Radiation of relativistic electrons moving along the arc of a circle

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The methods of classical theory are used to study the radiation of an electron moving along the arc of a circle. The main attention is concentrated on obtaining the exact analytical characteristics of the radiation in various cases.

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The radiation of electrons moving in systems of the "short-magnet" type has a number of remarkable features which, besides presenting theoretical interest, may turn out to be of practical importance. Some of these features have already been pointed out in several theoretical studies.[1–3] Here we attempt to give the most complete possible analysis of the properties of the radiation of an electron moving along the arc of a circle.

Motion along the arc of a circle is the simplest example of motion in a "short magnet" (for the case of a small arc length). On the one hand, many properties of the radiation generated in such motion can be calculated exactly in the classical theory and can be analyzed most completely in this way. On the other hand, all of the important features of the radiation which arises in motion in a short magnet of arbitrary structure occur also in this special case. Finally, synchrotron radiation is a particular case of the problem discussed here in the limit of an infinitely large number of revolutions of the electron along the arc.

1. GENERAL FORMULAS FOR THE RADIATION OF AN ELECTRON MOVING ALONG THE ARC OF THE CIRCLE

Let an electron with charge e move with a velocity $v = c\beta$ which is constant in magnitude (where c is the velocity of light), up to a certain moment of time in a straight line, then under the influence of external forces let it describe an arc of a circle of radius R and sector angle 2γ, and then let the electron again continue its motion along a straight line. The choice of the coordinate system is clear from Fig. 1.

The electronic vector $\mathbf{E}$ of the radiation field of the electron in the wave zone at a point with coordinates $\mathbf{r}$ at a time t, as is well known,[4] has the form

$$\mathbf{E}(\mathbf{r}, t) = (e/2\pi cr)\int_{-\infty}^{\infty} \mathbf{f}(\omega, \mathbf{n}) \exp[i\omega(t - r/c)]\,d\omega,$$
$$\mathbf{f}(\omega, \mathbf{n}) = \int_{-\infty}^{\infty} [\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]][1-(\mathbf{n}\boldsymbol{\beta})]^{-2}\exp\{-i\omega[t-(\mathbf{n}\mathbf{a})/c]\}\,dt, \tag{1}$$
$$\boldsymbol{\beta} = \mathbf{v}/c,\quad \dot{\boldsymbol{\beta}} = d\boldsymbol{\beta}/dt,\quad \mathbf{n} = \mathbf{r}/r = (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta).$$

Here $\mathbf{a} = \mathbf{a}(t)$ is the radius vector of the moving charge at a time t. From Eq. (1) we obtain the spectral and angular distribution of the total radiated energy

$$d\mathscr{E} = (e^2/4\pi^2 c)\,|\mathbf{f}(\omega, \mathbf{n})|^2 d\omega d\Omega,\quad d\Omega = \sin\theta d\theta d\varphi,\quad 0<\omega. \tag{2}$$

Introducing the unit vectors of a spherical coordinate system $\mathbf{n}, \mathbf{e}_\varphi, \mathbf{e}_\theta$, we see from Eq. (1) that

$$\mathbf{f} = \mathbf{e}_\varphi f_\sigma + \mathbf{e}_\theta f_\pi,\quad |\mathbf{f}|^2 = |f_\sigma|^2 + |f_\pi|^2, \tag{3}$$

where the quantities $f_\sigma = (\mathbf{f}\mathbf{e}_\varphi)$, and $f_\pi = (\mathbf{f}\mathbf{e}_\theta)$ characterize the σ and π components of the linear polarization of the electron radiation.[5,6]

In our case of motion along the arc of a circle it is easy to obtain the expressions[1)]

$$f_\sigma = \beta\int_{\varphi-\gamma}^{\varphi+\gamma} (\cos x - \mu)\, p^{-2}(x)\, e^{-iq\psi} dx,$$
$$f_\pi = \beta\cos\theta \int_{\varphi-\gamma}^{\varphi+\gamma} p^{-2}(x)\, e^{-iq\psi}\sin x\, dx, \tag{4}$$
$$p = p(x) = 1 - \mu\cos x,\quad \psi = \psi(x) = x - \mu\sin x,$$
$$\mu = \beta\sin\theta,\quad q = \omega/\omega_0,\quad \omega_0 = c\beta/R,$$

which are the starting point for analysis of the properties of the radiation.

