Change in the mass of an accelerated charge as a dynamical manifestation of the clock paradox

V. I. Ritus

P. N. Lebedev Physics Institute, Academy of Sciences of the USSR, Moscow
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The mass shifts of accelerated charges (the sources of massive vector and scalar fields) are studied in an approximation which is classical with respect to the motion of the charges but quantum with respect to the interaction of the charges with their self-fields. For uniformly accelerated charges, the mass shifts are expressed in terms of cylinder functions of the quantum parameter $\mu c^3/\hbar w_0$, which is essentially the ratio of the mass μ of the self-field quanta to the acceleration w_0 of the charge. For a finite positive value of this parameter, both the imaginary and real parts of the shifts are nonzero and negative. In the limit $\mu \to 0$, the real part of the mass shift of a vector charge tends to the classical value $-\alpha\hbar w_0/2c^3$, which was obtained by the author earlier [Sov. Phys. JETP **48**, 758 (1978) and **53**, 659 (1981)], while the mass shift for a scalar charge tends to zero. These and other properties of the shifts as functions of μ are considered on the basis of locality of the interaction, unitarity and causality. For the change $\Delta W(\mu^2)$ in the self-action of arbitrarily accelerated charges, dispersion relations in the variable μ^2 are established. It is shown that Im $\Delta W(\mu^2)$ is positive for real μ^2, and this makes it possible to interpret 2 Im $\Delta W(-k^2)$ as the probability of emission of virtual quanta with any value of the square k^2 of the 4-momentum. For $\mu > 0$, the real part Re $\Delta W(\mu^2)$, which determines the mass shift, must be nonzero and positive "on the average." A representation that directly expresses $\Delta W(\mu^2)$ in terms of the slowing down of the proper time of an accelerated charge compared with the proper time of propagation of the virtual quanta of its field is obtained.

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1. INTRODUCTION[1)]

By the clock paradox, we understand in this paper the well-known assertion of the special theory of relativity: The proper time of motion along a straight line between two timelike points is greater than the proper time of motion along any timelike curve between them. The interaction of a charge with itself can be represented as a process of exchange of virtual self-field quanta propagating along straight lines between the points of emission and absorption, whereas the charge itself, which is accelerated by external forces, moves along a curve. Because of the difference between the proper times of the motion of the charge and the quanta, the self-action energy, or the self-mass of the accelerated charge, is different from that of an unaccelerated charge moving along a straight line.

In the present paper, we consider the changes in the mass of classically moving accelerated charges whose self-field is massive (in contrast to Refs. 1 and 2) and has spin $n=0$ or 1. As a result, the mass shift of an accelerated charge is a complicated function of the dimensionless quantum parameter

$$\lambda=(\mu/w_0)^2\equiv(\mu c^3/\hbar w_0)^2, \tag{1}$$

which is the square of the ratio of the rest energy μc^2 of a field quantum to the characteristic kinetic energy $\hbar w_0/c$ of the quanta emitted by a charge with acceleration w_0. The estimate

$$[(\mu^2+k_\perp^2)^{1/2}-\mu]_{\text{eff}}\sim w_0 \tag{2}$$

of the effective kinetic energy of the quanta follows from the differential emission probability obtained by Nikishov and the present author[3] and generalized to the case $\mu\neq 0$.

The motion of a charge with massless self-field can be treated classically if the quantum parameter $\beta=w_0/m$, which is equal to the ratio of the characteristic energy $\hbar w_0/c$ of the emitted quanta to the rest energy mc^2 of the charge is small, i.e., if the condition $\beta\ll 1$ is satisfied (see Ref. 1, in which a quantum theory of the electron mass shift was developed). It is obvious that the corresponding condition for the motion of the source of a massive field will be smallness of the effective total energy of the emitted quanta compared with the mass of the charge, i.e., $\mu+w_0\ll m$; see (2). Therefore, the parameter β must, as before, be small, and the mass of the quanta must be small compared with the mass of the charge:

$$\beta\ll 1,\quad \mu/m\ll 1. \tag{3}$$

This means that the parameter λ must not be too small or too large: $\mu/m\ll\lambda^{1/2}\ll\beta^{-1}$. These are the conditions under which the results of the present paper are valid. In the region (3), which is classical with respect to the motion of the charge but quantum with respect to its interaction with the self-field, the results do not depend on the spin of the charge but do depend strongly on the spin of its field.

In the second section of this paper, we find the mass shifts of uniformly accelerated sources of massive vector and scalar fields. The shifts are expressed explicitly in terms of cylinder functions of the variable $\lambda^{1/2}$ and are analytic functions in the complex plane of λ with a cut along the positive real axis. The real parts of the shifts are nonzero and negative on the positive real axis $\lambda>0$ and vanish on the negative axis $\lambda<0$, having a discontinuity at the point $\lambda=0$ in the case of a vector field or a discontinuity of the derivative in the case of a scalar field. The imaginary parts of the shifts are negative not only on the half-axis $\lambda>0$, where -2 Im Δm is equal to the probability of emission

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of quanta in unit proper time, but also on the half-axis $\lambda<0$. This makes it possible to interpret $-2\ \mathrm{Im}\ \Delta m$ for any real λ as the probability of emission of virtual quanta with arbitrary square $k^2=-\lambda w_0^2$ of the momentum.

