Damping and frequency shift of phonons of a Wigner crystal interacting with a medium

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The renormalization of the spectrum of a two-dimensional Wigner crystal WC is considered. The renormalization is due to the interaction between the electrons that make up the crystal and the oscillations of the medium. This interaction is substantially nonlinear in the WC phonons. The long-wave phonons are therefore strongly renormalized (in contrast to the renormalization due to the internal anharmonicity of the crystal). The renormalization is calculated with allowance for the non-Born corrections. It is shown that above Wigner-crystal Debye temperature the relaxation has the same character as in the case of electrons that do not interact with one another. At low temperatures the expressions for the relaxation parameters contain explicitly the velocity of the transverse sound of the WC, and the long-wave high-frequency conductivity is inversely proportional to the cube of the frequency.

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Wigner[1] has indicated in 1934 that in the presence of a homogeneous neutralizing background the electrons can form a crystal. To this end the potential energy of their Coulomb repulsion must exceed greatly the kinetic energy. The latter reduces to two inequalities: $T \ll e^2/r_0$ and $r_0 \gg a_B$, where r_0 is the average distance between the electrons and a_B is the Bohr radius (the temperature is measured in energy units and it is assumed that the dielectric constant $\varepsilon = 1$).

It was noted in Refs. 2 and 3 that the crystallization conditions are easier to satisfy for electrons on the interface of liquid and gaseous helium placed in a capacitor, and on the semiconductor surfaces in metal-insulator-semiconductor (MIS) systems. The parameter r_0/a_B is very large ($r_0/a_B \gtrsim 10^4$ for electrons on helium). The neutralizing charge is located on the metallic electrode; its distribution adjusts itself to the electron distribution and is not rigid as in most other systems. Wigner crystallization on a helium surface has already been observed in experiment.[4] The thickness of the electron layer in the indicated systems is small and the transverse motion is quantized. At sufficiently low temperatures and densities, the electrons fill only the lowest energy level and behave in many respects like two-dimensional ($2D$) particles; in particular, they crystallize like $2D$ particles. We note that the two-dimensionality conditions were satisfied in Ref. 4. We shall therefore consider hereafter $2D$-electron systems.

The relaxation of a Wigner crystal (WC) is unique: it is not effected through the crystal boundaries, but is due to volume forces—the interaction of the electrons with the medium. Recognizing that $r_0 \gg a_B$ we can regard the medium (semiconductor or helium) as continuous. Interaction with the oscillations of the medium (MO) leads to damping and to a shift of the frequency of the natural modes of the MO. We analyze below the renormalization with respect to the long-wave modes. It can be experimentally investigated by measuring the absorption spectra of the electromagnetic field. Experiments of this kind were already performed for $2D$ plasma in MIS and on helium.[5,6]

The renormalization of the long-wave WC modes due to the interaction with the MO differs qualitatively from the renormalization due to the internal anharmonicity of the crystal, which decreases rapidly with decreasing wave vector for phonons of the acoustic type.[7] The difference is due to the following: first, the wave vector $\mathbf{q}$ of the MO can greatly exceed the maximum wave vector $k_{max} \sim r_0^{-1}$ of the WC phonons, since $r_0 \gg a_B$; second, since the interaction with the MO is three-dimensional,

$$H_i = \sum_{\mathbf{q}} V_{\mathbf{q}} c_{\mathbf{q}} \rho_{\mathbf{q}}, \quad \rho_{\mathbf{q}} = \sum_n \exp(i\mathbf{q}\mathbf{r}_n), \quad c_{\mathbf{q}} = b_{\mathbf{q}} + b_{-\mathbf{q}}^+ \tag{1}$$

($b_{\mathbf{q}}^+$ and $b_{\mathbf{q}}$ are the creation and annihilation operators of MO with wave vector $\mathbf{q}$, and $\mathbf{r}_n$ are the coordinates of the electrons that make up the WC), it can turn out to be essentially nonlinear in the WC phonons.

The coordinates $\mathbf{r}_n$ are expressed in terms of the phonon creation and annihilation operators $a_{\mathbf{k}j}^+$ and $a_{\mathbf{k}j}$ in the standard manner:

$$\mathbf{r}_n = \mathbf{R}_n + \sum_j \sum_{\mathbf{k}} \mathbf{e}_{\mathbf{k}j} x_{\mathbf{k}j} \exp(i\mathbf{k}\mathbf{R}_n), \quad x_{\mathbf{k}j} = A_{\mathbf{k}j} a_{\mathbf{k}j} + A_{-\mathbf{k}j}^* a_{-\mathbf{k}j}^+. \tag{2}$$

Here $\mathbf{e}_{\mathbf{k}j}$ is the polarization vector of the WC phonon of branch j with wave vector $\mathbf{k}$ (it is assumed that the medium is isotropic and that $|A_{\mathbf{k}j}| = |A_{-\mathbf{k}j}|$, $\mathbf{e}_{\mathbf{k}j} = \mathbf{e}_{-\mathbf{k}j}$).

The nonlinearity of H_i in $a_{\mathbf{k}j}$ and $a_{\mathbf{k}j}^+$ is obviously large if the mean value $q^2 u_n^2 (\mathbf{u}_n = \mathbf{r}_n - \mathbf{R}_n)$ at typical $\mathbf{q}$ is not small. For $2D$ crystals at finite temperatures, the mean squared displacement $\langle u_n^2 \rangle$ diverges, so that the nonlinearity of H_i in u_n must be taken into account in all orders of perturbation theory.[1)] The previous corresponding analysis was carried out in the Born approximation in H_i for the high-frequency branch of a $2D$ crystal in a quantizing magnetic field.[8] It was shown that the renormalization of the spectrum contains no divergences connected with the divergence of the mean squared displacement of the electrons. The authors of a recent paper[9] have reached an analogous conclusion. They have analyzed in the Born approximation the light-absorption peaks observed in Ref. 4 at frequencies close to the MO frequencies, with wave vectors equal to the

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WC reciprocal-lattice vectors. We do not consider these peaks below.

If the inequalities $T \ll e^2/r_0$ and $r_0 \gg a_B$ are satisfied, the relation between T and the characteristic Debye temperature Θ_D of the WC

$$\Theta_D=(e^2/r_0)\,(a_B/r_0)^{1/2} \quad (a_B=\hbar^2/me^2), \tag{3}$$

is obviously arbitrary. The physics of the relaxation processes differs in the cases $T \gg \Theta_D$ and $T \ll \Theta_D$. We propose here a method for going outside the framework of the Born approximation[8,9] and for calculating at different values of T/Θ_D the damping Γ_{kj} and the shift P_{kj} of the WC modes of not too low a frequency, such that

$$\Gamma_{kj},\ |P_{kj}| \ll \omega_{kj}. \tag{4}$$

The results for the long-wave high-frequency conductivity of the WC are valid also when (4) is violated.

In Sec. 1 we obtain a general expression for the Green's function of the WC oscillations. This equation is simplified in Sec. 2 for an actual case when the MO phase velocity is small compared with the speed of sound in the WC. In Secs. 3, 4, and 5 are determined in explicit form Γ_{kj}, P_{kj}, and the conductivity of the WC at $T \gg \Theta_D$ and $T \ll \Theta_D$, respectively. The results are discussed in Sec. 6.

1. CALCULATION OF THE GREEN'S FUNCTIONS FOR WC OSCILLATIONS

The total Hamiltonian of the $2D$ electrons that make up the WC and interact with the medium is of the form

$$\mathcal{H}=H_0+H_v+H_i, \quad H_0=\sum_{kj}\omega_{kj}a_{kj}^+a_{kj}, \quad H_v=\sum_q \omega_q b_q^+ b_q \quad (\hbar=1). \tag{5}$$

Calculation of the renormalization of the mode $\{kj\}$ reduces to an analysis of the equal-time Green's function

$$Q_{kj}(\omega)=\langle\langle x_{kj};\, x_{-kj}\rangle\rangle_\omega, \quad \langle\langle \hat{X}_1;\, \hat{X}_2\rangle\rangle_{\omega=i\nu}=-i\int_0^\infty dt e^{i\omega t}\langle[\hat{X}_1(t),\hat{X}_2(0)]\rangle \tag{6}$$

in the resonant region of frequencies $\omega \sim \omega_{kj}$. The function $Q_{kj}(\omega)$ is the analytic continuation, into the region of imaginary frequencies, of the temperature Green's function

$$G_{kj}(\omega_n)=-\int_0^{1/T} d\tau e^{i\omega_n\tau}\langle x_{kj}(-i\tau)x_{-kj}(0)\rangle, \quad \omega_n=2\pi nT, \quad G_{kj}(\omega_n)=Q_{kj}(i\omega_n). \tag{7}$$

It is also possible to express in terms of $Q_{kj}(\omega)$ the diagonal component of the long-wave conductivity of the WC (see, e.g., Ref. 10):

$$\mathrm{Re}\,\sigma_{xx}(\mathbf{k},\omega)=-\omega e^2N^2S\sum_j (e_{kjx})^2\,\mathrm{Im}\,Q_{kj}(\omega+i\varepsilon), \quad \varepsilon\to+0, \quad k\ll N^{1/2}, \tag{8}$$

where N is the electron density and S is the area of the system.

