The influence of correlation effects on phase transitions in quasi-two-dimensional semimetals in a strong magnetic field

I. V. Lerner and Yu. E. Lozovik

Institute of Spectroscopy, USSR Academy of Sciences
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The rearrangement of the ground state and the phase transitions in a quasi-two-dimensional semimetal in a strong transverse magnetic field are considered. The transitions are due to redistribution of the electrons between the Landau levels as a result of electron-hole pairing. It is shown that in the transition regions the electrons and holes form a nearly ideal exciton gas with Bose longwave properties, the nonideality being due only to virtual interlevel transitions. Without account of these transitions the ground state of the system is set up exactly: it is proven that any diagram correction to its energy is zero. The correlation effects at $T \neq 0$ are decisive for a phase transition which is of the nature of the topological transition usually encountered in two-dimensional degenerate systems. It is shown that many thermodynamic characteristics do not depend on the presence of a phase transition but are defined (even in the ideal gas approximation) by the regions of existence of short-range order which grow on decrease of the temperature.

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1. A two-dimensional semimetal in a strong magnetic field H oriented perpendicular to the plane of motion of the charge carriers is an interesting example of a system in which the bare spectrum of the charge carriers (electrons and holes) is completely discrete: the particle energy depends only on the Landau level number and is infinitely degenerate with respect to the projection of the angular momentum on the direction of H or, equivalently, the position of the center of the particle orbit.[1] Nevertheless the ground state of the noninteracting many-particle system is nondegenerate, as is indeed typical of a two-band [i.e., electron hole $(e-h)$] system in equilibrium. The point is that in such a system the Fermi level is of course independent of momentum and lies (for the case of equal masses of particles and holes) in the middle of the band overlap region. Therefore the Landau levels with energies below the Fermi energy are densely filled with electrons, while the rest are empty at zero temperature. [In the hole representation the Landau levels of the second (hole) band with energies above the Fermi energy are densely filled with holes.] Consequently, the state of the system is unique, so that the Coulomb interaction only shifts the energy of each level and cannot change this state substantially, provided the energy shift is small compared to the interval between Landau levels.

However, the Coulomb interaction does change the energy spectrum qualitatively; as we will show, discrete levels are transformed into bands with a continuous dependence on momentum. The lowest branch of the excitation spectrum is due to transitions of an electron in an upper filled level of one band into a low free level of the other band [where the filled level can be either in the first (electron) or the second (hole) band] and the binding of this electron with the hole so formed. The exciton formed as a result has a smooth spectrum with a continuous dependence on the effective momentum. The dispersion law is positive if the electron is in "its own" (the first) band and the hole in the second, negative (with a maximum at zero momentum) in the opposite case.

The gap in the spectrum is decreased (by comparison with the bare gap, which is equal to the spacing between neighboring unperturbed levels) by the binding energy of the exciton. When the binding energy (which is proportional to $H^{1/2}$) is equal to the spacing between levels (which changes linearly with H), the ground state of the system is reorganized: it becomes energetically favorable for electrons in the highest filled levels to make transitions into the lowest free levels. The loss of energy resulting from the change in population of the levels is compensated by the gain from $e-h$ pairing. It is clear that the bare ground state of the system is strongly degenerate in the regions where the levels are not densely filled.

In regions where there is no reorganization, the Coulomb interaction can be taken into account exactly at low temperatures[2] by the self-consistent field (Hartree-Fock) approximation, if we neglect the virtual transitions between empty and densely filled levels, whose contribution in strong fields is governed by the small parameter

$$r_H/a_\sigma \ll 1. \tag{1}$$

[Here $r_H = (c/eH)^{1/2}$, $a = 1/m_\sigma \tilde{e}^2$, m_σ is the bare effective mass in the electron ($\sigma = 1$) or hole ($\sigma = 2$) bands, $\tilde{e}^2 = e^2/\varepsilon$, and ε is the effective dielectric constant: we set $\hbar = 1$.]

In the regions where the ground state is reorganized, the Hartree-Fock approximation is insufficient. Moreover, in these regions the infinite degeneracy of the ground state means that the usual diagram technique is inapplicable. At the same time it is obvious that it is precisely in these regions, where the $e-h$ pairing is important and the number of filled levels is changing, that the most interesting physical effects occur.

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In the present paper we investigate the thermodynamic properties and excitation spectrum of a quasi-two-dimensional metal in a strong magnetic field [satisfying condition (1)], taking into account systematically correlation effects for arbitrary values of the parameters, and including the region of $e-h$ pairing. The construction of a convergent temperature diagram expansion turns out to be possible, if we take as our zero-order approximation the Hartree-Fock approximation including exciton pairing in the regions where it exists. We have previously developed an analogous method to discuss the nonequilibrium two-dimensional $e-h$ system in semiconductors in the lowest Landau level.[3] As well as the obvious physical differences between the equilibrium case and the nonequilibrium one, where the number of particles is determined by optical pumping, there are additional difficulties in the discussion of semimetal systems arising from the fact that the lowest Landau level is always densely filled and effects due to $e-h$ pairing are possible only in subsequent levels.

The correlation effects turn out to be extremely important; they determine not only the thermodynamic properties such as specific heat and compressibility at low temperatures, but also the nature of the phase transition. The phase transition turns out to be not of second order (as it would be in the Hartree-Fock approximation) but a topological transition of the kind characteristic of degenerate two-dimensional systems.[4,5] The point is that the system is equivalent to a weakly nonideal gas of excitons [the departure from ideality being indicated by the small parameter (1)] with long-wavelength properties of Bose type, and undergoes phase transitions of the same type as in a nonideal two-dimensional Bose gas. To the extent that we neglect inter-level transitions, the system is equivalent to an ideal gas, and its ground state can be found *exactly* in the regions of reorganization. (We show in Appendix I that at $T = 0$ all diagrammatic corrections to the energy of the groundstate defined by the Hartree-Fock approximation with $e-h$ pairing vanish.)

In an ideal Bose gas at rest there are no vortex excitations, since such a system does not undergo topological (or indeed any other) phase transitions. However, the analysis of correlation functions carried out here shows that in an ideal Bose gas there are regions of existence of short-range order. In the limit $T \to 0$ the correlation length defined by the dimensions of these regions tends to infinity. The presence of short-range order even in this approximation is responsible for a number of thermodynamic properties of the system.

In fields satisfying condition (1) the semimetal is found to be in the dielectric state (i.e. the linear response to an infinitesimal electric field is zero). The dielectric character of the regions with excitonic pairing is obvious; in the regions where there is no pairing all bands (Landau levels) are either densely filled or empty. The conductivity in a finite field E can occur only through tilting of the bands in the field and must have a nonlinear dependence on E.