In the third section, we show that these properties of the mass shifts of uniformly accelerated charges are not fortuitous but are a consequence of very general analytic properties of the variation of the action as a function of μ^2 for arbitrary motion of the charges. Namely, because of the causal properties of the propagation of the self-field, the change $\Delta W(\mu^2)$ in the action is analytic in the half-plane $\mathrm{Im}\ \mu^2<0$, and because the charge is pointlike (locality of the interaction) and its trajectory timelike (causality of the motion of the charge) the real part of the change in the action vanishes on the half-axis $\mu^2<0$, which leads to symmetry and analyticity of $\Delta W(\mu^2)$ in the entire complex plane of μ^2 cut along the half-axis $\mu^2\geqslant 0$. The dispersion relations for $\Delta W(\mu^2)$ and the positivity of its imaginary part for $\mu^2>0$ have the consequence that $\mathrm{Im}\ \Delta W(\mu^2)$ is also positive for $\mu^2<0$ and makes it possible to interpret this quantity as the emission probability of spacelike quanta. With regard to $\mathrm{Re}\ \Delta W(\mu^2)$, for $\mu>0$ it is necessarily nonzero and positive "on the Cauchy average."

In the fourth section, for trajectories with constant curvature and constant torsion (in particular, trajectories of an electric charge in an arbitrary constant electromagnetic field), the change in the self-action of a scalar field can be represented as the mean value of the relative slowing down $(z-u)/z$ of the proper time u of motion of the charge compared with the proper time z of propagation of the field quanta between the emission and absorption, and the change in the self-action of a vector charge can be represented as the mean value of the proper rate of change of the slowing down: $d(z-u)/du$. In other words, the changes in the self-action of the charges are a direct manifestation of the clock paradox, according to which the slowing down and the rate of slowing down of the proper time of an accelerated charge are positive:

$$z-u>0,\quad d(z-u)/du>0. \tag{4}$$

The function $\Delta W(\mu^2)$, determined for charges with massive self-field, is also important in quantum electrodynamics, and not only because it makes it possible to examine the phenomenon of mass shift from a more general physical position (dispersion with respect to μ). Although in electrodynamics the main contribution to the shift is given by the $\mu\to+0$ limit of $\Delta W_1(\mu^2)$, the polarization correction to it is determined by the function $\Delta W_1(\mu^2)$ as a whole due to the interaction between the virtual photons and pairs in the vacuum (see Sec. 5).

2. MASS SHIFTS OF UNIFORMLY ACCELERATED SOURCES OF MASSIVE FIELDS WITH SPIN UNITY AND ZERO

We proceed from the fundamental concept of the action W, which determines the amplitude e^{iW} for the vacuum persistance probability in the case of a given source.[4,5] The mass shift of a classically moving charge that is the source of a massive vector field is determined by the change in its self-action, i.e., the difference between the self-action of the charge in the external field and in the vacuum (symbol $|_0^F$):

$$\Delta W_1^{\mathrm{cl}}=-\Delta m_1^{\mathrm{cl}}\tau=\frac{i}{2}\int d^4x\,d^4x'\,j_\alpha(x)j_\alpha(x')\Delta^c(x-x',\mu)\big|_0^F$$
$$={}^1/_2 ie^2\int d\tau\,d\tau'\,\dot{x}_\alpha(\tau)\dot{x}_\alpha(\tau')\Delta^c(x(\tau)-x(\tau'),\mu)\big|_0^F. \tag{5}$$

Here, $j_\alpha(x)$ and $x_\alpha(\tau)$ are the current density and coordinate of the charge, and Δ^c is the causal propagator of the quanta of its field. For a uniformly accelerated charge, the expression (5) leads to the integral[1,2]

$$\Delta m_1^{\mathrm{cl}}=\frac{\alpha w_0}{2\pi}S_1(\lambda),\quad S_1(\lambda)=\int_0^\infty dz\,e^{-i\lambda/2z}\left[e^{iz}K_1(iz)-\left(\frac{\pi}{2iz}\right)^{1/2}\right], \tag{6}$$

where $\lambda=(\mu/w_0)^2$ and w_0 is the acceleration of the charge in its rest frame. Note that the factor $\alpha w_0/2\pi$ is purely classical; Planck's constant occurs in the parameter $\lambda\sim\hbar^{-2}$, since the propagation of the self-field, in contrast to the motion of the charge, is treated quantum mechanically.

Applying a direct and inverse Mellin transformation, it is possible to express the integral $S_1(\lambda)$ in terms of the modified cylinder functions $I_n(x)$ and $K_n(x)$:

$$\mathrm{Re}\,S_1(\lambda)=-\pi x^2\left[I_1(x)K_1(x)+\frac{1}{2}I_0(x)K_2(x)+\frac{1}{2}I_2(x)K_0(x)\right]+\pi x,$$
$$\mathrm{Im}\,S_1(\lambda)=x^2[K_1^2(x)-K_0(x)K_2(x)],\quad x=\lambda^{1/2}. \tag{7}$$

The function $S_1(\lambda)$ is analytic in the complex plane of λ cut along the real positive axis because of the logarithmic and root singularities at the point $\lambda=0$. For $S_1(\lambda)$, the real axis is a Riemann–Schwarz symmetry axis:

$$S_1(\lambda^*)=-S_1^*(\lambda). \tag{8}$$

Therefore, the values of $S_1(\lambda)$ below and above the cut differ by the signs of the real parts. The value given in (7) is the limit of $S_1(\lambda)$ as the cut is approached from below—it is this branch that determines the integral (6) in accordance with causality considerations ($\mu^2\to\mu^2-i\varepsilon$). On the negative real axis, the function $S_1(\lambda)$ is purely imaginary and equal to

$$S_1(\lambda)=i\{\tfrac{1}{4}\pi^2x^2[J_1^2(x)-J_0(x)J_2(x)+N_1^2(x)$$
$$-N_0(x)N_2(x)]-\pi x\}, \tag{9}$$