To calculate $Q_{kj}(\omega)$ with allowance for the interaction it is convenient to separate explicitly in H_i the terms containing the operators $x_{\pm kj}$:

$$H_i=H_i^{(1)}+H_i^{(2)}; \quad H_i^{(\alpha)}=\sum_q V_q c_q \rho_q^{(\alpha)}(\mathbf{kj}), \quad \alpha=1,2, \tag{9}$$

$$\left.\begin{aligned}
&\rho_q^{(1)}(\mathbf{kj})=i(\mathbf{qe}_{kj})\,[x_{kj}\rho_q'(\mathbf{kj})+x_{-kj}\rho_q'(-\mathbf{kj})];\\
&\rho_q^{(2)}(\mathbf{kj})=[1-(\mathbf{qe}_{kj})^2x_{kj}x_{-kj}]\rho_q''; \quad \rho_q'(\mathbf{kj})=\sum_n \exp(i\mathbf{qr}_n'+i\mathbf{kR}_n);\\
&\rho_q''=\sum_n \exp(i\mathbf{qr}_n'); \quad \mathbf{r}_n'=\mathbf{r}_n-\mathbf{e}_{kj}[x_{kj}\exp(i\mathbf{kR}_n)+\mathrm{H.c.}].
\end{aligned}\right\} \tag{10}$$

Representation of H_i in the form (9), (10) does not mean expansion in powers of the displacements of an electron from a WC site. The expansion is only in terms of one Fourier component of the displacement. Since the amplitude A_{kj} is small, $A_{kj} \propto S^{-1/2}$, the terms quadratic in x_{kj} have not been written out in (10) (except for the term $\sim x_{kj}x_{-kj}$) and terms of higher degree in x_{kj}; in the statistical limit $S\to\infty$ their contribution to $Q_{kj}(\omega)$ (as well as to any other Green's function) tends to zero in accordance with Wick's theorem.

The term $H_i^{(1)}$ describes the decay of the considered oscillation $\{\mathbf{kj}\}$ into MO, as well as, generally speaking, of several other WC modes. The term $H_i^{(2)}$ has a structure that is unusual for phonon-interaction problems: one of its terms is proportional to $x_{kj}x_{-kj} \propto S^{-1}$, and the other does not contain S in the denominator at all. Since these terms can be convoluted, the contribution from the term $\sim x_{kj}x_{-kj}$ turns out to be finite [the contribution of the term $\sim x_{kj}^2$ left out of (10) vanishes because the coefficient of x_{kj}^2 contains the momentum $\mathbf{k}$ explicitly, while ρ_q'' does not depend explicitly on $\mathbf{k}$].

The equation of motion for $Q_{kj}(\omega)$ can be easily obtained by differentiating $\langle[x_{kj}(t),x_{-kj}(0)]\rangle$ twice with respect to time, with allowance for (5), (9), and (10), and carrying out the Fourier transformation (6):

$$\left.\begin{aligned}
&(\omega^2-\omega_{kj}^2)Q_{kj}(\omega)=2\omega_{kj}[|A_{kj}|^2+A(\omega)+B(\omega)];\\
&A(\omega)=|A_{kj}|^2\langle\langle \hat{A};\, x_{-kj}\rangle\rangle_\omega; \quad \hat{A}=i\sum_q (\mathbf{qe}_{kj})V_q c_q \rho_q'(-\mathbf{kj});\\
&B(\omega)=|A_{kj}|^2\langle\langle \hat{B}x_{kj};\, x_{-kj}\rangle\rangle_\omega; \quad \hat{B}=-\sum_q (\mathbf{qe}_{kj})^2 V_q c_q \rho_q.
\end{aligned}\right\} \tag{11}$$

The functions $A(\omega)$ and $B(\omega)$ correspond to the terms $H_i^{(1)}$ and $H_i^{(2)}$ in (9), with ρ_q'' in the operator $\hat{B}$ replaced by ρ_q [this replacement, according to (9) and (10), adds to the function $B(\omega)$ terms of order A_{kj}^3 and A_{kj}^4, which vanish in the limit as $S\to\infty$]. The function $B(\omega)$ can be calculated exactly by using the fact that ρ_q is a sum over the electrons, i.e., is an operator additive with respect to the size of the system, and the operators a_{kj}^+ and a_{kj} do not depend on the dimensions of the system. The mean value of the product of the arbitrary additive operators $\hat{X}_{1,2}$ factors out[11]:

$$\langle \hat{X}_1\hat{X}_2\rangle=\langle \hat{X}_1\rangle\langle \hat{X}_2\rangle, \quad S\to\infty. \tag{12}$$

Relation (12) holds also if one of the operators is independent of the size of the system. Consequently

$$\left.\begin{aligned}
&B(\omega)=P_{kj}^{(2)}Q_{kj}(\omega), \quad S\to\infty;\\
&P_{kj}^{(2)}=|A_{kj}|^2\langle\hat{B}\rangle=-|A_{kj}|^2\sum_q (\mathbf{qe}_{kj})^2V_q\langle c_q\rho_q\rangle
\end{aligned}\right\} \tag{13}$$

[a proof of (13) is given in the Appendix].

Since $V_q c_q \rho_q = (V_{-q}c_{-q}\rho_{-q})^*$, the quantity $P_{kj}^{(2)}$ is real. Thus, the interaction $H_i^{(2)}$ leads only to a renormalization of the square of the natural frequency of the $\{\mathbf{kj}\}$ mode by an amount $2\omega_{kj}P_{kj}^{(2)}$.

In contrast to $H_i^{(2)}$, the interaction $H_i^{(1)}$ can be taken into account only approximately. When (4) is satisfied, we can confine ourselves to the second order in $H_i^{(2)}$, i.e., to the Born approximation

$$A(\omega)\approx|A_{kj}|^2\int_0^\infty dt e^{i\omega t}\int_0^t d\tau\langle[[H_i^{(i)}(\tau),\hat{A}^{(i)}(t)],x_{-kj}(0)]\rangle_i,$$

$$\hat{X}^{(i)}(t)=\exp(i\mathcal{H}_0t)\hat{X}\exp(-i\mathcal{H}_0t),\quad \mathcal{H}_0=H_0+H_v$$

($\langle\langle\ldots\rangle\rangle_i$ means that the averaging is carried out with the Hamiltonian $\mathcal{H}_0$). Taking only $H_i^{(1)}$ into account in H_i, we obtain

$$\begin{aligned}A(\omega)&\approx\Pi_{kj}^{(1)}(\omega)Q_{kj}(\omega),\quad \Pi_{kj}^{(1)}(\omega)\\&=-iNS|A_{kj}|^2\sum_q(\mathbf{qe}_{kj})^2|V_q|^2\int_0^\infty dt e^{i\omega t}\Pi_q(\mathbf{k},t),\\ \Pi_q(\mathbf{k},t)&=\zeta_q(\mathbf{k},t)\varphi(\omega_q,t)-\text{c.c.}\end{aligned}\tag{14}$$