Integrating over frequencies and angles in Eq. (2), we find the total radiated energy in the form

$$\mathscr{E} = W_0 T,\quad W_0 = \frac{2}{3}\frac{ce^2}{R^2}\beta^4\left(\frac{E}{mc^2}\right)^4,$$
$$E = \frac{mc^2}{(1-\beta^2)^{1/2}},\quad T = \frac{2\gamma}{\omega_0} = \frac{2\gamma R}{c\beta}. \tag{5}$$

Here W_0 is the total power of synchroton radiation of an electron moving along a circular arc of radius R,[5,6] and T is the time of motion along a circular arc with sector angle 2γ. Thus we have the obvious result that the total radiated energy is proportional to the time of motion along the circular arc. This simple fact shows that Eq. (2) can conveniently be rewritten in the form

$$d\mathscr{E} = W_0 T F(\beta, \gamma; \varkappa; \theta, \varphi)\, d\varkappa d\Omega,\quad F = F_\sigma + F_\pi,\quad \varkappa = q\gamma = \gamma\omega/\omega_0,$$
$$F_{\sigma,\pi}(\beta, \gamma; \varkappa; \theta, \varphi) = 3(1-\beta^2)^2(4\pi\gamma\beta)^{-2}|f_{\sigma,\pi}(\omega, n)|^2, \tag{6}$$

where the functions $F_{\sigma,\pi}$ characterize the spectral and angular distribution of the relative average power of polarized radiation in an arc with sector angle 2γ and

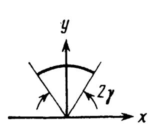

FIG. 1.

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we have the obvious property

$$W=W_\sigma+W_\pi=\int_0^\infty d\varkappa\oint d\Omega F=1,$$
$$W_\sigma=(6+\beta^2)/8,\quad W_\pi=(2-\beta^2)/8. \tag{7}$$

From Eq. (7) it follows in particular that the degree of linear polarization of the radiation (separated into σ and π components) coincides with the polarization of the synchrotron radiation. This, however, is a consequence of the general properties of the radiation which were proved in Refs. 7.

From Ref. 4 it follows that for $\omega\to\infty$ $f_{\sigma,\pi}$ fall off as ω^{-1} and consequently the spectral and angular distributions of the radiation fall off as ω^{-2}. This conclusion is valid only in our particular case—the acceleration in the motion considered is a discontinuous function of the time (it undergoes finite jumps). If the acceleration is continuous, then, as follows from the general theorems of the theory of Fourier integrals, the spectral and angular distributions fall off no slower than ω^{-4}. This is confirmed by the examples discussed in Ref. 2.

2. ANGULAR DISTRIBUTION OF THE RADIATION OF AN ELECTRON MOVING ALONG THE ARC OF A CIRCLE

The integration over frequency in Eq. (6) can be carried out exactly, and after simple but straightforward calculations we obtain

$$F_{\sigma,\pi}(\beta,\gamma;\theta,\varphi)=\int_0^\infty F(\beta,\gamma;\varkappa;\theta,\varphi)\,d\varkappa$$
$$=\Phi^0_{\sigma,\pi}+\Phi^1_{\sigma,\pi}(\varphi+\gamma)-\Phi^1_{\sigma,\pi}(\varphi-\gamma), \tag{8}$$

where Φ^0 characterizes the angular distribution of the synchrotron radiation averaged over a turn[5,6]:

$$\Phi_\sigma^0(\beta,\theta)=\frac{3(1-\beta^2)^2(4+3\mu^2)}{64\pi(1-\mu^2)^{7/2}},$$
$$\Phi_\pi^0(\beta,\theta)=\frac{3(1-\beta^2)^2(4+\mu^2)\cos^2\theta}{64\pi(1-\mu^2)^{7/2}}, \tag{9}$$

and the functions $\Phi^1_{\sigma,\pi}(x)$ are defined by the formulas

$$\Phi_\sigma^1(x)=Q\Big\{\frac{6\mu p^4(x)(4+3\mu^2)}{(1-\mu^2)^{1/2}}\operatorname{arctg}\frac{\mu\sin x}{p(x)+(1-\mu^2)^{1/2}}$$
$$+[(23\mu^2-2)p^3(x)+(9\mu^2-2)(1-\mu^2)p^2(x)-2(1-\mu^2)^2p(x)$$
$$+6(1-\mu^2)^3]\sin x\Big\}(1-\mu^2),$$
$$\Phi_\pi^1(x)=Q\Big\{\frac{6\mu p^4(x)(4+\mu^2)}{(1-\mu^2)^{1/2}}\operatorname{arctg}\frac{\mu\sin x}{p(x)+(1-\mu^2)^{1/2}}$$
$$+[(2+13\mu^2)p^3(x)+(3\mu^2+2)(1-\mu^2)p^2x+2(1-\mu^2)p(x)$$
$$-6(1-\mu^2)^3]\sin x\Big\}\cos^2\theta, \tag{10}$$
$$Q=(1-\beta^2)^2[128\pi\mu\gamma(1-\mu^2)^3p^4(x)]^{-1}.$$

Integration over the angle φ in Eq. (8) leads to the expressions

$$\int_0^{2\pi}F_{\sigma,\pi}(\beta,\gamma;\theta,\varphi)\,d\varphi=2\pi\Phi^0_{\sigma,\pi}(\beta,\theta), \tag{11}$$

which describe the angular distribution averaged over a turn of the synchrotron radiation. This property is also a consequence of the general statements proved in Ref. 7.

From Eqs. (8)–(10) it also follows that for $\gamma=\pi N$ ($N=1,2,\ldots$; the electron executes N complete turns)

$$F_{\sigma,\pi}(\beta,\gamma;\theta,\varphi)=\Phi^0_{\sigma,\pi}(\beta,\theta),$$

which again coincides with the angular distribution averaged over a turn of synchrotron radiation. For $\gamma\to 0$ we obtain from Eqs. (8)–(10)

$$F_{\sigma,\pi}(\beta,0;\theta,\varphi)=3(1-\beta^2)A^2_{\sigma,\pi}/8\pi p^5(\varphi),$$
$$A_\sigma=\mu-\cos\varphi,\quad A_\pi=\cos\theta\sin\varphi. \tag{12}$$

These are the instantaneous angular distributions of the polarized synchrotron radiation.[5]

Thus, Eqs. (8)–(10) with variation of γ in the range from 0 to π describe the continuous transition from the instantaneous angular distribution of synchrotron radiation to the average over a revolution.

3. GENERATION OF LOW-FREQUENCY RADIATION IN MOTION ALONG A CIRCULAR ARC

Letting $\omega\to 0$ in Eq. (6), we obtain

$$F_{\sigma,\pi}(\beta,\gamma;0;\theta,\varphi)=3\left[\frac{(1-\beta^2)B_{\sigma,\pi}\sin\gamma}{2\pi\gamma p(\varphi+\gamma)p(\varphi-\gamma)}\right]^2,$$
$$B_\sigma=\cos\varphi-\mu\cos\gamma,\quad B_\pi=\cos\theta\sin\varphi, \tag{13}$$

or after integration over the angles[2,3]

$$F_{\sigma,\pi}(\beta,\gamma;0)=\oint F_{\sigma,\pi}(\beta,\gamma;0;\theta,\varphi)\,d\Omega,$$
$$F_\sigma(\beta,\gamma;0)=3(1-\beta^2)^2(4\pi\gamma^2\beta^2)^{-1}\nu\ln(1+\nu)/(1-\nu),$$
$$F_\pi(\beta,\gamma;0)=3(1-\beta^2)^2(4\pi\gamma^2\beta^2\nu)^{-1}[\ln(1+\nu)/(1-\nu)-2\nu],$$
$$\nu=\beta\sin\gamma/(1-\beta^2\cos^2\gamma)^{1/2}. \tag{14}$$