where $x=(-\lambda)^{1/2}$, and $J_n(x)$ and $N_n(x)$ are Bessel and Neumann functions. In more detail, the analytic structure of the function $S_1(\lambda)$ can be represented by the everywhere convergent series

$$S_1(\lambda)=-\pi\left[1-\lambda^{1/2}+\sum_{k=1}^{\infty}\lambda^k(c_k\ln\lambda+c_k')\right]$$
$$-i\left[\ln\frac{4}{\gamma^2\lambda}-1-\frac{1}{2}\sum_{k=1}^{\infty}\lambda^k(c_k\ln^2\lambda+2c_k'\ln\lambda+c_k'')\right], \tag{10}$$

in which

$$c_k=\frac{\Gamma(k-{}^1/_2)}{2\pi^{1/2}\Gamma(k-1)\Gamma^2(k+1)},$$

$\gamma=1.781\ldots$, and the prime denotes the derivative with respect to k. The leading nonvanishing terms of this series in the limit $\lambda\to0$ were obtained in Refs. 1 and

2, in which the photon mass was regarded as a small parameter in the theory. They correspond to $k=0$ and are written out explicitly. At the point $\lambda=0$, the function S_1 has a discontinuity $-\pi$, which generates the classical shift $-\alpha\hbar w_0/2c^3$ of the mass of a uniformly accelerated charge in electrodynamics.

For $\lambda\gg 1$, it is however more convenient to use the asymptotic expansions

$$\operatorname{Re} S_1(\lambda)=-\frac{3\pi}{8\lambda^{1/2}}\left(1+\frac{5}{16\lambda}+\dots\right), \tag{11}$$

$$\operatorname{Im} S_1(\lambda)=-\frac{\pi}{2}e^{-2\lambda^{1/2}}\left(1+\frac{3}{4\lambda^{1/2}}-\frac{15}{32\lambda}+\dots\right). \tag{12}$$

Similarly, for negative λ when $-\lambda\gg 1$

$$S_1(\lambda)=-i\frac{3\pi}{8(-\lambda)^{1/2}}\left(1+\frac{5}{16\lambda}+\dots\right). \tag{13}$$

Besides the change in the mass of an accelerated source of a massive vector field, we consider the change in the mass of an accelerated source of a massive scalar field, which is determined by the change in the self-action (5) with replacement of the product of the current densities by the product of the charge densities and the product of the 4-velocities by unity:

$$j_\alpha(x)j_\alpha(x')\to\rho(x)\rho(x'),\quad \dot{x}_\alpha(\tau)\dot{x}_\alpha(\tau')\to 1. \tag{14}$$

Then the mass shift of a uniformly accelerated source of a massive scalar field is[6]

$$\Delta m_0^{\rm cl}=\frac{\alpha w}{2\pi}S_0(\lambda),\quad S_0(\lambda)=-\int\limits_0^\infty dz e^{-i\lambda/2z}\left[e^{iz}K_0(iz)-\left(\frac{\pi}{2iz}\right)^{1/2}\right] \tag{15}$$

where, as before, $\lambda=(\mu/w_0)^2$. Calculation of the integral S_0 by the same method leads to an expression that differs from (7) by a decrease in all the indices by unity and a change in the sign of the real part:

$$\begin{gathered}\operatorname{Re} S_0(\lambda)=\pi x^2[I_0(x)K_0(x)+I_1(x)K_1(x)]-\pi x,\\ \operatorname{Im} S_0(\lambda)=x^2[K_0^2(x)-K_1^2(x)],\quad x=\lambda^{1/2}.\end{gathered} \tag{16}$$

The function $S_0(\lambda)$ has the same analytic properties and symmetry as $S_1(\lambda)$, and on the negative real half-axis of λ is equal to

$$\begin{gathered}S_0(\lambda)=i\{-{}^1/_4\pi^2x^2[J_0^2(x)+J_1^2(x)+N_0^2(x)+N_1^2(x)]+\pi x\},\\ x=-\lambda^{1/2}.\end{gathered} \tag{17}$$

Its detailed analytic structure is represented by the everywhere convergent series

$$\begin{aligned}S_0(\lambda)=&-\pi\left[\lambda^{1/2}+\sum_{k=1}^\infty\lambda^k(c_k\ln\lambda+c_k')\right]\\ &-i\left[1-\frac{1}{2}\sum_{k=1}^\infty\lambda^k(c_k\ln^2\lambda+2c_k'\ln\lambda+c_k'')\right],\end{aligned} \tag{18}$$

in which

$$c_k=\frac{\Gamma(k-{}^1/_2)}{2\pi^{1/2}\Gamma^2(k)\Gamma(k+1)},$$

and the prime again denotes the derivative with respect to k.

As can be seen from the representations (10) and (18), both functions $S_n(\lambda)$ have at the point $\lambda=0$ logarithmic and square-root branch points, but, in contrast to S_1, the function S_0 is continuous at this point.