Here $NS\zeta_q(\mathbf{k},t)$ and $\varphi(\omega_q,t)$ are the time-dependent correlation functions for the electron-density operator and for the MO without allowance for the interaction:

$$\begin{aligned}\zeta_q(\mathbf{k},t)&=\sum_n\exp[i(\mathbf{q}-\mathbf{k})\mathbf{R}_n+W_q(\mathbf{R}_n,t)],\quad W_q(\mathbf{R}_n,t)\\&=-\sum_{kj}(\mathbf{qe}_{kj})^2|A_{kj}|^2\{(\bar{n}_{kj}+1)[1-\exp(-i\omega_{kj}t-i\mathbf{kR}_n)]\\&\quad+\bar{n}_{kj}[1-\exp(i\omega_{kj}t+i\mathbf{kR}_n)]\},\quad \bar{n}_{kj}\equiv\bar{n}(\omega_{kj});\\ \varphi(\omega,t)&=[\bar{n}(\omega)+1]e^{-i\omega t}+\bar{n}(\omega)e^{i\omega t},\quad \bar{n}(\omega)=[\exp(\omega/T)-1]^{-1}.\end{aligned}\tag{15}$$

From (11), (13) and (14) we obtain in the Born approximation in $H_i^{(1)}$ the relation

$$Q_{kj}(\omega)=2\omega_{kj}|A_{kj}|^2/\{\omega^2-\omega_{kj}^2-2\omega_{kj}[P_{kj}(\omega)-i\Gamma_{kj}(\omega)]\},\tag{16}$$

where

$$\begin{aligned}\Gamma_{kj}(\omega)&=-\operatorname{Im}\Pi_{kj}^{(1)}(\omega+i\varepsilon),\quad P_{kj}(\omega)=P_{kj}^{(1)}(\omega)+P_{kj}^{(2)},\\ P_{kj}^{(1)}(\omega)&=\operatorname{Re}\Pi_{kj}^{(1)}(\omega+i\varepsilon),\quad \varepsilon\to+0.\end{aligned}\tag{17}$$

If the parameters $\Gamma_{kj}(\omega)$ and $P_{kj}(\omega)$ are small in the resonance region $\omega\sim\omega_{kj}$ compared with ω_{kj} and depend smoothly on the frequency [the latter is needed also for the factorization (14) to be correct], it can be seen from (16) that the function $\operatorname{Im}Q_{kj}(\omega)$ is Lorentzian near the maximum:

$$\operatorname{Im}Q_{kj}(\omega)=\frac{-|A_{kj}|^2\Gamma_{kj}}{(\omega-\omega_{kj}-P_{kj})^2+\Gamma_{kj}^2},\quad \omega\sim\omega_{kj},\quad \Gamma_{kj}\equiv\Gamma_{kj}(\omega_{kj}),\quad P_{kj}\equiv P_{kj}(\omega_{kj}).\tag{18}$$

Equations (16) and (17) are valid also in the nonresonant region, where $|\omega-\omega_{kj}|\gg\Gamma_{kj}(\omega)$; this allows us to use them together with (8) to calculate the long-wave high-frequency conductivity of the WC. We note that the expressions for Γ_{kj} and P_{kj} do not contain any divergences connected with the divergence of the mean squared displacement of the electron from the crystal site.

2. ANALYSIS OF THE EXPRESSION FOR THE POLARIZATION OPERATOR

The calculation of $\Pi_{kj}^{(1)}(\omega)$ includes summation over the sites $\mathbf{R}_n$ and over $\mathbf{q}$. The summation is best carried out in one sequence or another, depending on the relation between ω and the frequency ω_{q_r}, where $q_r=\min\mathbf{G}$ ($\mathbf{G}$ is the reciprocal-lattice vector of the WC). We consider below the case $\omega\gg\omega_{q_r}$ and assume that the phase velocity of the WC phonons is much higher than the phase velocity of the MO, while the quantity $q^2|V_q|^2/\omega_q$ does not increase with decreasing q. It is more convenient in this case to sum first over $\mathbf{q}$ and then over $\mathbf{R}_n$.

We denote the term with $n=0$ in Eq. (15) for $\zeta_q(\mathbf{k},t)$ by $\bar{\zeta}_q(t)$

$$\bar{\zeta}_q(t)=\exp[q^2W(t)],\quad W(t)=-\frac{1}{2}\sum_{kj}|A_{kj}|^2[\varphi(\omega_{kj},t)-\varphi(\omega_{kj},0)].\tag{19}$$

The function $2W(t)$ is equal to the difference between the temporal correlation functions of the displacement of the electron from the WC site at the instants of time t and 0 (it is assumed that the symmetrical tensor made up by the correlator of the displacement components is diagonal). The terms contributing to $\Pi_{kj}^{(1)}(\omega)$ are $\bar{\zeta}_q(t)$ with $q\lesssim q_s=u_s^{-1}$, where u_s is the displacement of the electron during a time t_s equal to the duration of the scattering act, $u_s\sim[-W(t_s)]^{1/2}$. The contribution of the terms $q\lesssim r_0^{-1}\ll q_s$ is then small because the state densities of the corresponding MO are small. We note that a factor similar to $\exp[q^2W(t)]$ in (14) and (19) is well known in the theory of the Mössbauer effect. In our problem it appears because, in analogy with the γ-quantum momentum in the Mössbauer effect, the momentum of the MO is transferred in the scattering processes described by (14) and (19) to the electron lattice as a whole, and not to some particular mode of the WC.

The functions $W_q(\mathbf{R}_n,t)$ and $q^2|V_q|^2$ (the latter, by assumption) depend smoothly on $\mathbf{q}$ at $q\lesssim q_s$. If the duration t_s of the scattering process and the group velocity of the MO are small enough,

$$t_sq_s|\partial\omega_s/\partial q_s|\ll q_s^2/N,\quad \omega_s=\omega_{q_s}\quad (q_s\gg N^{1/2}),$$

the function $\varphi(\omega_q,t)$ in (14) also varies smoothly with q at $t\lesssim t_s$ and $q\sim q_s$. They are therefore averaged out upon summation over $\mathbf{q}$ in (14). Accurate to terms $\sim\exp(-\frac{1}{4}q_s^2/N)$, they can be neglected and $\Pi_q(\mathbf{k},t)$ in (14) can be replaced by the operator

$$\bar{\Pi}_q(t)=\bar{\zeta}_q(t)\varphi(\omega_q,t)-\text{c.c.}\tag{20}$$

In this replacement, no account is taken of the interference of the scattering processes of the short-wave ($q\gg N^{1/2}$) MO by various electron, and of the associated structure $\Pi_{kj}^{(1)}(\omega)$ near frequencies ω_G (the latter is responsible, in particular, for the light-absorption peaks observed in Ref. 4). In the region $\omega\gg\omega_{q_r}$ this structure is practically completely washed out at $T\neq0$.

In the calculation of $P_{kj}^{(2)}$ we shall assume the electron density to be low enough to make the restructuring of the MO spectrum by the interaction in the WC small (at any rate in the case of short-wave MO with $q\sim q_s$). The oscillations of the medium act then as a thermostat for the electrons, and all the terms $\sim NS|V_q|^4$ can be neglected compared with the terms $\sim|V_q|^2$, since their ratio is proportional to the ratio of the number of electrons to the number of the atoms of the medium. In this approximation

$$P_{kj}^{(2)}=|A_{kj}|^2\sum_q(\mathbf{qe}_{kj})^2|V_q|^2\int_0^{1/T}d\tau\varphi(\omega_q,-i\tau)\langle\rho_q(-i\tau)\rho_{-q}(0)\rangle.\tag{21}$$

In contrast to (14), the time-dependent density correlation function is calculated here with the total Hamiltonian $\mathscr{H}$.

If the interaction with the MO is weak enough, it suffices to calculate the correlator in (21) with the Hamiltonian $\mathscr{H}_0$:

$$\langle \rho_q^{(i)}(-i\tau)\rho_{-q}^{(i)}(0)\rangle_i = NS\bar{\zeta}_q(0,-i\tau). \tag{22}$$

Using the Kubo identity it is easy to show that the expression obtained in this manner for $P_{kj}^{(2)}$ is equal to $-\mathrm{Re}\Pi_{kj}^{(1)}(0)$. At $q_s \gg r_0^{-1}$ it takes the form

$$P_{kj}^{(2)} = \frac{1}{2}NS|A_{kj}|^2 \sum_q q^2|V_q|^2 \int_0^{1/T} d\tau \varphi(\omega_q, -i\tau)\bar{\zeta}_q(-i\tau). \tag{23}$$

From (14), (17), and (21) we have $\Gamma_{kj}, P_{kj} \propto |A_{kj}|^2$. The value of $|A_{kj}|^2$ increases rapidly with decreasing ω_{kj}. For phonons of sufficiently low frequency, the spectrum renormalization due to the interaction with the MO is therefore strong.