The results obtained in this paper should describe qualitatively the behavior of strongly anisotropic metals in a strong magnetic field. The binding energy of an exciton in such a system[6] is equal to the binding energy of a two-dimensional exciton in a strong field,[7] provided we neglect corrections involving the small parameter $(m_{xy}/m_z)^{1/3}$ due to the motion of the particle in the z-direction parallel to the field. The importance of taking the mass anisotropy into account when describing the transition of a semimetal to the excitonic phase is immediately clear from the fact that experimentally it is only anisotropic semimetals which undergo this transition.[8] There is also experimental support[9] for the idea that a two-dimensional description grasps the essence of important features of anisotropic objects: the magnetic freezing-out of electrons in an anisotropic semiconductor turns out to be proportional to the characteristically "two-dimensional" parameter $H^{1/2}$, rather than to $\ln H^2$ as would be expected in the isotropic case. It should be emphasized that in a strongly anisotropic system, in contrast to a purely two-dimensional system, there may exist long-range order, so that the phase transition has a different character. However, the low-temperature properties (and *a fortiori* the $T = 0$ properties), which are determined by the presence of short-range order, must be qualitatively the same in the two cases.

We note that the physical properties of the system considered here are qualitatively different from the properties of a three-dimensional system with $e-h$ pairing both in a strong magnetic field[10,11] and without a field.[12] The difference is due primarily to the zero dimensionality of the momentum space of the system in question. Mathematically the difference manifests itself in the choice of diagrams; the pole part of the Green's function in this case does not depend on the momentum ("quasi-zero-dimensionality"), so the choice of diagrams has nothing in common with that in systems with nonzero dimensionality in momentum space (see Appendix 1).

2. At $T = 0$ n levels are densely filled, and the remaining levels are empty, with

$$\xi_n<0, \quad \xi_{n+1}>0; \quad 2|\xi_n|>E_n, \quad 2\xi_{n+1}>E_{n+1}. \tag{2}$$

Here we have confined ourselves for simplicity to the case of equal masses, $m_1 = m_2 \equiv m_0$. We also set $\xi_n = n\omega_0 - \varepsilon_n - \mu$, where ξ_n is the energy of the n-th Landau level measured from the chemical potential $\mu = E_g/2$. [$E_g \equiv E_g(H) = E_g(0) - \omega_0 + \mu_B{}^*H$ is the overlap of bands in a field H, where $\mu_B{}^*$ is the effective Bohr magneton.] We also set $\omega_0 = eH/m_0c$ and denote by ε_n the normal Coulomb renormalization (i.e. that not associated with exciton pairing) of the energy of the n-th level.[2] The first pair of inequalities in (2) is the condition that exactly n Landau levels lie below the Fermi surface, the second ensures the impossibility of $e-h$ pairing in the groundstate in levels close to the Fermi level. With this condition the increase in energy E_n due to $e-h$ pairing (that is, the binding energy of an exciton in the n-th level[7]) cannot compensate the decrease due to the transition of the electron to a higher level; here $E_n = E_0 I_{nn}$, where $E_0 = (\bar{e}^2/r_H)(\pi/2)^{1/2}$ and analytic expressions for the coefficients I_{nn} were obtained in Refs. 2 and 7.

If the condition

$$2|\xi_n| \leqslant E_n \tag{3}$$

is fulfilled, a transition to the excitonic phase is possible in the n-th level. In that case the occupation of the upper level at $T=0$ changes linearly from zero (for $\xi_n = E_n/2$) to N_0 (for $\xi_n = -E_n/2$), where $N_0 = L^2/2\pi r_H^2$ is the degree of degeneracy of the Landau level (L is the linear dimension of the system). When the filling of the level is incomplete the state of the many-particle system neglecting interactions is strongly degenerate, and the diagram technique based on the usual separation of the Hamiltonian into $\mathscr{H}_0$ and $\mathscr{H}_{int}$ becomes useless. A self-consistent field approximation which takes into account the possibility of $e-h$ pairing is inadequate to describe the properties of the system for $T \neq 0$, but, as will be shown below, it is exact for $T=0$ if we neglect inter-level transitions, and moreover is a suitable initial approximation for the construction of a convergent perturbation theory at arbitrary T; all diagrams below will be constructed on the basis of the Green's functions given by this approximation.

For future convenience we write the Green's functions of the excitonic state in the Hartree-Fock approximation, which are obtained from the solution of equations[2] analogous to the Gor'kov-Éliashberg equations[13] in the theory of superconductivity, in the following form:

$$G_{\sigma\sigma'}(\omega, p_x; y, y') = \sum_{j=0}^{\infty} g^j_{\sigma\sigma'}(\omega)\chi_j(p_x, y)\chi_j(p_x, y');$$

$$g^j_{\sigma\sigma'} = g_{\sigma\sigma'}(\eta_j)g_j^+(\omega) + g_{\sigma\sigma'}(-\eta_j)g_j^-(\omega), \tag{4}$$

$$g_j^{\pm}(\omega) = (i\omega \pm \eta_j)^{-1}, \quad g_{\sigma\sigma'}(\eta_j) = \frac{1}{2\eta_j}\begin{bmatrix} \eta_j+\xi_j & -\Delta_j \\ -\Delta_j & \eta_j-\xi_j \end{bmatrix}.$$

Here $\chi_j(p_x, y)$ are the oscillator wave functions[1] with orbit centers at the points $p_x r_H^2$, Δ_j is the order parameter characterizing the $e-h$ pairing in the j-th level, and the quantity $\eta_j \equiv (\xi_j^2 + \Delta_j^2)^{1/2}$ satisfies the following self-consistency condition:

$$\Delta_j = E_0 \sum_{i=0}^{\infty} \frac{\Delta_i J_{ij}}{2\eta_i} \operatorname{th}\left(\frac{\eta_i}{2T}\right). \tag{5}$$

If we neglect corrections of order of the parameter (1), it is sufficient to keep only the term with index $i=n$, where n is the number of the uppermost of the filled levels. In this approximation the transition temperature is obtained from (5) in the form

$$T_n = \xi_n \ln^{-1}\left(\frac{E_n+2\xi_n}{E_n-2\xi_n}\right) \leqslant \frac{E_n}{4}. \tag{6}$$