For $\lambda\gg 1$, the functions $S_0(\lambda)$ have the asymptotic expansion

$$\operatorname{Re} S_0(\lambda)=-\frac{\pi}{8\lambda^{1/2}}\left(1+\frac{9}{16\lambda}+\dots\right), \tag{19}$$

$$\operatorname{Im} S_0(\lambda)=-\frac{\pi}{2}e^{-2\lambda^{1/2}}\left(1-\frac{1}{4\lambda^{1/2}}+\frac{9}{32\lambda}-\dots\right). \tag{20}$$

and for negative λ when $-\lambda\gg 1$ the asymptotic behavior

$$S_0(\lambda)=-i\frac{\pi}{8(-\lambda)^{1/2}}\left(1+\frac{9}{16\lambda}+\dots\right). \tag{21}$$

Note that $\operatorname{Im} S_n(\lambda)$ decreases exponentially as $\lambda\to+\infty$ with coefficient and argument of the exponential which do not depend on n, the spin of the self-field, whereas $\operatorname{Re} S_n(\lambda)$ in the limit $\lambda\to+\infty$ and $\operatorname{Im} S_n(\lambda)$ as $\lambda\to-\infty$ decrease as $|\lambda|^{-1/2}$ with the same coefficient proportional to $2n+1$, the number of polarization states of the self-field. Since the spin of the field has a quantum origin, it is natural that the dependence on it appears only in conjunction with powers of the parameter $\lambda^{-1/2}$, which is proportional to $\hbar$. In the quasiclassical region $\lambda\gg 1$, this dependence on the spin of the field is weak, while in the quantum region, $\lambda\ll 1$, it is strong and qualitative [see the series (10) and (18)]. For example, for $\lambda\gg 1$

$$\operatorname{Re}\Delta m_n\approx-\frac{2n+1}{16}\frac{\alpha w_0^2}{\mu}, \tag{22}$$

while for $\lambda\ll 1$

$$\operatorname{Re}\Delta m_1\approx-{}^1/_2\alpha w_0,\quad \operatorname{Re}\Delta m_0\approx-{}^1/_2\alpha\mu. \tag{23}$$

As can be seen from the expressions we have obtained and Fig. 1, the functions $S_n(\lambda)$ on the real λ axis have the following general properties: 1) $\operatorname{Re} S_n(\lambda)=0$ for $\lambda<0$; 2) $\operatorname{Re} S_n(\lambda)<0$ for $\lambda>0$; 3) $\operatorname{Im} S_n(\lambda)<0$ for both $\lambda>0$ and $\lambda<0$. We shall show that these global properties are a reflection of fundamental physical properties of the motion of the charge, its field, and the interaction between them, namely, the locality of the interaction, unitarity, and causality. These properties are expressed concretely in the pointlike nature of the charge, the timelike nature of its trajectory, conservation of the curent, and the decrease in the proper time of motion along a timelike curve compared with the proper time along a straight line.

3. CONNECTION BETWEEN THE REAL AND IMAGINARY PARTS OF THE CHANGE IN THE SELF-ACTION

The change (5) of the self-action can be represented in terms of the Fourier transforms of the causal pro-

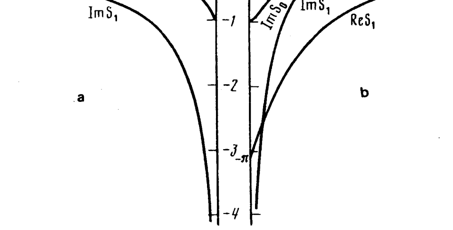

FIG. 1.

pagator and the current or charge density in the form of the Hilbert transform of the function $\Delta w(k^2) \equiv w(k^2)|_0^F$, i.e., the change in the function $w(k^2)$:

$$\Delta W_n(\mu^2)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dk^2\Delta w_n(k^2)}{k^2+\mu^2-i\varepsilon}, \quad (24)$$

$$w_1(k^2)=\int\frac{d^3k}{(2\pi)^3 2k_0}|j_\alpha(k)|^2, \quad w_0(k^2)=\int\frac{d^3k}{(2\pi)^3 2k_0}|\rho(k)|^2. \quad (25)$$

The function $w(k^2)$ (the spectrum of the source) is obtained by invariant integration of the square of the current or charge density over a 3-hypersurface with given k^2, i.e., with respect to 3-vectors k satisfying the condition $\mathbf{k}^2 \geqslant k^2\theta(k^2)$; by k_0 we understand $k_0=(\mathbf{k}^2-k^2)^{1/2} \geqslant 0$. For uniformly moving (um) vector and scalar charges it is

$$w_{1,0}^{\text{um}}(k^2)=\mp\alpha(k^2)^{1/2}\theta(k^2)\tau, \quad (26)$$

differing from zero only for spacelike momenta $k^2>0$, since a uniformly moving charge does not emit real quanta. Therefore, $\Delta w(k^2)=w(k^2)$ for $k^2<0$. The spectrum $w(k^2)$ does not contain an ultraviolet divergence and by virtue of the subtraction its change, i.e., the function $\Delta w(k^2)$, decreases at large spacelike momenta $k^2\to+\infty$, so that the change in the action is finite.[2)]

The imaginary part of the change in the action is equal to the change in the spectrum at the point $k^2=-\mu^2$, and the real part is determined by the integral of $\Delta w(k^2)$ over all real $k^2\neq-\mu^2$:

$$\operatorname{Im}\Delta W(\mu^2)=\frac{1}{2}\Delta w(-\mu^2), \quad \operatorname{Re}\Delta W(\mu^2)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dk^2\Delta w(k^2)}{k^2+\mu^2}. \quad (27)$$

The function $w(k^2)$ for $k^2<0$ must be positive due to conservation of probability: The probability that during the entire time τ the source does not emit free quanta is equal to $\exp[-2\operatorname{Im}\Delta W(\mu^2)]$ and must be less than unity. Therefore and in accordance with (27)

$$w(-\mu^2)=2\operatorname{Im}\Delta W(\mu^2)>0,$$

this being valid for any positive μ^2, since the form of the spectrum $w(k^2)$ does not depend on the mass of the free quanta. If allowance is made for conservation of the current, $k_\alpha j_\alpha(k)=0$, then it does indeed follow from (25) that for timelike k_α, for $k^2<0$, the function $w_1(k^2)$ is positive. For a scalar charge, $w_0(k^2)$ is manifestly positive for all k^2. Therefore, for $k^2<0$ the change in the spectrum is positive, $\Delta w(k^2)=w(k^2)>0$, for both vector and scalar charges.