3. RENORMALIZATION OF WC SPECTRUM AT HIGH TEMPERATURES

In the case of high temperatures $T \gg \Theta_D, \omega_s$, the integral with respect to time of (14) with allowance for (20) is evaluated by the saddle-point method. The integrand $\bar{\Pi}_q(t)$ has a maximum near $t = t_s = i/2T$. Since $|t_s| \ll \omega_{kj}^{-1}$, it suffices to expand $W(t)$ in powers of $\omega_{kj}t$ up to quadratic terms. Then

$$\bar{\zeta}_q(t) \approx \exp\left[-\frac{q^2}{2m}(it+Tt^2)\right], \quad T \gg \Theta_D, \quad t \ll \Theta_D^{-1} \tag{24}$$

(we have used here the relation $\sum_{kj}|A_{kj}|^2\omega_{kj} = m^{-1}$).

Substituting (24) in (20) and then in (14) and (17) we obtain for the damping

$$\Gamma_{kj}(\omega) = \omega m NS|A_{kj}|^2\gamma, \quad \gamma = \left(\frac{\pi}{2mT}\right)^{1/2}\sum_q q|V_q|^2\omega_q^{-1}\exp\left(-\frac{q^2}{8mT}\right), \quad \omega, \omega_s \ll T. \tag{25}$$

We have discarded here the corrections $\sim \omega_s/T$ due to the inelasticity of the scattering by the MO [it is seen from (25) that $q_s = (8mT)^{1/2}$]. It is important that many ($\sim T/\Theta_D \gg 1$) WC modes take part in the scattering, and the main contribution is made by the short-wave modes whose state density is high.

The very fact of Wigner crystallization of the electrons is reflected in (15) only by the coefficient $|A_{kj}|^2$. It is easy to verify that γ is equal to the average frequency of the collisions with the MO for a gas of nondegenerate electrons. The reason is that the scattering by the MO is due to the electron motion, but at $T \gg \Theta_D$ the momentum distribution of the electrons that constitute the WC is Maxwellian, as in a gas.[2)]

It follows from (18) and (25) that in the absence of a magnetic field the damping of the $\{kj\}$ mode does not depend on k or j:

$$\Gamma_{kj} = {}^1/_2\gamma, \quad \gamma \ll \omega_{kj}. \tag{25a}$$

In the approximation (20), (23), and (24) the formula (17) for the frequency renormalization reduces to the form

$$P_{kj}(\omega) = \frac{m\omega^2}{2T}NS|A_{kj}|^2\sum_q |V_q|^2\omega_q^{-1}\left\{1+i\left(\frac{2\pi mT}{q^2}\right)^{1/2} \times\left(\frac{q^2}{4mT}-1\right)\exp\left(-\frac{q^2}{8mT}\right)\mathrm{erf}\left[i\left(\frac{q^2}{8mT}\right)^{1/2}\right]\right\},$$

$$\omega, \omega_s \ll T, \quad \mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp(-z^2)\,dz. \tag{26}$$

At $q^2 \gg 8mT$ the expression in the curly bracket is proportional to q^{-4}. It is assumed throughout that $|V_q|^2/\omega_q$ increases with q more slowly than q^2, so that there is no short-wave divergence in (26). The contribution of the small q to $P_{kj}(\omega)$ does not diverge if $q^2|V_q|^2/\omega_q$ is finite as $q \to 0$ [strictly speaking, if $q^2|V_q|^2/\omega_q$ does not tend to zero as $q \to 0$, then (26) diverges logarithmically, but this divergence must be cut off at $q^2 \sim m\omega^2/T$, since Eq. (26) was derived under the assumption that $\omega^2 \ll q^2T/m$].

It is seen from a comparison of (25) and (26) that at $T \gg \Theta_D$ the damping of the WC phonons exceeds considerably the renormalization of their frequency. In this case Γ_{kj} does not depend on ω_{kj}, and $P_{kj} \propto \omega_{kj}$. For low ω_{kj}, the criterion (4) is violated, the renormalization of the spectrum is large, and Eq. (18) does not hold. If $\omega \gg \gamma$ and $\omega \gg \omega_{kj}$, we obtain for the conductivity of the WC from (8), (16), and (25)

$$\mathrm{Re}\,\sigma_{xx}(\omega) \approx \frac{e^2N}{m}\frac{\gamma}{\omega^2}, \quad T \gg \omega \gg \gamma. \tag{27}$$

This equation coincides with the expression for the high-frequency conductivity of nondegenerate and noninteracting electrons quasielastically scattered by MO.

4. RENORMALIZATION OF WC SPECTRUM AT LOW TEMPERATURES

In the region $T \ll \Theta_D$, the Wigner crystallization is manifest not only in the presence of an additional oscillation mode compared with a $2D$ plasma, but also in the character of the mode renormalization on account of the interaction with the MO. We consider the renormalization of phonons for which

$$T < \omega_{kj} \ll \Theta_D, \quad \exp(\omega_{kj}/T) \gg 1. \tag{28}$$

At $\omega > T$ the duration of the scattering act [the region of integration with respect to t in (14)] is $t_s \lesssim \omega^{-1} < T^{-1}$. The ratio $\nu(t_s)$ of the contribution of the thermal fluctuations to the mean squared displacement of an electron during a time t_s to the contribution of the quantum fluctuations is

$$\nu(t_s) \approx \sum_{kj}|A_{kj}|^2\bar{n}_{kj}\omega_{kj}^2t_s^2 \Big/ \sum_{kj}|A_{kj}|^2, \quad t_s < T^{-1}.$$

Using the dispersion law for long-wave longitudinal ($j = l$) and transverse ($j = t$) WC modes:

$$\omega_{kl} \approx \omega_p(k/N^{1/2})^{1/2}, \quad \omega_{kt} \approx c_t k, \quad k \ll N^{1/2}; \quad \omega_p = (2\pi e^2N^{3/2}/m)^{1/2}; \quad c_t \sim \omega_p/N^{1/2}, \tag{29}$$

it is easy to show that $\nu(t_s) \sim T^3t_s^2/\omega_p < T/\omega_p \sim T/\Theta_D \ll 1$. Accordingly, accurate to corrections $\sim T^3t^2/\omega_p$,

$$W(t) = -\eta + w(t), \quad t \lesssim \omega^{-1} < T^{-1}, \quad \eta = \frac{1}{2}\sum_{kj}|A_{kj}|^2,$$

$$w(t)=\frac{1}{2}\sum_{kj}|A_{kj}|^2\exp(-i\omega_{kj}t),\quad \eta\equiv w(0), \tag{30}$$

i.e., the contribution of the thermal fluctuations to $W(t)$ can be completely neglected at $\omega_p \gg \omega > T$.

To calculate $\Pi^{(1)}_{kj}(\omega)$ we must substitute (30) in (19) and then in (20) and (14), and expand $\zeta_q(t)$ in a series in $q^2w(t)$. The terms of zeroth and first order yield respectively $\Pi^{(1)}_{0,kj}(\omega)$ and $\Pi^{(1)}_{1,kj}(\omega)$, where

$$\Pi^{(1)}_{0,kj}(\omega)=NS|A_{kj}|^2\sum_q q^2|V_q|^2\exp(-\eta q^2)\omega_q/(\omega^2-\omega_q^2);$$

$$\Pi^{(1)}_{1,kj}(\omega)=\tfrac{1}{2}NS|A_{kj}|^2\sum_q q^4|V_q|^2\exp(-\eta q^2)\sum_{k'j'}|A_{k'j'}|^2 \tag{31}$$
$$\times\{\bar n^q+1)(\omega_{k'j'}+\omega_q)[\omega^2-(\omega_{k'j'}+\omega_q)^2]^{-1}+\bar n_q(\omega_{k'j'}-\omega_q)/[\omega^2-(\omega_{k'j'}-\omega_q)^2]\},\quad \bar n_q\equiv\bar n(\omega_q).$$

The imaginary part of $\Pi^{(1)}_{0,kj}(\omega+i0)$ determines the damping of the $\{kj\}$ mode on account of the direct decay into MO. The corresponding damping does not depend on temperature and is equal to

$$\Gamma_{0,kj}(\omega)=-\operatorname{Im}\Pi^{(1)}_{0,kj}(\omega+i0)=\frac{\pi}{2}NS|A_{kj}|^2\sum_q q^2|V_q|^2\exp(-\eta q^2)\delta(\omega-\omega_q). \tag{32}$$