By carrying out an analysis of the vertex diagrams, which in the region of reorganization ($T < T_n$) is analogous to the diagram analyses for the nonequilibrium $e-h$ system in semiconductors in the lowest Landau level,[3] we can verify that the contribution of the particles in the uppermost filled level to any non-ladder diagram is proportional to the parameter

$$\alpha_n = \frac{E_n}{4T}\left[1 - \operatorname{th}^2\left(\frac{\eta_n}{2T}\right)\right] \leqslant 1, \tag{7}$$

which for $T \ll E_n$ becomes

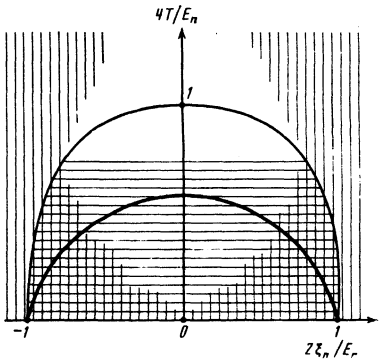

FIG. 1. The phase diagram in a semimetal with n filled levels. The upper curve is the Hartree-Fock temperature T_n; the lower curve is the temperature of the topological phase transition as calculated in the small T region (cf. footnote 4) (the extension of this curve into the region $2\xi_n/E_n \sim 1$ is an extrapolation). In the vertically shaded region one can choose a diagram expansion corresponding to the usual separation of the Hamiltonian into $\mathscr{H}_0$ and $\mathscr{H}_{int}$. In the horizontally shaded region a convergent expansion is obtained if exciton pairing is taken into account in the zeroth approximation, with the expansion parameter $\alpha < 1/2$. In the unshaded region we have $1/2 \leqslant \alpha \leqslant 1$.

$$\alpha_n = \frac{E_n}{T}\exp\left(-\frac{E_n}{2T}\right) \ll 1. \tag{8}$$

For $T > T_n$ (when $\Delta = 0$) the parameter (7) goes over into

$$\alpha_n = \frac{E_n}{T} f_n(1-f_n), \tag{9}$$

where $f_n \equiv f_n(T)$ are the Fermi occupation coefficients of the n-th level. In the region (2) for $T=0$ the parameter (9) vanishes, since the n-th level is either completely filled or empty; for sufficiently low nonzero temperatures the parameter (9) is exponentially small.[1)]

At high T this parameter is small because of the factor E_n/T. Thus, over a wide region of the parameters (see Fig. 1) it is sufficient to restrict ourselves to a consideration of the ladder diagrams, while the non-ladder diagrams are of the order of the small parameters[2)](7)–(9). This can be confirmed by estimates of the higher-order diagrams, which can be carried out as in Ref. 3. We note that if we were to construct our diagrams for region (3) with the bare Green's functions, or equivalently with the Green's functions of the Hartree-Fock approximation without account of $e-h$ pairing, then the contribution of the non-ladder diagrams would be everywhere proportional to the parameter (9), which for $T < T_n$ is small only in an insignificant part of region (3) (see Fig. 1).

We proceed to sum the ladder diagrams in the channel corresponding to scattering of an electron from the first band with a hole from the second. To sum these diagrams it is convenient to transfer the χ_j-functions to the Coulomb interaction lines and take as Green's functions the quantities $g^j_{\sigma\sigma'}(\omega)$. In region (3), in the expressions (4), it is adequate to restrict ourselves to the term with $j=n$; the contribution of the other terms is governed by the small parameter (1). In this representation the bare Coulomb vertex is

$$\gamma_n(p_x, p_x+q_x; p_x+k_x, p_x+q_x+k_x)\equiv\gamma_n(k_x, q_x)=\int dy\,dy'\,d\varkappa\,\frac{\exp[i\varkappa(y-y')]}{(\varkappa^2+q_x^2)^{1/2}}$$
$$\times\chi_n(p_x, y)\chi_n(p_x+q_x, y)\chi_n(p_x+k_x, y')\chi_n(p_x+k_x+q_x, y'). \tag{10}$$

The Bethe-Salpeter equation for the complete vertex is simply solved in the transverse exciton momentum representation which was introduced for the three-dimensional case in the work of Gor'kov and Dzyaloshinskiĭ,[14] and for the many-particle problem in the work of Brazovskiĭ.[10] The solution reduces to the solution of the following set of equations for the complete 16-component vertex $\Gamma_{\mu\nu}^{\sigma\tau}(n;\varepsilon,\mathbf{k})$

$$\Gamma_{\mu\nu}^{\sigma\tau}(n;\ \varepsilon,\ \mathbf{k})=-E_n\gamma_n(\mathbf{k})\{\delta_{\sigma\tau}\delta_{\mu\nu}+\Xi_{\beta\nu}^{\sigma\alpha}\Gamma_{\mu\beta}^{\alpha\tau}(n;\ \varepsilon,\ \mathbf{k})\}. \tag{11}$$

Here and below a summation over repeated band indices is implied, $\delta_{\sigma\tau}$ is the Kronecker symbol, ε is the even (Bose) "frequency," and all momenta are measured in units of $1/r_H$; $\gamma_n(k)\equiv\gamma_n$ is the bare vertex in the transverse momentum representation:

$$E_n\gamma_n(k)=\tilde{e}^2\int_0^\infty dq\exp\left(-\frac{q^2}{2}\right)\left[L_n\left(\frac{q^2}{2}\right)\right]^2 J_0(kq), \tag{12}$$

where L_n is the Laguerre polynomial.

In (11) we have introduced the following notation for the frequency sums:

$$\Xi_{\mu\nu}^{\sigma\tau}=\Xi_{\mu\nu}^{\sigma\tau}(n;\varepsilon)=T\sum_{\omega}g_{\sigma\tau}^{n}(\omega)g_{\mu\nu}^{n}(\omega+\varepsilon). \tag{13}$$

The calculation of these sums is straightforward. Then solving the system of equations (11), we represent the result in a form suitable for subsequent calculations:

$$\Gamma_{\mu\nu}^{\sigma\tau}(n;\varepsilon,k)=\frac{E_n\gamma_n^2}{2\eta_n}\{\Gamma_n^{+}(\varepsilon,k)\gamma_{\mu\nu}^{\sigma\tau}(\eta_n)-\Gamma_n^{-}(\varepsilon,k)\gamma_{\mu\nu}^{\sigma\tau}(-\eta_n)\}, \tag{14}$$

where

$$\Gamma_n^{\pm}(\varepsilon,k)=(i\varepsilon\pm2\eta_n[1-\gamma_n(k)])^{-1};$$
$$\gamma_{11}^{\alpha\beta}(\eta_n)=-\begin{bmatrix}\Delta_n^2 & \Delta_n(\eta_n+\xi_n)\\ \Delta_n(\eta_n+\xi_n) & (\eta_n+\xi_n)^2\end{bmatrix};$$
$$\gamma_{22}^{\alpha\beta}(\eta_n)=-\begin{bmatrix}(\eta_n-\xi_n)^2 & \Delta_n(\eta_n-\xi_n)\\ \Delta_n(\eta_n-\xi_n) & \Delta_n^2\end{bmatrix},$$
$$\gamma_{12}^{\alpha\beta}(\eta_n)=\gamma_{21}^{\alpha\beta}(\eta_n)=\begin{bmatrix}(\eta_n-\xi_n)\Delta_n & \Delta_n^2\\ \Delta_n^2 & (\eta_n+\xi_n)\Delta_n\end{bmatrix}. \tag{15}$$