We show that for $k^2>0$ the function $\Delta w(k^2)$, which contains the nonvanishing subtractional term (26), is also positive. For this, we go over in (5) to the representation

$$i\Delta^c(x-x',\mu)=\frac{1}{4\pi}\delta(z^2)-\frac{\mu}{8\pi z}H_1^{(2)}(\mu z), \quad (28)$$

where $H_1^{(2)}$ is a Hankel function, and z is the interval equal to $[-(x-x')^2]^{1/2}$ for $(x-x')^2<0$ and $-i[(x-x')^2]^{1/2}$ for $(x-x')^2>0$, i.e., it is equal to the proper time or proper length (multiplied by $-i$) of the segment of the 4-line joining the points x and x'. Then we obtain

$$\Delta W_1(\mu^2)=\frac{1}{2}e^2\int d\tau\,d\tau'\dot{x}_\alpha(\tau)\dot{x}_\alpha(\tau')\left[-\frac{\mu}{8\pi z}H_1^{(2)}(\mu z)\right]_0^F, \quad (29)$$

where the interval $z=[-(x_\alpha(\tau)-x_\alpha(\tau'))^2]^{1/2}$ is real and positive on a timelike trajectory of the charge. The term with $\delta(z^2)$ is omitted, since it makes a vanishing contribution on subtraction. Continuing this expression analytically to negative values of μ^2, and using the connection $H_1^2(-ix)=(-2/\pi)K_1(x)$ between the Hankel and MacDonald functions, and denoting $\mu^2=-\varkappa^2$, $\mu=-i\varkappa$, $\varkappa>0$ we obtain the purely imaginary function

$$\Delta W_1(-\varkappa^2)=\frac{1}{2}e^2\int d\tau\,d\tau'\dot{x}_\alpha(\tau)\dot{x}_\alpha(\tau')\left[-\frac{i\varkappa}{4\pi^2 z}K_1(\varkappa z)\right]_0^F. \quad (30)$$

For a scalar charge, $\Delta W_0(-\varkappa^2)$ is also purely imaginary, since it differs from (30) by the substitution $\dot{x}\dot{x}'\to1$.

Thus, $\operatorname{Re}\Delta W(\mu^2)=0$ for real $\mu^2<0$. For this very important property, the pointlike nature of the charge and the fact that its trajectory is timelike are essential; for it is in this case that the interval z takes only real positive values.

Formula (24) determines $\Delta W(\mu^2)$ as an analytic function in the lower half-plane of the complex μ^2. This analyticity is an explicit manifestation of the causal propagation of the field quanta. If we now note that $\operatorname{Re}\Delta W(\mu^2)=0$ on the real half-axis $\mu^2<0$, the function $\Delta W(\mu^2)$ can be analytically continued through the negative μ^2 axis into the upper half-plane of μ^2 by means of the Riemann–Schwarz principle, i.e., the relation

$$\Delta W(\mu^{2*})=-\Delta W^*(\mu^2). \quad (31)$$

Thus, the function $\Delta W(\mu^2)$ is analytic in the entire complex μ^2 plane with cut along the positive μ^2 axis, on each side of which it has the same imaginary part and real parts of opposite signs.

For such a function, we have the dispersion representations ($\operatorname{Im}\mu<0$)

$$\Delta W(\mu^2)=\frac{2i}{\pi}\int_0^{\infty}\frac{dx\,x\operatorname{Re}\Delta W(x^2)}{x^2-\mu^2},$$

$$\Delta W(\mu^2)=-\frac{2\mu}{\pi}\int_0^{\infty}\frac{dx\operatorname{Im}\Delta W(x^2)}{x^2-\mu^2}, \quad (32)$$

which enable us to recover the function $\Delta W(\mu^2)$ in the complex μ^2 plane from its real or imaginary part given in the lower edge of the cut, which is the real positive axis. In the limit $\operatorname{Im}\mu\to0$, the representations (32) go over into the Kramers–Kronig relations between the real and imaginary parts of $\Delta W(\mu^2)$ on the real axis of μ.

In accordance with the second of the representations (32), $\operatorname{Im}\Delta W(\mu^2)$ is positive in the entire complex plane of μ^2 (or the lower half-plane of μ) and, in particular, on the negative half-axis $\mu^2=-\varkappa^2$, $\mu=-i\varkappa$, $\varkappa>0$, where

$$\Delta W(-\varkappa^2)=i\frac{2\varkappa}{\pi}\int_0^{\infty}\frac{dx\operatorname{Im}\Delta W(x^2)}{x^2+\varkappa^2}. \quad (33)$$

The positive-definiteness of $\operatorname{Im}\Delta W(\mu^2)$ in the complex plane is a consequence of its positivity on the real positive half-axis $\mu^2>0$, the vanishing of $\operatorname{Re}\Delta W(\mu^2)$ on the real negative half-axis $\mu^2<0$, and, of course, the analyticity of $\Delta W(\mu^2)$ for $\operatorname{Im}\mu^2<0$ (unitarity, locality, causality).

Since (27) holds for any real μ^2, it follows from (33) that the change in the spectrum for spacelike k_α is not only positive but is also related to the change in the spectrum for timelike k_α:

$$\Delta w(k^2)=\frac{2(k^2)^{1/2}}{\pi}\int_0^\infty\frac{dx\Delta w(-x^2)}{x^2+k^2}>0,\quad k^2>0. \tag{34}$$

This connection cannot be seen from the representation (24) and (25) and arises as a result of allowance for the pointlike nature of the charge and the fact that its trajectory is timelike. Such locality and causality properties can be formulated naturally in the "coordinate" representation; they are manifested in the analytic properties of the functions that occur in the "momentum" representation.