The functions $F_{\sigma,\pi}(\beta,\gamma;0)$ are maximal for $\gamma=0$ and at the maximum they have the form

$$F_\sigma(\beta,0;0)=3/4F(\beta,0;0),\quad F_\pi(\beta,0;0)=1/4F(\beta,0;0),$$
$$F(\beta,0;0)=2(1-\beta^2)/\pi. \tag{15}$$

For $\gamma=\pi N$ (an integer number of revolutions) the radiation at zero frequency disappears. Consequently the relative contribution of low frequencies to the total radiation is greatest in a "short magnet" and decreases with increase of the arc angle 2γ. This indicates that with increase of γ ($0<\gamma<\pi$) the maximum in the radiation spectrum shifts to the short-wavelength region.

Equation (14) also permits us to obtain a distinct criterion of smallness of γ, namely that γ can be considered small if $\nu\ll 1$, which is equivalent to the inequality

$$\gamma\ll(1-\beta^2)^{1/2}=mc^2/E. \tag{16}$$

The criterion (16) characterizes a "short magnet."

It follows from Eq. (15) that on division into σ and π components in a short magnet the radiation at zero frequency is polarized in the ratio $\frac{3}{4}$ and $\frac{1}{4}$, regardless of the electron energy. However, calculation of $\mathbf{f}(0,\mathbf{n})$ shows that the radiation at zero frequency for arbitrary γ and β has complete linear polarization, and the vector $\mathbf{E}$ of the radiation field at zero frequency is parallel to the vector $\mathbf{l}(\gamma)$:

$$\mathbf{l}(\gamma)=(\cos\varphi-\mu\cos\gamma)\mathbf{e}_\varphi+\cos\theta\sin\varphi\,\mathbf{e}_\theta. \tag{17}$$

4. RADIATION OF AN ELECTRON MOVING IN A SHORT MAGNET

As we already noted, the angle γ can be considered small if it satisfies the criterion (16). In this case, expanding the expressions (4) and (6) in series in γ and retaining only the first nonzero terms, we find that the radiation in a short magnet is completely linearly polarized for any β and ω, and the vector $\mathbf{E}$ is parallel to the vector $\mathbf{l}(\gamma=0)$ defined by Eq. (17). The spectral and angular distribution of the radiation in a short magnet has the form

$$F_{\sigma,\pi}(\beta,0;\varkappa;\theta,\varphi)=3\left[\frac{(1-\beta^2)A_{\sigma,\pi}\sin\varkappa p(\varphi)}{2\pi\varkappa p^3(\varphi)}\right]^2. \tag{18}$$

Integration over frequency in Eq. (18) leads to the expressions (12) which characterize the instantaneous angular distribution of synchrotron radiation.

It is evident from Eq. (18) that the maximum in the radiation spectrum of an electron in a short magnet corresponds to $\omega=0$, and on integration over angle for $\omega=0$ we obtain Eq. (15), in accordance with the conclusions of Sec. 3. With increase of ω, as follows from Eq. (18), the spectral and angular distribution falls off slowly to zero (as ω^{-2}), and the effective width of the spectrum is $\Delta\varkappa=\pi/2(1-\beta^2)$ or, converting to frequency,

$$\Delta\omega=\pi\omega_0/2\gamma(1-\beta^2)=\pi c\beta/l(1-\beta^2)=(\pi c\beta/l)(E/mc^2)^2, \tag{19}$$