Here we have omitted exponentially small terms proportional to α_n. As a result of a trivial analytic continuation of the vertex we find the elementary excitation spectrum:

$$\mathscr{E}_n(k)=\pm E_n(1-\gamma_n(k)). \tag{16}$$

Here $\gamma_n(k)$ is given by expression (12), and we have used the fact that at low temperatures the parameter η_n is equal to $E_n/2$ within terms of the order of the small parameter (8). The energy spectrum (16) agrees with the dispersion relation for a single exciton in the n-th Landau level as calculated from the binding energy[7]; this indicates the equivalence of the system to an ideal gas of such excitons (when inter-level transitions are neglected).[3)] The double-valued character of the dispersion law (16) is due to the symmetry of the system relative to the replacement $\xi_n\to-\xi_n$. The point $\xi_n=0$ corresponds to a half-filled n-th Landau level. For $\xi_n>0$, when the level is less than half filled, the excitations are the usual excitons, corresponding to the plus sign in (16). For $\xi_n<0$, when the number of particles in the n-th level is greater than $N_0/2$, it is convenient to measure the energy from the energy of a completely filled level ($-N_0E_n$); then the elementary excitations are "antiexcitons" with a negative dispersion (16), formed by a hole from the first, electron, band and an electron from the second, hole, band.

The dispersion law (16) is nonmonotonic[7] in k; there are n additional subsidiary minima. However, it is clear that for small T the principal contribution to the thermodynamics comes from excitations with small momenta $k\ll1$. We carry out an expansion of $\mathscr{E}_n(k)$ for small k and also an asymptotic expansion for $k\gg1$, the latter important for subsequent estimates of integrals:

$$\mathscr{E}_n(k)=\begin{cases}k^2/2M_n, & k\ll1\\ E_n-(\tilde{e}^2/r_H)k^{-1}, & k\gg1.\end{cases} \tag{17}$$

Here the effective mass of an exciton in the n-th level is given for small momentum by

$$\frac{1}{2M_n}=\frac{E_0r_H^2}{4}\left\{\left[\frac{(2n-1)!!}{(2n)!!}\right]^2(2n^2+4n+1)\right.$$
$$\left.-\sum_{j=0}^{n-2}\left[\frac{(2j+1)!!}{(2j)!!}\right]^2\frac{(2n-2j-3)!!}{(2n-2j)!!}\right\}.$$

We note that the function (12), and hence the spectrum (16), may be calculated analytically. The result is expressed in the form of some combination of modified Bessel functions and exponentials and is very complicated, so we shall not give it, since it is not needed in what follows. The corresponding expressions for some low levels were given in Ref. 7.

Expression (14) determines the vertex part for $T\leq T_n(\xi_n)$. We give the expression for the $e-h$ vertex for $T\geq T_n$, which is obtained by direct summation of the ladder diagrams:

$$\Gamma_{22}^{11}(n;\varepsilon,k)$$
$$=-E_n\gamma_n(k)\left\{1-\frac{E_n\gamma_n(k)\operatorname{th}(\xi_n/2T)}{i\varepsilon-2\xi_n+E_n\gamma_n(k)\operatorname{th}(\xi_n/2T)}\right\}. \tag{18}$$

[For $T=T_n$ expressions (14) and (18) are of course identical.] The vertex component Γ_{11}^{22} is obtained from (18) by the replacement $\xi_n\to-\xi_n$, and the other components are zero in region (2). The pole of the expression (18) for $\xi_n>0$ gives, after analytic continuation into the upper half-plane, an excitonic excitation spectrum which for $T=0$ differs from the spectrum (16) by the presence of a gap $2\xi_n-E_n>0$:

$$\mathscr{E}_n(k)=2\xi_n-E_n+E_n(1-\gamma_n(k)). \tag{19}$$

In this region the pole of the vertex $\Gamma_{11}^{22}(\varepsilon,k)$ defines an analogous "antiexcitonic" spectrum with a negative dispersion law and gap, corresponding to the excitation of a pair composed of an electron from the hole band and a hole from the electron band. (In the region $-E_n/2<\xi_n<0$ the role of the vertices Γ_{11}^{22} and Γ_{22}^{11} is reversed.)

From this point of view the reorganization of the ground state corresponds to a "softening" of the branch (19) of oscillations, and a Bose condensation of the oscillations corresponding to it, the excitons. (It will be shown below that the long-wavelength excitons are of Bose type.) We note that the Hartree-Fock transition

temperature (6) can be obtained from the condition that a pole appears at $\varepsilon = k = 0$ in the vertex (18).

3. Let us now calculate the correlation free energy. A fairly straight-forward analysis shows that the diagrams of the random phase approximation (that is, the ring diagrams constructed from one-loop polarization operators) give a contribution to the energy which is determined by the small parameter α_n. It is clear that ladder corrections to the simple polarization operator must be important. In Appendix 1 we prove that the principal contribution to the correlation free energy is given by the sum of diagrams of Fig. 2(a), just as in the case of a nonequilibrium $e-h$ system in the lowest Landau level.[3] The analytic sum of these diagrams is given by the following expression:

$$F_{corr} = \frac{1}{2} N_0 E_n T \sum_{\omega} \int_0^1 \frac{d\lambda}{\lambda} \int dk k \lambda \gamma_n(k) \Pi_n(\varepsilon, k; \lambda). \tag{20}$$

Here the ladder polarization operator [Fig. 2(b)] is

$$\Pi_n(\varepsilon, k) = T^2 \sum_{\omega,\omega'} \{g_{\sigma\nu}^n(\omega) g_{\mu\sigma}^n(\omega+\varepsilon) g_{\beta\tau}^n(\omega')$$
$$\times g_{\tau\alpha}^n(\omega'+\varepsilon) [\Gamma_{\nu\mu}^{\alpha\beta}(n; \omega-\omega', k) - E_n \gamma_n(k) \delta_{\alpha\beta}\delta_{\nu\mu}]\}. \tag{21}$$

The dependence of the polarization operator on the coupling constant λ in (20) is given by multiplying by λ all the bare vertices γ_n in the final expression for $\Pi_n(\varepsilon, k)$. To carry out the sum over band indices σ and τ in (21) it is convenient to use the following easily verified summation formula:

$$g_{\mu\sigma}^n(\omega) g_{\sigma\nu}^n(\omega') = g_n^+(\omega) g_n^+(\omega') g_{\mu\nu}(\eta_n) + g_n^-(\omega) g_n^-(\omega') g_{\mu\nu}(-\eta_n). \tag{22}$$