The function $\Delta w(k^2)$ can be regarded as the mean number of virtual photons with momentum square k^2 emitted (and absorbed) by the source during the entire time over and above the quanta emitted (and absorbed) by the charge during this time in uniform motion. The positivity of $\Delta w(k^2)$ means that acceleration of the source leads to an increase in the rate of emission and absorption of quanta at all k^2.

It can now be seen from the representation (27) that the enhancement of the exchange of spacelike quanta always makes a positive contribution to Re $\Delta W(\mu^2)$, i.e., it decreases the mass of the charge. In contrast, the contribution due to the exchange of timelike quanta may have any sign, but its absolute magnitude is evidently less than the contribution of the spacelike quanta due to the exponential decrease of $\Delta w(k^2)$ as $k^2\to-\infty$.

In accordance with the first of the representations (32), on the negative half-axis $\mu^2=-\varkappa^2$, $\mu=-i\varkappa$, $\varkappa>0$, we have

$$\Delta W(-\varkappa^2)=\frac{2i}{\pi}\int_0^\infty\frac{dxx\,\mathrm{Re}\,\Delta W(x^2)}{x^2+\varkappa^2}. \tag{35}$$

Equating the right-hand sides of formulas (33) and (35), we obtain the important equation

$$\frac{2}{\pi}\int_0^\infty\frac{dxx\,\mathrm{Re}\,\Delta W(x^2)}{x^2+\varkappa^2}=\frac{2\varkappa}{\pi}\int_0^\infty\frac{dx\,\mathrm{Im}\,\Delta W(x^2)}{x^2+\varkappa^2}=\mathrm{Im}\,\Delta W(-\varkappa^2)>0, \tag{36}$$

which shows that Re $\Delta W(\mu^2)$ cannot be equal to zero or negative for all $\mu>0$. In accordance with (36), the functions x Re $\Delta W(x^2)/\varkappa$ and Im $\Delta W(x^2)$ have the same positive mean values, equal to Im $\Delta W(-\varkappa^2)$, as a result of averaging over x with probability density $2\pi^{-1}\varkappa(\varkappa^2+x^2)^{-1}$ (Cauchy distribution[7]). The positivity of the function Re $\Delta W(x^2)$ "on the average", which is equivalent to positivity of Im $\Delta W(-\varkappa^2)$, is a consequence of the unitarity and causality. We recall that the positivity of Im $\Delta W(\mu^2)$ for $\mu^2>0$ is a consequence of unitarity alone.

If Im $\Delta W(x^2)$ is bounded at the origin, then, letting $\varkappa$ tend to zero in the relation (36), we obtain

$$0\leqslant\frac{2}{\pi}\int_0^\infty\frac{dx}{x}\mathrm{Re}\,\Delta W(x^2)=\mathrm{Im}\,\Delta W(0)<\infty. \tag{37}$$

This means that the function Re $\Delta W(x^2)$ vanishes fairly rapidly as $x\to0$ and Re $\Delta W(0)=0$. But if Im $\Delta W(x^2)$ in the limit $x\to0$ tends to infinity logarithmically,

$$\mathrm{Im}\,\Delta W(x^2)=a\ln x^{-2}+b(x^2), \tag{38}$$

[$a>0$, and $b(x^2)$ is a function bounded at the origin], then it follows from the relation (36) that Re $\Delta W(x^2)$ tends in the limit $x\to0$ to a nonvanishing positive quantity:

$$\mathrm{Re}\,\Delta W(0)=\pi a>0. \tag{39}$$

These two types of behavior are, as it happens, exhibited by the changes in the action of uniformly accelerated sources of scalar and vector fields, for which

$$\Delta W_n(\mu^2)=-\tau\frac{\alpha w_0}{2\pi}S_n(\lambda),\quad \lambda=\left(\frac{\mu}{w_0}\right)^2. \tag{40}$$

4. CHANGE IN THE SELF-ACTION AND THE CLOCK PARADOX

With a view to a further elucidation of the physical mechanism which is the basis of the integral (dispersion) relations of the previous section, we return to the space-time description of the change in the self-action of an accelerated charge. We restrict ourselves to trajectories for which the "distance" z between any two points is a function of only the "arc length" $u=\tau-\tau'$ between them:

$$z(\tau,\tau')\equiv[-(x(\tau)-x(\tau'))^2]^{1/2}=z(\tau-\tau'). \tag{41}$$

The trajectories of an electric charge in an arbitrary constant electromagnetic field have this property, for example (see Ref. 2). Then instead of (30) we obtain

$$\Delta W_1(-\varkappa^2)=i\tau\frac{\alpha\varkappa}{\pi}\int_0^\infty du\left[\frac{1}{2}\frac{d^2z^2}{du^2}\frac{K_1(\varkappa z)}{z}-\frac{K_1(\varkappa u)}{u}\right]. \tag{42}$$

Since

$$\int_0^\infty du\left[\frac{K_1(\varkappa u)}{u}-\frac{dz}{du}\frac{K_1(\varkappa z)}{z}\right]=0, \tag{43}$$

the expression (42) can be written in the form

$$\Delta W_1(-\varkappa^2)=i\tau\frac{\alpha\varkappa}{\pi}\int_0^\infty du\frac{d}{du}\left[z\left(\frac{dz}{du}-1\right)\right]\frac{K_1(\varkappa z)}{z}. \tag{44}$$

Hence, after integration by parts, we obtain

$$\Delta W_1(-\varkappa^2)=i\tau\frac{\alpha\varkappa^2}{\pi}\int_0^\infty dz\left(\frac{dz}{du}-1\right)K_2(\varkappa z). \tag{45}$$

Since $K_2(\varkappa z)>0$ and by virtue of the clock paradox $z'(u)>1$ [see (4) and below], for negative μ^2 the imaginary part of the change in the action is positive. Moreover, since $x^2K_2(x)$ is a monotonically decreasing function of x, it follows that Im $\Delta W_1(-\varkappa^2)$ decreases monotonically with increasing $\varkappa^2$.