In the actual case $\omega > \omega_s$ [it is seen from (31) that $q_s = \eta^{-1/2}$] direct decays have low probability and the main contribution to the damping is made by the term $\Pi^{(1)}_{1,kj}(\omega)$ in $\Pi^{(1)}_{kj}(\omega)$:

$\Pi^{(1)}_{1,kj}(\omega)$ в $\Pi^{(1)}_{kj}(\omega)$:

$$\Gamma_{kj}(\omega)\approx\Gamma_{1,kj}(\omega),\quad \Gamma_{1,kj}(\omega)=-\operatorname{Im}\Pi^{(1)}_{1,kj}(\omega+i0)$$
$$=\frac{NS|A_{kj}|^2}{16mc_t^2N}\sum_q q^4|V_q|^2e^{-\eta q^2}(2\bar n_q+1), \tag{33}$$
$$\omega_p\gg\omega\gg\omega_s.$$

The damping $\Gamma_{1,kj}=\Gamma_{1,kj}(\omega_{kj})$ is due to the decay of the $\{kj\}$ oscillation into a transverse acoustic phonon of the WC with wave vector $k'=\omega_{kj}/c_t$. This decay is induced by the interaction with the medium and is accompanied by creation or annihilation of an MO quantum. Decay into longitudinal WC phonons has low probability because of the low phonon state density, and adds to Γ_{kj} an increment of the order of $(\omega_{kj}/\omega_p)^2\Gamma_{kj}$. Also little likely is decay in a large number of phonons, described by the next terms of the expansion of $\exp[q^2W(t)]$ in (14), (19), and (20) in powers of $w(t)$. In particular, decay into two transverse phonons gives rise to a correction $\sim(\omega_{kj}/\omega_p)\Gamma_{1,kj}$.

Since $\tau \leqslant 1/T$ in (23), Eqs. (19) and (30) can be used also to calculate $\bar P^{(2)}_{kj}$. It is convenient to separate here the contribution of the WC phonons with relatively low frequencies $\omega_{k'j'} < T$. It is small ($\sim T/\omega_p \ll 1$) and can be calculated by perturbation theory. Retaining in the expansion of $\zeta_q(-i\tau)$ in terms of the "high-frequency" part $w(-i\tau)$ only two terms [just as in (31), we obtain for the shift $P_{kj}(\omega)$ from (17), (23), and (31)]

$$P_{kj}(\omega)\approx P_{0,kj}(\omega)+P_{1,kj}(\omega),\quad P_{0,kj}(\omega)=NS|A_{kj}|^2\sum_q\frac{q^2|V_q|^2}{\omega_q}\cdot$$
$$\times\exp(-\eta q^2)\operatorname{Re}\frac{\omega^2}{\omega^2-\omega_q^2+i0}, \tag{34}$$

$$P_{1,k}(\omega)=NS|A_{kj}|^2(8\pi mc_t^2N)^{-1}\sum_q q^4|V_q|^2(2\bar n_q+1)\cdot$$
$$\times\exp(-\eta q^2)\ln(\omega/\bar\omega),\quad \bar\omega=\max(\omega_s,T)$$

[the term $P_{1,kj}(\omega)$ was written out for the case $\omega \gg \omega_s$].

It is seen from (18) and (34) that $P_{1,kj}/P_{0,kj}\sim(\bar\omega/\omega_p)\times\ln(\omega_{kj}/\bar\omega)\ll 1$, i.e., the main contribution to the shift of the peak $Q_{kj}(\omega)$ and to the renormalization of ω_{kj} is made by the term $P_{0,kj}$. It is due to the virtual transitions at which only the number of the MO changes. The term $P_{1,kj}$ is due to transitions with creation or annihilation of an MO quantum and a WC acoustic phonon with frequency between $\bar\omega$ and ω_{kj}. The contribution of the transitions with participation of longitudinal phonons, as well as of high-frequency phonons ($\omega_{k'j'}\sim\omega_p$) whose state density is large, is small [$\sim(\omega_{kj}/\omega_p)^2P_{kj}$]. The last circumstance ensures rapid convergence of the expansion of P_{kj} in the number of phonons that participate in the virtual transition. The cutoff parameter $\bar\omega$ enters in P_{kj} under the logarithm sign. It can be shown that at $T \gg \omega_s$ and at $T \ll \omega_s$ there are in P_{kj} no terms of the order of $(\bar\omega/\omega_p)P_{0,kj}$ that are not proportional to $\ln(\omega_{kj}/\bar\omega)$. At $T\sim\omega_s$ the inequality $\ln(\omega_{kj}/\bar\omega) \gg 1$ must be satisfied if (34) is to be valid.

According to (18), (33), and (34), in the region (28) the broadening Γ_{kj} of the peak $Q_{kj}(\omega)$ is small compared with the shift P_{kj}. The renormalization of the WC crystal corresponds then to dynamic "hardening" of the phonons: the increment $2\omega_{kj}P_{kj}(\omega)$ to ω^2_{kj} is positive at $\omega_p \gg \omega \gg \omega_s$; it is particularly significant and relatively large for low-frequency phonons. We note that the static phonon Green's functions $Q_{kj}(0)$ are not renormalized in practice: it is seen from (34) that $P_{0,kj}(0)=0$ [it can be shown that when terms with $n \neq 0$ are taken into account in (15) we have $P_{kj}(0)\propto|A_{kj}|^2k^2$ in the region of small k, the proportionality coefficient being $\sim\exp(-q_s^2/4N)$].

For the nonresonant long-wave high-frequency conductivity of WC we obtain at $\omega_p \gg T$ from (8), (16), and (33), in the absence of a magnetic field, the expression

$$\operatorname{Re}\sigma_{zz}(\omega)=\frac{e^2N}{m}\frac{\bar\gamma^2}{\omega^3},\quad \bar\gamma=\Big[(16m^2c_t^2N)^{-1}\sum_q q^4|V_q|^2\exp(-\eta q^2)(2\bar n_q+1)\Big]^{1/4}, \tag{35}$$
$$\omega_p\gg\omega\gg\bar\omega.$$

This expression, just as the classical equation (17), yields a power-law frequency dependence of the conductivity, but the exponents are different in the quantum and classical cases.

5. ALLOWANCE FOR THE NON-BORN CORRECTIONS

It is seen from (31)–(34) that the renormalization of the WC modes increases rapidly (like $|A_{kj}|^2$) with decreasing ω_{kj}. Equation (18) does not hold for modes of sufficiently low frequency: the shift P_{kj} (or the damping Γ_{kj} if $T \gg \omega_p$) exceeds the bare frequency ω_{kj}. The characteristic frequency Ω_0 that separates the regions of strongly and weakly renormalizable phonons is equal at $\omega_p \gg T$ to the frequency of a phonon $\{k_0j\}$ for which

the shift and the bare frequency coincide:

$$\Omega_0 \equiv \omega_{\mathbf{k}sj}, \quad \omega_{\mathbf{k}sj} = P_{\mathbf{k}sj}, \quad \omega_p \gg T. \tag{36}$$

In the case $\Omega_0 \gg \omega_s$ of greatest interest, the parameter Ω_0 in the absence of a magnetic field [when $|A_{\mathbf{k}j}|^2 = 1/(2mNS\omega_{\mathbf{k}j})$] is proportional according to (34) and (36) to the first power of the constant of the coupling to MO:

$$\Omega_0 \approx (\omega_p P_0)^{1/2}, \quad P_0 = \frac{1}{2m\omega_p}\sum_{\mathbf{q}} q^2 |V_q|^2 \exp(-\eta q^2)\omega_q^{-1}, \quad \omega_p \gg T. \tag{36a}$$

At $T \gg \omega_p$ it is necessary to equate Ω_0 to the damping γ; Ω_0 is here quadratic in the interaction.