We note that a simple generalization [Eq. (33)] of formula (22) to the case of an arbitrary number of Green's functions serves as a basis for the analysis of the high-order non-ladder diagrams. The remaining summations over the indices ν, μ, α, β reduce to multiplication of matrices of type (4) and (15). As a result we get the following expression:

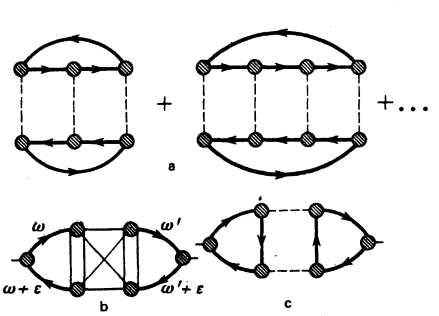

FIG. 2. a) Diagrams giving the principal contribution to the correlation free energy. Full lines denote the Green's functions $g_{\sigma\sigma'}(\omega)$, broken ones the Coulomb interaction (10). At the vertices summation is carried out over the band indices $\sigma = 1, 2$. b) The polarization operator in terms of which the sum of the diagrams a is expressed. The square denotes the vertex $\Gamma_{\mu\nu}^{\sigma\tau}$, with indices from a single pair joined by double lines. c) For clairity we show the first approximation to the polarization operator b.

$$\Pi_n(\varepsilon, k) = 2E_n \eta_n \gamma_n^2(k) T^2 \sum_{\omega,\omega'} \{g_n^+(\omega) g_n^+(\omega+\varepsilon)$$
$$\times \Gamma_n^-(\omega-\omega', k) g_n^-(\omega') g_n^-(\omega'+\varepsilon) - g_n^-(\omega) g_n^-(\omega+\varepsilon)$$
$$\times \Gamma_n^+(\omega-\omega', k) g_n^+(\omega') g_n^+(\omega'+\varepsilon)\}.$$

After summation over frequencies and some elementary transformations we finally get

$$\Pi_n(\varepsilon, k) = -\frac{2\alpha_n T \gamma_n}{\varepsilon^2 + 4\eta_n^2 \gamma_n^2} \frac{\mathrm{sh}[\eta_n \gamma_n / T]}{\mathrm{sh}[\eta_n (1-\gamma_n)/T]}$$
$$+ \frac{\delta_{\varepsilon 0} \alpha_n}{4} \left\{ \frac{2}{\eta_n} \mathrm{cth} \left[\frac{\eta_n (1-\gamma_n)}{T} \right] - \frac{2-\alpha_n \gamma_n}{E_n} \right\}. \tag{23}$$

Substituting (23) in (20), we get after summation over frequencies and integration over the coupling constant

$$F_{corr} = N_0 T \int_0^\infty dk k \left\{ \ln \left[\frac{\mathrm{sh}(\eta_n(1-\gamma_n)/T)}{\mathrm{sh}(\eta_n/T)} \right] + \frac{E_n \gamma_n}{4T} \left(1 + \frac{4\eta_n^2}{E_n^2} \right) + \frac{\gamma_n^2 \alpha_n^2}{8} \right\}. \tag{24}$$

Using the asymptotic form (17) we can convince ourselves that for large k the integrand is proportional to $(E_n/T)\alpha_n^2 k^{-3}$, i.e. the integral converges and the contribution to the result from its short-wavelength part is exponentially small (which is quite natural, since states with large momentum are occupied with exponentially small probability at low T). The principal contribution to the integral (24) comes from states with $k \ll 1$. As a result of straightforward calculations we get

$$F_{corr} = -4N_0 \eta_n \int_0^1 \left[\exp\left(\frac{2\eta_n u}{T} \right) - 1 \right]^{-1} u du = -\frac{\pi^2 T^2 N_0}{3E_n}. \tag{25}$$

It is important that the above integral depends only exponentially weakly on its upper limit, provided that this limit is substantially larger than the quantity $T/E_n \ll 1$; this dependence is omitted in Eq. (25).

As is the case of a nonequilibrium $e-h$ system in the lowest Landau level, the energy (25) is the free energy of an ideal gas of excitons whose momentum distribution for $k \ll 1$ is of Bose type (for $k \gtrsim 1$ the distribution function is considerably different from the Bose one). The expression (25) determines the correlation specific heat C_v, which is linear in T, the correlation pressure, which is proportional to T^2, and so on; the corresponding Hartree-Fock quantities are exponentially small at low temperatures.[2] However, the principal contribution to quantities which depend weakly on temperature (such as the magnetic susceptibility in a strong field) is given by the Hartree-Fock approximation, while the contribution of correlation effects to these is determined by the small parameter $(T/E_n)^2 \ll 1$.

Inter-level transitions make the exciton gas weakly nonideal, so that its low-temperature properties are substantially changed. The calculation of the corrections proportional to powers of the parameter (1) at $T = 0$ is straightforward in principle, though extremely cumbrous. The corrections to the purely Hartree-Fock results are obtained by retaining in Eq. (5) all terms, not just the term with $i = n$. We note that the normal renormalization of the Landau levels do not change up to first order in r_H/a_0. Moreover, there are correlation corrections to the free energy; the contribution of the first correction in r_H/a_0 is given at $T = 0$ by the two second-order vacuum diagrams, just as in the semiconductor case.[15] We will not carry out the correspond-

ing calculations here, since in this context the difference between the system in question and the nonequilibrium exciton gas in semiconductors[3] is purely quantitative. The only point of importance for what follows is that the system is equivalent to a weakly nonideal gas of excitons with long-wavelength Bose properties, the departure from ideality being determined by the small parameter (1).

4. The thermodynamic properties of the system in question may be calculated assuming the existence of a condensate. However, since the system is equivalent to an ideal (or, if we take account of inter-level transitions, weakly nonideal) Bose gas, a condensate cannot exist for $T \neq 0$; it is destroyed by density fluctuations. Nevertheless, these fluctuations give a small contribution to the free energy, so that Eq. (25) remains true. The point is that it is sufficient for the validity of this formula that short-range order be formed in the system. We shall now demonstrate that for an ideal two-dimensional Bose gas at sufficiently low temperatures regions of short-range order are formed, the dimensions of which increase without limit as $T \to 0$; we shall also discuss the effects associated with weak nonideality.