The property Im $\Delta W_1(\mu^2)>0$ for $\mu^2<0$ is a consequence of the "clock paradox," according to which the proper time of motion along a straight line between two timelike points is greater than the proper time of motion along a curve between them: $z(u)>u$. It can also be said that

$$z(u+\Delta u)>z(u)+z(\Delta u), \tag{46}$$

since the proper time of motion along the straight line which subtends the arc $u+\Delta u$ is greater than the total proper time of motion along the broken curve consisting of the two straight lines subtending the arcs u and Δu.

Since for any trajectory and sufficiently small Δu

$$z(\Delta u)=\Delta u+{}^1/_{24}a^2(\Delta u)^3+\ldots, \tag{47}$$

where a^2 is the square of the 4-acceleration, it follows from (46) that $z'(u)>1$, which leads to positivity of $\operatorname{Im}\Delta W_1(\mu^2)$ for $\mu^2<0$. Note that formula (47) is important for the integral transformations made above.

For a scalar charge, we now obtain instead of (42)

$$\Delta W_0(-\varkappa^2)=i\tau\frac{\alpha\varkappa}{\pi}\int_0^\infty du\left[-\frac{K_1(\varkappa z)}{z}+\frac{K_1(\varkappa u)}{u}\right]. \tag{48}$$

Since $z(u)>u$ by the clock paradox, and the function $x^{-1}K_1(x)$ decreases monotonically, the integrand is positive and, therefore, $\operatorname{Im}\Delta W_0(-\varkappa^2)>0$.

However, it is convenient to represent formula (48) for a scalar charge in a firm similar to (45). For this, we transform it by means of the relation (43) and integration by parts, obtaining

$$\Delta W_0(-\varkappa^2)=i\tau\frac{\alpha\varkappa^2}{\pi}\int_0^\infty dz\left(1-\frac{u}{z}\right)K_2(\varkappa z). \tag{49}$$

It again follows from this formula that the function $\operatorname{Im}\Delta W_0(-\varkappa^2)$ is positive, because $z>u$, and that it also decreases monotonically with increasing $\varkappa^2$.

Returning to the positive axis of μ^2 by means of analytic continuation of the representations (45) and (49), the contour passing below the point $\mu^2=0$, we obtain

$$\Delta W_n(\mu^2)=\tau\frac{\alpha\mu^2}{2}\int_0^\infty dz f_n(z)H_2^{(2)}(\mu z), \tag{50}$$

$$f_0(z)=1-u/z,\quad f_1(z)=dz/du-1. \tag{51}$$

Thus, the changes in the action for sources of scalar and vector fields can be regarded as certain "averages" of the relative slowing down of the time, $(z-u)/z$, and the rate of slowing down of the time, $d(z-u)/du$; these are the two quantities that characterize the clock paradox integrally and differentially. More precisely, $\Delta W_n(\mu^2)$ are the integral Meijer transforms (or K transforms)[8,9] of the functions $f_n(z)$, which are the relative slowing down and the rate of slowing down of the time on the trajectories of the charges.

Since the Bessel and Neumann functions forming the real and imaginary part of the function

$$H_2^{(2)}(\mu z)=J_2(\mu z)-iN_2(\mu z)$$

oscillate, it is difficult to draw from the representation (50) any conclusions about the signs of the real and imaginary parts of $\Delta W(\mu^2)$, although the imaginary part of ΔW must be positive for $\mu^2>0$ by unitarity, and it can be seen from the representation (25) that this is so. Of course, the functions $\Delta W(\mu^2)$, which are determined by the expressions (45), (49), and (50), satisfy the general relations of the preceding section, so that for them all the consequences that flow from these relations, in particular, positivity of $\operatorname{Re}\Delta W(\mu^2)$ "on the average," holds for them.

For a uniformly accelerated charge, the functions $f_n(z)$ are very simple:

$$f_0(z)=1-\frac{2}{w_0z}\operatorname{Arsh}\frac{w_0z}{2},\quad f_1(z)=\left[1+\left(\frac{w_0z}{2}\right)^2\right]^{1/2}-1. \tag{52}$$

In the integral

$$\operatorname{Re}\Delta m_n=-\frac{1}{2}\alpha\mu^2\int_0^\infty dz f_n(z)J_2(\mu z) \tag{53}$$

it is values $z\sim\mu^{-1}$ which are effective for the real part of the mass shift. From this, we can readily obtain the limiting cases (22) and (23) and the qualitative estimate

$$\operatorname{Re}\Delta m_n\sim-{}^1/_2\alpha\mu f_n(c\mu^{-1}),\quad c\sim1.$$

It can be seen that the linearity of the shift in w_0 and the fact that it is independent of μ arise only if the "perturbation" $f_n(z)$ has a linearly increasing asymptotic behavior over the z interval. Such an asymptotic behavior is possible only for $f_1(z)$, i.e., for the source of a vector field, since $f_0(z)$ is always less than unity.