The strong renormalization of the WC modes with frequency $\omega_{\mathbf{k}j} \lesssim \Omega_0$ does not violate the conditions for the applicability of Eq. (16) for the phonon Green's function in the frequency region $\omega \gg \Omega_0$. Indeed, only the factorization (14) is approximate in this formula. It is valid if the interaction with the MO does not manage to "manifest itself" within a characteristic time t_s equal to the duration of the scattering processes and determining the size of the region of integration with respect to t in (14) (i.e., the individual scattering processes are sequential in time and are not superimposed on one another). From the definitions of Ω_0 and t_s it follows that in the cases considered in the preceding sections we have $t_s \ll \Omega_0^{-1}$ ($t_s = \omega^{-1}$ and $t_s = T^{-1}$ at $\omega_p \gg T$ and $\omega_p \ll T$, respectively). The contribution made to $W(t)$ by phonons with frequencies lower than Ω_0 in the region $t \lesssim t_s$ is negligibly small: of the order of Ω_0^2/ω_p^2 at $T \gg \omega_p$ and of the order of $\Omega_0^2 t_s/\omega_p$ at $T \ll \omega_p$ [we use here the bare phonon-dispersion law (29)]. The Born approximation (14) is valid precisely because the contribution to the polarization operator $\Pi_{\mathbf{k}j}^{(1)}(\omega)$ of the strongly renormalizable low-frequency phonons is small.

It can be verified that when account is taken of the renormalization the contribution of the low-frequency phonons to $W(t)$ remains small. To this end we compare $W(t)$ with the function $\tilde{W}(t)$, which is obtained from $W(t)$ as a result of "dressing"

$$W(t) = \frac{1}{2}\langle [\mathbf{u}_n(t) - \mathbf{u}_n(0)]\mathbf{u}_n(0)\rangle \tag{37}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega (1 - e^{-i\omega t})[\bar{n}(\omega)+1]\sum_{\mathbf{k}j} \operatorname{Im} Q_{\mathbf{k}j}(\omega + i\varepsilon), \quad \varepsilon \to +0 \quad (\mathbf{u}_n \equiv \mathbf{r}_n - \mathbf{R}_n).$$

Let us estimate $\tilde{W}(t)$ at $\omega_p \gg T$ (the case $\omega_p \ll T$ is easier and can be considered similarly) in the actual time interval $\omega_p^{-1} \ll t \ll \Omega_0^{-1}$. The contribution to $\tilde{W}(t)$ from the weakly renormalizable WC phonons with frequency $\omega_{\mathbf{k}j} \gg \Omega_0$ obviously agrees, accurate to terms $\sim \Omega_0^2$, with their contribution to $W(t)$ and is described by Eq. (30). To estimate the contribution of the low-frequency ($\omega_{\mathbf{k}j} \lesssim \Omega_0$) phonons, we break up the region integration with respect to ω in (37) into the sections $|\omega| \geq \omega_1$ and $|\omega| < \omega_1$. The frequency ω_1 is chosen such that $\Omega_0 \ll \omega_1 < t^{-1}$. At $|\omega| \geq \omega_1 \gg \Omega_0, \omega_{\mathbf{k}j}$ the Green's function of the low-frequency phonon takes the form (16) and

$$\operatorname{Im} Q_{\mathbf{k}j}(\omega) \approx -4\omega_{\mathbf{k}j}^2 |A_{\mathbf{k}j}|^2 \Gamma_{\mathbf{k}j}(\omega)/\omega^4.$$

It follows therefore that their contribution, connected with the region $|\omega| \geq \omega_1$, to $\tilde{W}(t)$ amounts to $\sim \Omega_0^4/\omega_p\omega_1^3 \propto \Omega_0^4$, i.e., is exceedingly small. In the region $|\omega| < \omega_1 < t^{-1}$ we can expand $\exp(-i\omega t)$ in (37) in a series. Applying the relation

$$0 < -\int_{-\omega_1}^{\omega_1} \omega d\omega \operatorname{Im} Q_{\mathbf{k}j}(\omega + i0) < -\int_{-\infty}^{\infty} \omega d\omega \operatorname{Im} Q_{\mathbf{k}j}(\omega + i0) = 2\pi\omega_{\mathbf{k}j}|A_{\mathbf{k}j}|^2$$

to the linear term of the expansion, we find that the increment proportional to t made to $\tilde{W}(t)$ by the low-frequency phonons and connected with the region $|\omega| < \omega_1$ is of the order of $\Omega_0^2 t/\omega_p$. The increments proportional to higher powers of t contain, besides $\Omega_0^2 t/\omega_p$, and additional factor $\sim (\omega_1 t)^n < 1$.

Thus the renormalization of the WC phonons, accurate to terms $\sim \Omega_0^2$, does not influence the function $W(t)$ in the region $t \ll \Omega_0^{-1}$. Consequently, the scattering processes of duration $t_s \ll \Omega_0^{-1}$ can be treated in the Born approximation, i.e., the factorization (14) and expression (16) for the phonon Green's function are valid at $\Omega_0 \ll \omega$.

Unlike the polarization operator $\Pi_{\mathbf{k}j}^{(1)}(\omega)$, which is determined by phonons of frequency $\omega_{\mathbf{k}'j'} \gtrsim \omega \gg \Omega_0$, the shift $P_{\mathbf{k}j}^{(2)}$, as seen from (34), contains a contribution of phonons of relatively low frequency $\sim \bar{\omega}$. At $\bar{\omega} < \Omega_0$ the latter are strongly renormalized, therefore to calculate $P_{\mathbf{k}j}^{(2)}$ we must use the rigorous expression (21) rather than the formula (22) obtained in the Born approximation. At $q_s \gg N^{1/2}$ we can confine ourselves in the density correlator in (21) to the "single-site" mean values

$$\tilde{\xi}_{\mathbf{q}}(-i\tau) = \langle \exp[i\mathbf{q}\mathbf{r}_n(-i\tau)]\exp[-i\mathbf{q}\mathbf{r}_n(0)]\rangle.$$

Expanding $\exp(i\mathbf{q}\mathbf{r}_n)$ in terms of the normal coordinates $x_{\mathbf{k}j}$ and expressing the mean values $\langle x_{\mathbf{k}j}(-i\tau)x_{-\mathbf{k}j}(0)\rangle$ in terms of the temperature Green's functions (7), we obtain

$$\tilde{\xi}_{\mathbf{q}}(-i\tau) \approx \exp[q^2 W(-i\tau)], \quad W(-i\tau) = T\sum_{n=0}^{\infty}[1-\cos(\omega_n\tau)]\sum_{\mathbf{k}j} G_{\mathbf{k}j}(\omega_n), \quad \tau \geq 0. \tag{38}$$

We have discarded here mean values of the type $\langle x_{\mathbf{k}_1 j_1} \ldots x_{\mathbf{k}_n j_n}\rangle \delta_{\mathbf{k}_1 + \ldots + \mathbf{k}_n}$ with $n > 2$. They lead to corrections $\sim (\Omega_0/\omega_p)^2 \ll 1$.

According to (7), (16), (31), and (34) we have, accurate to within the corrections $\sim T/\omega_p$ and ω_s/ω_p,

$$G_{\mathbf{k}j}(\omega_n) \approx -2\omega_{\mathbf{k}j}|A_{\mathbf{k}j}|^2/[\omega_n^2 + \omega_{\mathbf{k}j}^2 + 2\omega_{\mathbf{k}j}P_{0,\mathbf{k}j}(i\omega_n)]. \tag{39}$$

The value of $\tilde{W}(-i\tau)$ can be easily calculated from (38) and (39) if $T \gg \omega_s$. At $\omega_n = 2\pi nT \gg \omega_s$, as seen from (34) and (36) we have $\omega_{\mathbf{k}j}P_{0,\mathbf{k}j}(i\omega_n) \approx \Omega_0^2$ and, accurate to terms $\sim \Omega_0/\omega_p$ and T/ω_p, the "dressing" of the phonons reduces to a shift, independent of ω_n, of their frequency. Therefore the function $\tilde{W}(-i\tau)$ has the same structure as in the absence of the interaction with the MO [cf. (19)]:

$$W(-i\tau) \approx \frac{1}{2}\sum_{\mathbf{k}j}(\omega_{\mathbf{k}j}/\tilde{\omega}_{\mathbf{k}j})|A_{\mathbf{k}j}|^2[\varphi(\tilde{\omega}_{\mathbf{k}j}, -i\tau) - \varphi(\tilde{\omega}_{\mathbf{k}j}, 0)], \tag{38a}$$

$$\tilde{\omega}_{\mathbf{k}j} = (\omega_{\mathbf{k}j}^2 + 2\Omega_0^2)^{1/2}, \quad T \gg \omega_s.$$

We shall show that the "hardening" of the low-frequency modes, which is obvious from (38a), leads to some de-

crease of the frequency shift of the high-frequency modes. The principal term $P_{0,kj}$ in (34) is not changed here [this allows us to use Eq. (38a), but the term $P_{1,kj}$ is renormalized.

