\varphi_1)I_2'+\Big(\varphi_1^2\frac{c^2}{c_l^2}-2\varphi_1-2\gamma\Big)\times I_3'+2\varphi_1\frac{c}{c_l}I_4'+I_5'+\varphi_1\frac{P+E_0}{\rho c_l^2}\Big(\frac{c_l}{c}(\varphi_1+\gamma)I_2'-\varphi_1I_3'-\frac{c}{c_l}I_4'\Big)\Big].$$

We have written out here the contribution from one mode of spin waves with velocity c, and introduced the notation

$$\varphi_1=-\frac{\partial\ln T}{\partial u_{ii}},\quad \varphi_2=\frac{\partial\ln\chi}{\partial u_{ii}},\quad \gamma=\frac{\partial\ln c_l}{\partial u_{ii}},$$
$$I'=\dot I\Big(\frac{c}{c_l}\Big),\quad I'=I\Big(\frac{c}{c_l}\Big).$$

The corrections to the imaginary part of the spectrum are given by[4)]:

$$\delta\alpha_l=-\frac{\omega}{c_l^2}\delta c_l,\quad \delta\alpha_t=-\frac{\omega}{c_t^2}\delta c_t. \tag{47}$$

Here α is the coefficient of ω^2 in the imaginary part of the wave vector.

The expressions for the fluctuation contributions in the frequency region (38) are given in (43). Substituting them in (27) and linearizing with respect to the variation of the variables in the sound wave, we can obtain the corrections to the dispersion laws of the longitudinal and transverse sound waves. The corrections to the sound-wave velocities take in this limit the form

$$\delta c_t = \frac{\omega^{3/2} T}{240\pi\rho c_t}\left[\frac{1}{\Gamma_\theta^{\parallel 3/2}} + \frac{2}{\Gamma_\theta^{\perp 3/2}} + \frac{4\sqrt{2}}{(\Gamma_\theta^{\parallel}+\Gamma_\theta^{\perp})^{3/2}}\frac{(c_\parallel^2-c_\perp^2)^2}{c_\parallel^2 c_\perp^2}\right],$$

$$\begin{aligned}\delta c_l = \frac{\omega^{3/2} T}{16\pi\rho c_l}\Bigg\{&\frac{1}{4}(\varphi_2-\varphi_1)(1+\varphi_2-\varphi_1)\left(\frac{1}{\Gamma_p^{\parallel 3/2}}+\frac{2}{\Gamma_p^{\perp 3/2}}\right)\\ &+\frac{1}{4}\left[(\varphi_1+\varphi_2+2\gamma_\parallel)^2-\frac{1}{3}(\varphi_1+\varphi_2+2\gamma_\parallel)+\frac{2}{15}\right]\frac{1}{\Gamma_\theta^{\parallel 3/2}}\\ &+\frac{1}{2}\frac{1}{\Gamma_\theta^{\perp 3/2}}\left[(\varphi_1+\varphi_2+2\gamma_\perp)^2-\frac{1}{3}(\varphi_1+\varphi_2+2\gamma_\perp)+\frac{2}{15}\right]\\ &+\frac{4\sqrt{2}}{15(\Gamma_\theta^{\parallel}+\Gamma_\theta^{\perp})^{3/2}}\frac{(c_\parallel^2-c_\perp^2)^2}{c_\parallel^2 c_\perp^2}+\frac{\sqrt{2}}{3}\varphi_1\left[\frac{c_\parallel^2}{c_\perp^2}\frac{\varphi_1}{(\Gamma_p^{\perp}+\Gamma_\theta^{\perp})^{3/2}}\right.\\ &\left.+2\frac{c_\perp^2}{c_\parallel^2}\frac{\varphi_1}{(\Gamma_p^{\perp}+\Gamma_\theta^{\perp})^{3/2}}+\frac{(c_\parallel^2-c_\perp^2)^2}{c_\parallel^2 c_\perp^2}\frac{1}{(\Gamma_p^{\perp}+\Gamma_\theta^{\parallel})^{3/2}}(\varphi_1-2\gamma_\perp+2\gamma_\parallel)\right]\\ &-\frac{\sqrt{2}}{3}\varphi_1^2\frac{P+E_0}{\rho c_l^2}\left[\frac{1}{(\Gamma_p^{\parallel}+\Gamma_\theta^{\parallel})^{3/2}}+\frac{2}{(\Gamma_p^{\perp}+\Gamma_\theta^{\perp})^{3/2}}\right]\Bigg\}.\end{aligned} \quad (48)$$

The imaginary part of the spectrum is now obtained in accord with (47).

We see that the denominators of (48) contain the coefficients Γ. At least for $\Gamma_0^\perp$ and $\Gamma_0^\parallel$ it can be stated that they tend to zero as the antiferromagnetic transition point is approached. The fluctuation corrections thus increase when the transition point is approached, and this obviously does not depend on the model chosen by us and takes place in any case. This increase gives grounds for hoping to observe the effect in experiment.

[1)] The group of transformations of the crystal represents the atomic planes, and it can therefore be assumed that x_μ with respect to the index μ is the basis of the representation of the crystal point group.

[2)] The energy density E should be an invariant of the crystal point group, and this determines the possible contractions with respect to the index μ; E should also be an invariant of the rotation group, and this determines the possible contractions with respect to the spatial indices. We note that in the presence of vector variables the dependence of E on ∇x_μ does not reduce, generally speaking, to a dependence on the strain tensor $u_{\mu\nu}$.

[3)] It must be borne in mind here that besides that main contribution (16) to the energy and momentum densities, the fluctuation make also contributions in the form of gradients.

[4)] It is assumed in (46) and (47) that $c \ll c_t$ and $c \ll c_l$.

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