where l is the length of the circular arc. In this way we arrive at the conclusion that in a short magnet "white noise" is actually generated with an effective spectral width $\Delta\omega$. If we assume $l\sim 10-100$ cm, then for an electron with energy ~1 GeV we have $\Delta\omega\sim 10^{17}-10^{16}$ sec^{-1}, i.e., white noise is generated with an effective spectrum width from zero up to the ultraviolet, and the power of the signal is determined by the power of the synchrotron radiation, but the signal itself has the nature of a narrow ray bundle (with an apex angle $\Delta\Omega\sim mc^2/E$) directed along the electron velocity.

Integrating over angle in Eq. (18), we can find the spectral distribution of the radiation in the form

$$\begin{gathered}F_{\sigma,\pi}(\beta,\varkappa)=\oint F_{\sigma,\pi}(\beta,0;\varkappa;\theta,\varphi)\,d\Omega\\ =3(1-\beta^2)^2(4\pi\varkappa^2\beta^3)^{-1}\int_{1-\beta}^{1+\beta}x^{-6}Q_{\sigma,\pi}\sin^2 x\varkappa\,dx,\\ Q_\sigma=\beta^2(x^2+2x+\beta^2-3)+2\beta|1-x|,\quad Q_\pi=x^2-2x+\beta^2+1-2\beta|1-x|.\end{gathered} \tag{20}$$

The integrals in the right-hand side of Eq. (20) can obviously be expressed by elementary transformations in terms of the sine and cosine integrals, but the resulting expressions are much more cumbersome than Eq. (20) and we shall not give them here.

5. RADIATION OF A NONRELATIVISTIC ELECTRON

Exact results can be obtained also for a nonrelativistic electron ($\beta=0$). The spectral and angular distribution for $\beta=0$ has the form

$$\begin{gathered}F_\sigma(0,\gamma;\varkappa;\theta,\varphi)=3(S_1+2S_2\cos 2\varphi),\\ F_\pi(0,\gamma;\varkappa;\theta,\varphi)=3(S_1-2S_2\cos 2\varphi)\cos^2\theta,\\ S_1=\frac{\sin^2(\varkappa-\gamma)}{16\pi^2(\varkappa-\gamma)^2}+\frac{\sin^2(\varkappa+\gamma)}{16\pi^2(\varkappa+\gamma)^2},\quad S_2=\frac{\sin(\varkappa+\gamma)\sin(\varkappa-\gamma)}{16\pi^2(\varkappa+\gamma)(\varkappa-\gamma)}.\end{gathered} \tag{21}$$

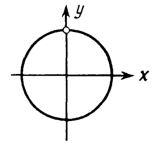

FIG. 2.

After integration over angle we obtain the spectral distribution in the form

$$F_\sigma(0,\gamma;\varkappa)=3F_\pi(0,\gamma;\varkappa)=12\pi S_1. \tag{22}$$

Further, if we integrate over frequency in Eq. (21), the angular distribution of the radiation of a nonrelativistic electron has the form

$$\begin{gathered}F_\sigma(0,\gamma;\theta,\varphi)=\frac{3}{16\pi}\left(1+\frac{\sin\gamma}{\gamma}\cos\gamma\cos 2\varphi\right),\\ F_\pi(0,\gamma;\theta,\varphi)=\frac{3\cos^2\theta}{16\pi}\left(1-\frac{\sin\gamma}{\gamma}\cos\gamma\cos 2\varphi\right).\end{gathered} \tag{23}$$

The same result follows for $\beta=0$ from Eq. (8). From Eq. (22) it follows that the main contribution to the radiation is from the region of frequencies $\varkappa\sim\gamma$ with an effective width $\sim 1/\gamma$.