To calculate P_{kj} in the "single site" approximation we must substitute (38a) in [Eq. (38) for $\bar{\xi}_q(-i\tau)$ and then substitute $NS\bar{\zeta}_q(-i\tau)$ in (21) in place of the electron-density correlator. If $\Omega_0 > T$, it is convenient to calculate the integral with respect to τ in (21) by expanding $\bar{\xi}_q(-i\tau)$ in a series in

$$\tilde{w}(-i\tau)=\tfrac{1}{2}\sum_{kj}(\omega_{kj}/\bar{\omega}_{kj})|A_{kj}|^2\varphi(\bar{\omega}_{kj},-i\tau).$$

The zeroth term of the series yields

$$P_{0,kj}^{(1)}=NS|A_{kj}|^2\sum_q q^2|V_q|^2\exp(-\tilde{\eta}q^2)\omega_q^{-1},$$

$$\tilde{\eta}=\tilde{w}(0)\quad(\exp(\Omega_0/T)\gg 1).$$

Combining this expression with $\mathrm{Re}\,\Pi_{0,kj}^{(1)}(\omega)$ and recognizing that $\tilde{\eta}\approx\eta$ accurate to Ω_0/ω_p, we obtain for $P_{0,kj}(\omega)$ the expression given in (34). We can similarly calculate the contribution from the series term that is linear in $\tilde{w}(-i\tau)$. The corresponding increment to the shift, with allowance for $\Pi_{1,kj}^{(1)}(\omega)$ takes the form $P_{1,kj}(\omega)$, except that $\bar{\omega}$ in (34) must be replaced by Ω_0. Since the corrections $\sim\Omega_0/\omega_p$ have been discarded from $P_{0,kj}(\omega)$ the term $P_{1,kj}(\omega)$ need be taken into account only if $\ln(\omega/\Omega_0)\gg 1$. Obviously $\ln(\omega/\Omega_0)<\ln(\omega/T)$ at $\omega \gg \Omega_0 > T$, i.e., in the case $\Omega_0 \gg T$ the shift of $P_{kj}(\omega)$ decreases because of the hardening of the low-frequency phonons of the WC. In the case $T \gg \Omega_0$ the shift remains practically unchanged. This is seen from (38a): the contribution of the phonons with $\bar{\omega}_{kj} < T$ to $\bar{W}(-i\tau)$ is small.

If $\omega_s \gtrsim T$, then (38a) is valid only in the regions $\tau \ll \omega_s^{-1}$ and $T-\tau \ll \omega_s^{-1}$. However, it is precisely the regions $\tau \lesssim \omega_s^{-1}$ and $T-\tau \lesssim \omega_s^{-1}$ which contribute to the integral (21). Therefore at $\omega_s > T$ and $\ln(\omega/\bar{\omega})\gg 1$ we likewise obtain for $P_{kj}(\omega)$ the formula (34), in which now $\bar{\omega} = \max(\Omega_0, \omega_s)$. Consequently Eq. (34) is valid at arbitrary ω_s/T if the parameter $\bar{\omega}$ is defined with the aid of the relation

$$\bar{\omega}=\max(\Omega_0,\ \omega_s,\ T). \tag{40}$$

It is assumed here that $\ln(\omega/\bar{\omega})\gg 1$, and the corrections $\sim\bar{\omega}/\omega_p$ not proportional to $\ln(\omega/\bar{\omega})$ are discarded.

The analysis in this section shows that the strong renormalization of the low-frequency modes of the WC leads to a non-Born correction to the polarization operator at high frequencies. The correction arises at $\Omega_0 > \omega_s$ and $\Omega_0 > T$. It plays therefore an important role at low temperatures and in interactions with low-frequency MO, even if the interaction is weak, i.e., the WC phonons of relatively high frequency are weakly renormalized. According to (36a), (34), and (40) Ω_0 is proportional to the first power of the coupling constant with the MO, and enters in the final expressions under the logarithm sign.

6. CONCLUSIONS

It is seen from the result of the paper that the mechanisms of the damping and of the mode shifts of WC at high ($T \gg \Theta_D$) and low ($T \ll \Theta_D$) temperatures are different. In the former case many short-wave phonons of the WC participate in the scattering, and the damping exceeds the shift appreciably. The relaxation has here the same character and is described by the same parameter as for an electron gas. In the latter case, on the contrary, the damping at $\omega_{kj} \gg \omega_s$ is due to decay, induced by the MO, into transverse phonons, and is consequently essentially connected with the Wigner crystallization. Thus, at $\omega_s \ll \Theta_D$ and $T \ll \Theta_D$ the crystallization can be revealed even by the spectrum of the longitudinal oscillations, and the transverse sound velocity c_t can be determined. The parameter c_t enters also in expression (35) for the nonresonant conductivity of a WC. The latter has a unique frequency dependence in the quantum case.

The parameters Γ_{kj} and P_{kj} increase with decreasing ω_{kj} and the low-frequency phonons are strongly renormalized. These are the causes of the main non-Born corrections: if the low frequency modes influence the weak renormalization of the considered mode of relatively higher frequency, the strong renormalization of the low-frequency modes must be taken into account.

The peaks of the conductivity of the WC near the natural frequencies have, as seen from (8) and (18), a Lorentz shape. Explicit expressions for the broadening and the shift of the peaks can be easily obtained in the absence of a magnetic field [$|A_{kj}|^2 = 1/(2mNS\omega_{kj})$] and if

$$|V_q|^2/\omega_q = S^{-1}Dq^{2n} \tag{41}$$

[for electrons that interact on helium with capillary waves (ripplons) we have $n=-1$ in a wide range of parameters]. At high temperatures, according to (25), the broadening Γ_{kj} and the average collision frequency γ are given by

$$\Gamma_{kj}=\frac{1}{2}\gamma,\quad \gamma=\frac{D}{2\sqrt{\pi}}(8mT)^{n+1}\left(n+\frac{1}{2}\right)!,\quad T\gg\Theta_D. \tag{42}$$

At low temperature (but at $T \gg \omega_s$), according to (33) and (34),

$$\Gamma_{kj}=\frac{(n+2)!DT}{64\pi m^2c_t^2N\omega_{kj}}\eta^{-(n+3)},\quad P_{kj}=\frac{(n+1)!D}{8\pi m\omega_{kj}}\eta^{-(n+2)} + \frac{(n+2)!DT\ln(\omega_{kj}/\bar{\omega})}{32\pi^2m^2c_t^2N\omega_{kj}}\eta^{-(n+3)}, \tag{43}$$

$$\Theta_D\gg\omega_{kj}\gg T\gg\omega_s.$$

Since $\eta\sim(m\omega_p)^{-1}\propto N^{-3/4}$, we have at low temperatures $\Gamma_{kj}\propto TN^{3/4(n+1)}$ and the first and second terms in P_{kj} are $\propto N^{(3/4)(n+2)}$ and $\propto TN^{3/4(n+1)}\ln(\omega_{kj}/\bar{\omega})$ respectively (the parameter $\bar{\omega}$ can depend on the temperature, density, and the coupling constant). We note that the damping and the shift of the mode $\{kj\}$ depend on k and j only via the frequency ω_{kj}.

Equations (42) and (43) can be used directly for electrons on helium. In the actual density region $N\sim 10^8$ cm^{-2} and at temperatures $T \lesssim 0.5$ K the frequency ω_s

of the MO that are important for the scattering is $\sim 10^8$ sec^{-1}. Usually $T \gg \hbar\omega_s$, and the inequality $\omega_{kj} \gg \omega_s$ is satisfied for longitudinal phonons at $k > 1$ cm^{-1}. A stronger lower bound on k is imposed by the condition $\Gamma_{kj} \ll \omega_{kj}$, therefore the resonant absorption must be investigated at frequencies > 100 MHz, just as for a 2D plasma (cf. Ref. 5). It is of interest also that, both at $T \gg \Theta_D$ and at $T < \hbar\omega_{kj} \ll \Theta_D$ the broadening Γ_{kj} for electrons on helium in the absence of a magnetic field is practically independent of the density. The use of relations (33) and (34) for WC in a magnetic field permits[10] a qualitative explanation of the experiments[12] on cyclotron resonance of 2D electrons on helium.