The chemical potential of an ideal two-dimensional Bose gas is

$$\zeta = T \ln [1 - \exp(-T_0/T)], \tag{26}$$

which is exponentially small for $T \ll T_0$, where $T_0 = 2\pi\rho/M$ is the temperature at which, as will become clear below, short-range order is formed (here ρ is the two-dimensional density and M the mass of the bosons). The free energy is

$$F = -\frac{MT^2}{2\pi} g_2\left(-\frac{\zeta}{T}\right) - \zeta N,$$

where for $T \ll T_0$ goes over, apart from exponentially small terms ($\sim\zeta$), into formula (25). Here we used the asymptotic expression[16] for the Bose integral

$$g_2(x) = \sum_{s=1}^{\infty} \exp(-sx)\, s^{-2}.$$

We now calculate the two-particle correlation function of an ideal Bose gas:

$$K(\mathbf{r}) = \int \frac{d^2k}{(2\pi)^2} \frac{\exp(i\mathbf{kr})}{\exp(k^2/2MT - \zeta/T) - 1}. \tag{27}$$

In Appendix 2 it is shown that for $T \lesssim T_0$ we have

$$K(r) = \begin{cases} \rho \dfrac{T}{T_0} \ln(Ar_0/r), & (\pi\rho_0)^{-1} T/T_0 \ll r \ll r_0 \quad (28a) \\ \rho \dfrac{T}{T_0} \left(\dfrac{2\pi r_0}{r}\right)^{1/2} \exp\left(-\dfrac{r}{r_0}\right), & r \gg r_0, \quad (28b) \end{cases}$$

where the correlation radius is given by

$$r_0 = (-2M\zeta)^{-1/2} = (2MT)^{-1/2} \exp(T/2T_0). \tag{29}$$

The condition for existence of the second asymptotic expression in (28), namely $r_0 \gg (2MT)^{-1/2}$, is automatically fulfilled for $T \lesssim T_0$. Thus, for $T \sim T_0$ regions of short-range order (28a) are formed in the system, in which correlation plays an essential role. The dimensions (29) of these regions grow without limit as $T \to 0$. However, the very long-distance asymptotic, (28b), is exponentially decreasing, since in this case, in contrast to the nonideal two-dimensional Bose gas,[4,5,17] there is no phase transition associated with a change in the asymptotic behavior of the correlation functions.

It is well known that the temperature T_{cr} for such a transition in the weakly nonideal two-dimensional Bose gas does not depend on the nonideality parameter (the speed of sound c_0) but is determined only by the mass M and density ρ of the bosons.[4)] Thus we must find the correct procedure for taking the limiting transition from the nonideal gas to the ideal one.

Both in the weakly nonideal and in the ideal gas there exists in regions of finite dimension l a "bare" condensate, that is a macroscopic number of particles are condensed in momentum states with $p \leq 1/l$. In the nonideal gas there is no long-range order, since the phases of the order parameter in different regions are uncorrelated. However, the modulus of the order parameter (i.e., the condensate density) in the nonideal gas does not change on further increase of the dimensions of the regions, once we reach sufficiently large regions (with dimensions $l \gg 1/Mc_0$). It is just this feature which ensures the possibility of a transition from the Bose gas to the equivalent XY model. In an ideal Bose gas the density of the bare condensate decreases continuously with increasing dimension of the regions, if $l \gg (MT)^{-1/2}$, and the transition to the XY model with constant spins is impossible. It is now clear how to take the limiting transition $c_0 \to 0$ from the nonideal gas to the ideal one. For arbitrarily small c_0 the very-long-range asymptotic form for temperatures $T < T_{cr}$ is of power-law form, but the dimensions of the regions in which this asymptotic form appears increase without limit as c_0 decreases.

Formally, the fundamental difference between the critical behavior of a nonideal and an ideal Bose gas lies in the fact that in the first case the long-range order is destroyed for $T \neq 0$ by fluctuations of the phase of the order parameter, while in the second case fluctuations of the density are also important. It is easy to convince oneself of this by going over in the usual way[17] from the Bose ψ_0 operators to "density-phase" variables. In these variables the correlation function can be written in the form of an expansion:

$$\langle \psi_0(\mathbf{r}) \psi_0(\mathbf{r}') \rangle = -\rho_0(k_0) \langle e^{i(\varphi - \varphi')} \rangle + \frac{1}{4\rho_0(k_0)} \langle \pi\pi' e^{i(\varphi - \varphi')} \rangle + \ldots, \tag{30}$$

where $\pi(\mathbf{r})$ is the operator of deviations of the density from the bare condensate density $\rho_0(k_0)$; here primed operators refer to the point $\mathbf{r}'$, unprimed ones to the point $\mathbf{r}$. For a Bose gas with a spectrum $\varepsilon(k)$ the "density-density" Green's function in the momentum representation is

$$\langle \pi(k, \omega) \pi(-k, -\omega) \rangle = \frac{\rho k^2/M}{\omega^2 + \varepsilon^2(k)}. \tag{31}$$

In the nonideal Bose gas with a linear spectrum the correlation $\langle \pi(\mathbf{r}) \pi(\mathbf{r}') \rangle$ is convergent and small if we choose the value of the cutoff k_0 correctly. Thus in the expan-

sion (30) it is sufficient to keep only the first term, which gives the usual power-law asymptotic form. In an ideal gas with a quadratic spectrum the correlation function of the density fluctuations is

$$\langle \pi(\mathbf{r})\pi(\mathbf{r}')\rangle = -\frac{T}{L^3}\sum_{|k|<k_0}\frac{\rho k^2/M}{\omega^2+(k^2/2M)^2}e^{ik(r-r')}\sim\int_0^{k_0}\frac{dk}{k}$$

which is logarithmically divergent. Hence it is necessary to sum all terms in the expansion (30), which must lead to the correct formula (28).

It is important that the principal contribution to all the asymptotics discussed comes from the long-wavelength limit. Consequently, all results are applicable also the the $e-h$ system, which is equivalent in this limit to a gas of excitons with Bose statistics. However, in this system there may occur effects associated with the nonmonotonic behavior of the dispersion law for $k\sim 1$. For sufficiently large T states with large k are occupied, and when the temperature is decreased, it may be possible to form a metastable condensate of excitons in one of the subsidiary minima. In this state of the system each exciton has a dipole moment $\sim kr_H^2$, and there may occur in the gas of interacting dipoles a transition to a metastable ferroelectric state.