In the representation (53), the mass shift of a vector charge acquires an ever greater formal similarity with the shift of a parity-degenerate level of the hydrogen atom under the influence of an external electromagnetic field ε (we have the matrix element $V_{12}\sim e\varepsilon a$, where a is the diameter of the atom) and a perturbation that lifts the parity degeneracy (matrix elements V_{11} and V_{22}):

$$\Delta E=\frac{V_{11}+V_{22}}{2}\mp\left[\left(\frac{V_{11}-V_{22}}{2}\right)^2+V_{12}^2\right]^{1/2}, \tag{54}$$

see also formula (55.8) in Ref. 10. Indeed, for a uniformly accelerated vector charge

$$\operatorname{Re}\Delta m_1\sim-\frac{1}{2}\alpha[(\mu^2+w_0^2)^{1/2}-\mu], \tag{55}$$

so that the role of V_{12} and $(V_{11}\pm V_{22})/2$ are played, respectively, by w_0 amd μ multiplied by α. It is however difficult to relate linearity of the asymptotic behavior of $f_1(z)$ to parity degeneracy.

5. THE FUNCTION $\Delta W(\mu^2)$ IN ELECTRODYNAMICS

Although the mass of a real photon in electrodynamics is zero, the function $\Delta W(\mu^2)$ arises directly when one considers the radiation corrections. For example, if one takes into account the polarization correction of second order to the photon propagator, then instead of the propagator, $-i(k^2-i\varepsilon)^{-1}$ it is necessary to use

$$-i\left(\frac{1}{k^2-i\varepsilon}+\int_{4m_1^2}^\infty\frac{d\mu^2\rho_2(\mu^2)}{k^2+\mu^2-i\varepsilon}\right),\quad \rho_2(x)=\frac{\alpha}{3\pi}\frac{1}{x}\left(1+\frac{2m_1^2}{x}\right)\left(1-\frac{4m_1^2}{x}\right)^{1/2} \tag{56}$$

[see Ref. 11, Eq. (35.10)]. Here, m_1 is the mass of one of the particles of a lepton pair formed by a virtual photon in the intermediate state, and $\rho_2(x)$ is the Källén–Lehmann spectral density. Then for the change in the self-action of an accelerated charge we obtain instead of (24)

$$\Delta W_1(0)+\int_{4m_1^2}^\infty d\mu^2\rho_2(\mu^2)\Delta W_1(\mu^2), \tag{57}$$

where μ^2 is the square of the mass of the lepton pair produced by the virtual photon. Thus, in electrodynamics $\mu=0$ in the leading term, but the second, polarization term contains the function $\Delta W(x^2)$.

In the integral (57), it is values $\mu^2\sim4m_1^2$ that are effective. Therefore, the parameter λ, on which in reality [in the approximation (3)] the function $\Delta W(x^2)$ depends, is of order $(m_1/w_0)^2$. If $m_1=m$, then $\lambda_{eff}\sim\beta^{-2}$ and we are at the limit of applicability of the classical treatment of the motion of an accelerated charge; see (3). But if the accelerated particle is significantly heavier than the particles of the virtual pair, as is the case, for example, for an accelerated muon and an electron–positron pair then $\lambda\sim(m_1/w_0)^2\ll\beta^{-2}$ and classical description of the motion of the heavy particle is valid.

The polarization term in (57) has real and imaginary parts, which are determined by the corresponding parts of the function $\Delta W_1(\mu^2)$. Twice its imaginary part is the probability of formation of a lepton pair by an accelerated charge. For a uniformly accelerated charge, such a probability was found by Nikishov.[12] In the general case, it can be represented by the form

$$\int_{4m_1^2}^{\infty} dx\rho_2(x)\Delta w_1(-x) \qquad (58)$$

[cf. formula (6.37) of Schwinger's Ref. 13].

When allowance is made for the higher radiative corrections in the relations (56)–(58), the function $\rho_2(x)$ must be replaced by the exact Källén–Lehmann spectral density $\rho(x)$, for which the minimal value of the square of the mass of the intermediate state is equal to zero and not $4m_1^2$, as for $\rho_2(x)$. Therefore, in electrodynamics the function $\Delta W_1(\mu^2)$ always has physical meaning in the region of sufficiently small positive μ^2.

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Note Added in Proof (March 24, 1982). In Ref. 2, formulas (37), (39), and (40) were obtained for the electromagnetic mass of a charge, the mass being determined by the state of the self-field at the time t. The last of these formulas is also valid for the field mass of a charge with a massive vector self-field ($\mu\neq 0$):

$$m_1^{\text{field}}=\frac{1}{2}\gamma\left[-\int dV A_\alpha j_\alpha+\int dV\,\partial_\alpha(F_{\alpha\beta}A_\beta)\right].$$

The variable mass shift of the charge $\Delta m_1^{\text{cl}}=m_1^{\text{field}}\big|_0^F$ for the electromagnetic field is completely given by the divergence term [see (51) in Ref. 2], and for a massive vector field by the first term, which reduces at large t to (5) if the potential is expressed in terms of the current.

[1)]We use a system of units in which $\hbar=c=1$ except when it is desirable to emphasize the quantum and relativistic nature of the quantities. The charge e is measured in Heaviside units, $\alpha=e^2/4\pi\hbar c$. We use the notation $a_\alpha=(\mathbf{a}, ia_0)$, $a_\alpha b_\alpha=\mathbf{a}\cdot\mathbf{b}-a_0b_0$. For the sources of the vector and scalar fields we use the expressions "vector charge" and "scalar charge."

[2)]The self-action of a uniformly moving charge determined by the function (26) diverges linearly with respect to the momentum and leads to a divergent field mass $\pm\alpha k_{\max}^{1/2}/\pi$, which is usually expressed in the form $\pm\alpha/2a$ in terms of the cutoff radius a. In a quantum treatment of the motion of a charge, the divergence of the field mass is different and depends on the spin of the charge. However, in both the classical and the quantum approach these divergences disappear from the physical observables, which are differences.

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