APPENDIX

To prove (13) it suffices to show that the function

$$\tilde{\Lambda}_1(\omega) = \langle\langle \Lambda_1 x_{kj}|, \ \Lambda_1 = \hat{B} - \langle \hat{B} \rangle \ (\langle\langle \hat{X}| \equiv \langle\langle \hat{X}; x_{-kj} \rangle\rangle_\omega) \tag{A1}$$

vanishes in the limit as $S \to \infty$. The equation of motion for $\tilde{\Lambda}_1(\omega)$ takes the form

$$(\omega^2 - \omega_{kj}^2)\tilde{\Lambda}_1(\omega) = \lambda_1 + \langle\langle \Lambda_2 x_{kj}| + 2\omega_{kj}|A_{kj}|^2 \langle\langle \Lambda_1 \hat{A}|, \tag{A2}$$

where

$$\lambda_1 = 2\omega_{kj}|A_{kj}|^2\langle\Lambda_1\rangle - i\langle[(\dot{\Lambda}_1 + i\omega\Lambda_1), x_{-kj}]x_{kj}\rangle + i\langle[\Lambda_1, x_{-kj}]\dot{x}_{kj}\rangle,$$
$$\Lambda_2 = \ddot{\Lambda}_1 + 2i\omega\dot{\Lambda}_1 + 2\omega_{kj}|A_{kj}|^2\hat{B} \tag{A3}$$

(the dot denotes differentiation with respect to time, $\dot{\hat{X}} \equiv -i[\hat{X}, \hat{\mathscr{H}}]$).

It follows from (A1) and (11) that the operator Λ_1 is additive, since it is a sum of single-electron operators, and $\langle\Lambda_1\rangle = 0$. From the expression for Λ_1 it is seen that $\lambda_1 = 0$ in the limit as $S \to \infty$. For the functions $\langle\langle \Lambda_2 x_{kj}|$ and $\langle\langle \Lambda_1 \hat{A}|$ in (A2) we can likewise set up equations of motion such as (A2) and express them in terms of other Green's functions, for which, in turn, we can set up equations of motion, etc. We shall show that all the inhomogeneous terms, i.e., the terms that are not Green's functions, vanish in the corresponding chain of equations. The system of linear equations for $\tilde{\Lambda}_1(\omega)$ is then homogeneous and has the trivial solution $\tilde{\Lambda}_1(\omega) = 0$.

The first of the Green's functions in the right-hand side of (A2) is similar in structure to $\tilde{\Lambda}_1(\omega)$. The general form of the equation for functions of this type in the chain of equations of motion for $\tilde{\Lambda}_1(\omega)$ is:

$$(\omega^2 - \omega_{kj}^2)\tilde{\Lambda}_p(\omega) = \lambda_p + 2\omega_{kj}|A_{kj}|^2\langle\langle \Lambda_p \hat{A}| + \tilde{\Lambda}_{p+1}(\omega), \quad p = 1, 2, \ldots$$
$$\tilde{\Lambda}_p(\omega) \equiv \langle\langle \Lambda_p x_{kj}|, \quad \Lambda_{p+1} = \ddot{\Lambda}_p + 2i\omega\dot{\Lambda}_p + 2\omega_{kj}|A_{kj}|^2\Lambda_p\hat{B}, \tag{A4}$$
$$\lambda_p = 2\omega_{kj}|A_{kj}|^2\langle\Lambda_p\rangle - i\langle[(\dot{\Lambda}_p + i\omega\Lambda_p), x_{-kj}]x_{kj}\rangle + i\langle[\Lambda_p, x_{-kj}]\dot{x}_{kj}\rangle.$$

According to (11), (A1), and (A4) all the Λ_p are linear combinations of the products of the additive operators, and since $\langle\Lambda_1\rangle = 0$, it follows with allowance for (12) that $S^{-1}\langle\Lambda_p\rangle = 0$ for all p. The operators Λ_p do not contain the momentum k explicitly. Therefore the mean values of the commutators in (A4) are zero. Indeed, expressing x_{kj} in terms of the displacements $\mathbf{u}_n = \mathbf{r}_n - \mathbf{R}_n$, we obtain for the k-independent additive operator $\hat{X} = \sum_n \hat{X}_n$ (n is the number of the electron)

$$\langle[\hat{X}, x_{-kj}]x_{kj}\rangle = (NS)^{-2}\sum_{n_1 n_2 n_3}\langle[\hat{X}_{n_1}, \mathbf{e}_{kj}\mathbf{u}_{n_3}]\mathbf{e}_{kj}\mathbf{u}_{n_2}\rangle$$
$$\times \exp[ik(\mathbf{R}_{n_3} - \mathbf{R}_{n_2})] \sim S^{-1}\langle\hat{X}_n\rangle \sim S^{-1} \tag{A5}$$

(the operator $\hat{X}_n$ for the n-th electron depends only on the coordinates and momenta of the given electron and of some of its neighbors). Obviously, the estimate (A5) is valid if x_{kj} is replaced by $\dot{x}_{kj}$, and also if $\hat{X}$ is replaced by a product of k-independent operators and allowance is made for (12). Consequently $\lambda_p = 0$ as $S \to \infty$.

The chain of equations for the functions $|A_{kj}|^2\langle\langle \Lambda_p \hat{A}|$ in the right-hand side of (A4) is of the form

$$\omega\tilde{\Lambda}_{p,q}(\omega) = \lambda_{p,q} + \tilde{\Lambda}_{p,q+1}(\omega); \quad q = 1, 2, \ldots; \quad \tilde{\Lambda}_{p,q}(\omega) = \langle\langle \Lambda_{p,q}|;$$
$$\lambda_{p,q} = \langle[\Lambda_{p,q}, x_{-kj}]\rangle; \quad \Lambda_{p,q+1} = [\Lambda_{p,q}, \hat{\mathscr{H}}], \quad \Lambda_{p,1} = |A_{kj}|^2\Lambda_p\hat{A}. \tag{A6}$$

The operators $\Lambda_{p,q}$ constitute a linear combination of products of additive operators, one of which is of the form $\sum_n \hat{X}_n \exp(-i\mathbf{k}\mathbf{R}_n)$, and the others are independent of $\mathbf{k}$. An estimate of the type (A5) with allowance for (12) shows that $\lambda_{p,q} = 0$ in the limit as $S \to \infty$.

The quantities λ_p and $\lambda_{p,q}$ determine all the inhomogeneous terms in the chain of equations for $\tilde{\Lambda}_1(\omega)$. Their vanishing proves that $\tilde{\Lambda}_1(\omega) = 0$.

[1)] The internal anharmonicity of a WC is determined by the relative electron displacements $\mathbf{u}_n - \mathbf{u}_{n'}$. Inasmuch as $\langle(\mathbf{u}_n - \mathbf{u}_{n'})^2\rangle \ll (\mathbf{R}_n - \mathbf{R}_{n'})^2$ far from the melting point, the Hamiltonian that describes this anharmonicity can be expanded in powers of $\mathbf{u}_n = \mathbf{u}_{n'}$, and the corresponding small renormalization of the long-wave phonons can be obtained in standard fashion by perturbation theory.

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[1] E. P. Wigner, Phys. Rev. **46**, 1002 (1934).
[2] R. S. Crandall and R. Williams, Phys. Lett. **34A**, 404 (1971).
[3] A. V. Chaplik, Zh. Eksp. Teor. Fiz. **62**, 746 (1971) [Sov. Phys. JETP **35**, 395 (1971)].
[4] C. C. Grimes and G. Adams, Phys. Rev. Lett. **42**, 795 (1979).
[5] C. C. Grimes and G. Adams, *ibid.* **36**, 145 (1976).
[6] T. N. Theis, J. P. Kotthaus, and P. J. Stiles, Sol. St. Commun. **25**, 603 (1978).
[7] E. M. Lifshitz and L. P. Pitaevskii, Physical Kinetics, Pergamon, 1980, Chap. 7.
[8] M. I. Dykman, Sol. St. Commun. **35**, 753 (1980).
[9] A. G. Eguiluz, A. A. Maradudin, and R. J. Elliott, Phys. Rev. **B23**, 197 (1981).
[10] M. I. Dykman, IMF Preprint No. 12, 1981.
[11] L. D. Landau and E. M. Lifshitz, Statistical Physics, Part 1, Pergamon, 1980, §2.
[12] V. S. Édel'man, Zh. Eksp. Teor. Fiz. **77**, 673 (1979) [Sov. Phys. JETP **50**, 338 (1979)].

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