APPENDIX 1

We shall prove that in the limit $T\to 0$ any diagram describing corrections to the energy of the exciton condensate vanishes. We will carry out the analysis in the region (3) of incomplete occupancy of the n-th level. Every vacuum diagram is composed of j closed fermion loops ($j=1,2,3\ldots$) connected in a definite way by interaction lines. (Of course, we do not have to consider one-loop diagrams within which the interaction lines do not cross; these diagrams have been included in the initial Hartree-Fock approximation. In the remaining diagrams there should be no separate Hartree-Fock elements.) Let us choose, in an arbitrary diagram, a closed loop of l-th order. In it, independently of the position of the interaction lines and the other loops, we can carry out a summation over the band indices and at least one isolated summation over frequency[5]:

$$\Sigma_l = T\sum_{\omega} g^n_{\sigma_1\sigma_2}(\omega)\,g^n_{\sigma_2\sigma_3}(\omega+\varepsilon_1)\ldots g^n_{\sigma_l\sigma_1}(\omega+\varepsilon_{l-1}). \tag{32}$$

Here ω is an odd frequency and $\varepsilon_1,\ldots,\varepsilon_{l-1}$ are even frequencies. If all interaction lines connect the loop in question with other loops, and there are no internal lines, then all the frequencies ε are independent [the requirement of energy conservation, that the sum of all outgoing frequencies be equal to zero, is automatically fulfilled with the notation (32)]. In the presence of internal interaction lines (obviously, there cannot be less than two of them) certain definite relations are imposed on the frequencies ε, but these relations do not contain ω; this allows us to carry out the summation over ω in (32) independently of the subsequent summations over the ε. For the summation over repeated band indices in (32) we use the following formula:

$$\begin{aligned} &g_{\sigma\sigma_1}(\omega_1)g_{\sigma_1\sigma_2}(\omega_2)\ldots g_{\sigma_j\sigma'}(\omega_j)\\ &=g^+(\omega_1)\ldots g^+(\omega_j)g_{\sigma\sigma'}(\eta)+g^-(\omega_1)\ldots g^-(\omega_j)g_{\sigma\sigma'}(-\eta). \end{aligned}\tag{33}$$

This formula is a generalization of formula (22) which is easily proved by induction [the basis of induction for $j=0$ is the representation of the Green's functions $g_{\sigma\sigma'}(\omega)$ in the form of the expansion (4)], using the following properties of the matrix $g_{\sigma\sigma'}(\eta)$:

$$g_{\sigma\sigma_1}(\eta)g_{\sigma_1\sigma'}(\eta)=g_{\sigma\sigma'}(\eta),\quad g_{\sigma\sigma_1}(\eta)g_{\sigma_1\sigma'}(-\eta)=0. \tag{34}$$

Setting $\sigma=\sigma'$ in (33) and carrying out the remaining sum over σ [remembering that $g_{\sigma\sigma}(\eta)=\mathrm{Tr}\,g=1$] we get from (32) after the substitution of $g^{\pm}(\omega)$:

$$\begin{aligned}\Sigma_l = T\sum_{\omega}\Big\{&\frac{1}{i\omega+\eta_n}\frac{1}{i(\omega+\varepsilon_1)+\eta_n}\cdots\frac{1}{i(\omega+\varepsilon_{l-1})+\eta_n}\\ &+\frac{1}{i\omega-\eta_n}\cdots\frac{1}{i(\omega+\varepsilon_{l-1})-\eta_n}\Big\}.\end{aligned}\tag{35}$$

For $T=0$ the sums (35) are replaced by integrals according to the usual prescription:

$$T\sum_{\omega}\to\int(d\omega/2\pi).$$

Since all poles of the integrand lie in the right (or left) half-plane and the expression converges sufficiently fast at infinity (at least as ω^{-4}), the integral (35) vanishes. This shows that every correction to the ground-state energy of the exciton condensate (defined in the Hartree-Fock approximation with $e-h$ pairing taken into account, which is the analogue of the BCS approximation for our system) vanishes.

For $T=0$ the sums in (35) are replaced by integrals of the type

$$\int\frac{d\omega}{2\pi i}\Big\{\frac{1}{\omega+\eta_n}\cdots\frac{1}{\omega+i\varepsilon_{l-1}+\eta_n}+\frac{1}{\omega-\eta_n}-\frac{1}{\omega+i\varepsilon_{l-1}-\eta_n}\Big\}\,\mathrm{th}\frac{\omega}{2T} \tag{36}$$

with an integration contour which includes the real axis. The first-order residues cancel one another [they depend on temperature like $\tanh(\eta/2T)$, and at $T=0$ one gets zero]. The higher-order residues are proportional to the various derivatives:

$$\frac{d^l}{d\eta^l}\Big(\mathrm{th}\frac{\eta_n}{2T}\Big)\propto\Big(1-\mathrm{th}^2\frac{\eta_n}{2T}\Big)\frac{1}{T^l}\propto\frac{1}{T^{l-1}}\alpha_n.$$

Consequently, for $T\neq 0$ any diagram is proportional to at least the first power of the parameter α. It is evident that every loop in the diagram gives some nonzero power of the small parameter α. As was shown by a detailed analysis of the vertex diagrams for the $e-h$ system with fixed particle number,[3] the small parameter α arises not only as a result of closing of the fermion lines, but also for any crossing of the interaction lines. Since in closed-loop diagrams there must necessarily be internal crossings (the closed diagrams without crossings were already taken into account in the zeroth Hartree-Fock approximation) any correction diagram must be proportional to α^j with $j\geq 2$. Thus, in any order the leading diagrams are either the one-loop diagrams with a single crossing of the Coulomb lines, or the two-loop ladder diagrams of Fig. 2; these are proportional to α^2. The sum of all diagrams of the first type is also proportional[3] only to α^2. However, when the diagrams of Fig. 2(a) are summed up the exponential smallness disappears, so that they give the leading finite-temperature contribution to the free energy.

We emphasize once again that the vanishing takes place as a result of mutual compensation of all diagrams with the same topological structure and all possible distributions of the band indices $\sigma = 1, 2$ over the vertices of the diagram. (The diagrams of Fig. 2 constitute just such sums; in them it is assumed that the summation over band indices has been carried out.) On the other hand diagrams with fixed band indices, when taken separately, have no particular order of smallness. This circumstance makes preliminary estimates of the diagrams much more difficult. The standard order-of-magnitude estimates are empty of content; the exact compensation of identical diagrams with different band indices can be demonstrated only by an exact calculation based on Eqs. (33)–(35).

Some confusion may be caused by inadequate analogies between the system considered here and other $e-h$ systems. As already emphasized, the distinguishing characteristic of the system considered here (as of the nonequilibrium system of Ref. 3) is its "quasi-zero-dimensionality," which from a mathematical point of view manifests itself in the lack of dependence of the pole parts of the Green's functions (4) on momentum. Had the quantities η in the denominators of the $g_{\sigma\sigma'}(\omega)$ depended on p, formulae (22) and (33) would be inapplicable.