6. RADIATION OF AN ELECTRON EXECUTING AN INTEGER NUMBER OF REVOLUTIONS. FORMATION OF THE SYNCHROTRON RADIATION SPECTRUM

Now let an electron complete a finite number N of complete revolutions ($\gamma=\pi N$, $N=1,2,\ldots$). In this case it is convenient to fix the point of entry and exit of the electron, choosing it to be located, for example, on the y axis. The choice of the coordinate system in this case is made clear by Fig. 2. Equation (6) is now written in the form

$$\begin{gathered}d\mathscr{E}=W_0Tg_N(q)(G_\sigma+G_\pi)\,dq\,d\Omega,\qquad q>0,\\ G_\sigma=[3(1-\beta^2)^2/4\pi^3]\left|\left\{q\int_0^{\varphi}\sin x\,e^{-iq\psi(x)}dx\right.\right.\\ \left.\left.-ie^{-iq\psi(\varphi)}p^{-1}(\varphi)\sin\varphi+\pi q\mathbf{E}_q'(q\mu)\right\}\sin\pi q+\pi q\mathbf{J}_q'(q\mu)\cos\pi q\right|^2,\\ G_\pi=[3(1-\beta^2)^2/4\pi^3\beta^2]\operatorname{ctg}^2\theta\left|\left\{q\int_0^{\varphi}e^{-iq\psi(x)}dx\right.\right.\\ \left.\left.-ie^{-iq\psi(\varphi)}p^{-1}(\varphi)+i\pi q\mathbf{E}_q(q\mu)\right\}\sin\pi q+i\pi q\mathbf{J}_q(q\mu)\cos\pi q\right|^2,\\ g_N(q)=(N\sin^2\pi q)^{-1}\sin^2(\pi qN).\end{gathered} \tag{24}$$

Here $\mathbf{J}_\alpha(x)$ and $\mathbf{E}_\alpha(x)$ are the Anger and Weber functions,[8] and the prime designates their derivatives with respect to the entire argument. Thus we see that the number of revolutions N enters only in the factor $g_N(q)$, and the remaining part of the expressions in Eq. (24) do not depend on N and coincide with the spectral and angular distribution of radiation of an electron executing a single turn, $g_1(q)=1$. The spectral-angular distribution depends on the angle φ. It is physically obvious that the asymmetry of the distribution in the angle φ is due to the presence of the point of entry (exit) of the electron. Integration over frequency in Eq. (24) leads to the formulas of Sec. 2 and removes the asymmetry in φ.

The formulas (24) permit us to trace how the synchrotron radiation spectrum is formed as $N\to\infty$. It is easy to see that

$$\lim_{N\to\infty} g_N(q) = \sum_{n=-\infty}^{\infty} \delta(q-n), \qquad (25)$$

and here if $N \gg 1$ the effective width of the spectral line $q=n$ is $\Delta q \sim 1/N$. From Eq. (24) we find for $N\to\infty$:

$$dW = \lim_{N\to\infty} d\mathscr{E}/T = (ce^2\beta^2/2\pi R^2) \sum_{n=1}^{\infty} n^2[\beta^2 J_n'^2(n\mu) + \mathrm{ctg}^2\,\theta J_n^2(n\mu)]\,d\Omega. \qquad (26)$$

Here we have taken into account that $\mathbf{J}_n(x) = J_n(x)$, where $J_n(x)$ is a Bessel function on integer order n. This is the well known Schott formula—the classical expression for the spectral-angular distribution of synchrotron radiation.[5,6]

As $N\to\infty$ the terms in Eq. (24) which produce the asymmetry in φ drop out, and for $N \gg 1$ they are $1/N$ times smaller than the terms which do not depend on φ.

Thus, it is possible to demonstrate the nature of the formation of the classical spectrum of synchrotron radiation in the case when the electron executes a finite but very large number of revolutions on a circular path.

[1)]Since Eq. (2) involves $|\mathbf{f}|^2$, the expressions (4) for f_σ and f_π have been written with accuracy to a common phase factor, which is unimportant for discussion of the properties of the radiation. This phase factor will be omitted everywhere in what follows.

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