It may seem strange that it is possible to determine exactly the characteristics of a system which undergoes a phase transition (at $T = 0$, in the "external field" ξ). At first sight there occur, in the natural expansion of the energy in Δ near the transition point (i.e., at $T = 0$ in the neighborhood of the points $\xi_n = \pm E_n/2$), diagrams which do not vanish and have no particular order of smallness. Indeed these diagrams do exist, but here as elsewhere the whole point lies in the mutual compensation of (substantial numbers of) diagrams. This compensation is far from simple to see in terms of an expansion in Δ. The graphical expansion of the Green's functions is realized according to Fig. 3, in which the index 1(2) refers to the bare Green's functions of an electron from the first (second) band. Even the simplest diagram of second order in the interaction (the "envelope" diagram) corresponds to 2^4 diagrams with fixed band indices (of which 6 are substantially different). The sum of these diagrams, as was shown above, is equal to zero, but each by itself corresponds to a very cumbrous expression. Of course, the sum of all diagrams obtained from those given above at any order of the expansion in powers of Δ must also vanish. However, even in fourth order in Δ we get 70 diagrams (of which 14 are substantially different). Thus, an estimate of any one particular diagram actually gives no information.

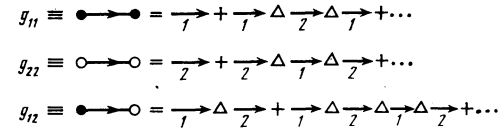

FIG. 3. Expansion of the Green's functions in powers of Δ near the transition point.

In higher orders in the interaction a calculation of diagrams with fixed band indices in isolation and subsequent summation of them becomes totally unrealistic. The possibility of estimating an arbitrary class of topologically equivalent diagrams is based on summing first over the band indices [by formula (33)], and immediately thereafter over the frequencies and momenta.

From a physical point of view, the possibility of an exact description of the ground state of the system considered even near the point at which reorganization is based on its equivalence (neglecting inter-level transitions) to an ideal gas, whose Bose condensation can be described exactly.

APPENDIX 2

Decomposing the denominator of expression (27) in a series, we get after integration

$$K(r)=\frac{MT}{2\pi}\sum_{j=1}^{\infty} j^{-1}\exp\left[\frac{j\zeta}{T}-\frac{r^2MT}{2j}\right]. \tag{37}$$

The expression under the summation sign is a maximum when

$$j\sim j_0=\frac{2b}{(1+4ab)^{1/2}-1}=\begin{cases}a^{-1}(ab)^{1/2}, & ab\gg 1, \quad (38a)\\ a^{-1}, & ab\ll 1, \quad (38b)\end{cases}$$

where we have introduced the dimensionless parameters

$$a=-\zeta/T=-\ln\left[1-\exp\left(-T/T_0\right)\right],\quad b=r^2MT/2.$$

For $T\ll T_0$ the quantity a is exponentially small. Then in (38) we have $j_0\gg 1$ in both cases, so that the sum (37) can be replaced by an integral:

$$K(r)=\frac{MT}{2\pi}\int_1^{\infty}\exp\left(-ax-\frac{b}{x}\right)\frac{dx}{x} \tag{39}$$

with a negligible dependence on the lower limit. In the case (38a) the integral can be calculated by the method of steepest descents. The result is

$$K(r)=\frac{MT}{2\pi}\left(\frac{2\pi r_0}{r}\right)^{1/2}\exp\left(-\frac{r}{r_0}\right), \tag{40}$$

where r_0 is given by expression (29). To calculate $K(r)$ in the intermediate asymptotic region $1\ll b\ll a^{-1}$ it is convenient to represent the integral (39) in the form

$$K(r)=\frac{MT}{2\pi}\left\{\int_0^{\infty}\exp\left(-ax-\frac{b}{x}\right)\frac{dx}{x}-\sum_{n=0}^{\infty}\frac{(-ab)^n}{n!}\Gamma(-n,b)\right\}. \tag{41}$$

Using the well-known expression for the asymptotic form of the incomplete Γ-function, $\Gamma(-n,b)=b^{-n-1}\exp -b$ for $b\gg 1$, we find that the sum in (41) is given by

$$\sum_{n=0}^{\infty}\frac{(-ab)^n}{n!}\Gamma(-n,b)\sim\frac{e^{-b}}{b}\sum_{n=0}^{\infty}\frac{(-a)^n}{n!}=\frac{e^{-ab}}{b}\sim\frac{1}{b}\ll 1.$$

The remaining integral can be evaluated exactly:

$$K(r)=\frac{MT}{2\pi}2K_0(\sqrt{2ab})\approx\frac{MT}{2\pi}\ln\frac{Ar_0}{r}. \tag{42}$$

Here $A=2^{3/2}e^{\gamma}$, γ is Euler's constant and K_0 is the Macdonald function. Combining formulae (40) and (42), we obtain expression (28) given in the main text.

[1)]The contribution of the levels $i \neq n$, which are remote from the Fermi energy by an amount $\sim |i-n|\omega_0$, are exponentially small, being controlled by the parameter $\exp(-\omega_0/T) \ll \alpha_n$.

[2)]For $T \neq 0$ the parameter (7) vanishes, since the ground state of the system is exactly described (in the absence of inter-level transitions) by the Hartree-Fock approximation taking excitonic pairing into account. This assertion is proved in Appendix I with the help of an analysis of the vacuum diagrams.

[3)]In such an interpretation one should represent the ground state energy in the form $E = -N_n E_n + 2N_n\xi_n$ (for $\xi_n > 0$). Here the first term is the total energy of an ideal gas of excitons, while the second is the decrease in energy due to the appearance of particles above the Fermi level. Minimizing this expression with respect to the number N_n of particles in the n-th level (which depends on ξ_n) we find

$$E = -N_0 E_n (1 - 2\xi_n/E_n)^2, \quad N_n = N_0(1/2 - \xi_n/E_n),$$

which agrees with the rigorously derived expressions.[2]

[4)]In the system considered here the mass and density of particles is determined by the magnetic field: $M \propto H^{1/2}$ (formula (17)] and the effective density $\rho_{max} = (N_n/L^2)_{max} \propto H$ (see footnote 3). From the known formula for T_{cr} in a nonideal two-dimensional Bose gas[5] we obtain, for the system in question, $T_{cr} \propto H^{1/2}$. For the analogous system with fixed particle number we have[3] $[T_{cr} \propto H^{-1/2}]$.

[5)]We do not take into account here the contribution to the thermodynamics from inter-level transitions, which vanishes with the small parameter (1). Hence we include in the analysis not the complete Green's functions (4) but their components $g^n_{\omega\omega'}(\omega)$ which describe the distribution of particles within only the n-th Landau level.

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