Two- and three-dimensional solitons in weakly dispersive media

E. A. Kuznetsov and S. K. Turitsyn

Institute for Automation and Electrometry, Siberian Division of the Academy of Sciences of the USSR
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We show that in weakly dispersive media two-dimensional sound solitons are stable against two-dimensional perturbations, while three-dimensional ones are unstable. With respect to three-dimensional bending perturbations both two-dimensional and one-dimensional solitons [B. B. Kadomtsev and V. I. Petviashvili, Sov. Phys. Dokl. **15**, 539 (1970)] turn out to be unstable.

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INTRODUCTION

It is well known what an important role is played in the dynamics of nonlinear waves by solitary waves—solitons. Their properties can be most completely studied in integrable models.[1] As a rule the solitons behave in such models like particles: when solitons interact with one another there occurs elastic scattering of the solitons with the non-soliton part; this leads to definite shifts in the centers of the solitons, thus indicating their stability. When one analyzes the problem of the stability even of one-dimensional solitons in models which do not allow the application of exact methods, there arise certain difficulties. The situation becomes even more complicated when we study many-dimensional solitons. So far there are no general analysis methods in this region and solutions of soliton type have been constructed analytically[2,3] (again, though, using the inverse scattering method) for some nonlinear equations, but the problem of their stability remains unresolved. One should add that by using the traditional approach based upon a linearization of the original equations one can only in exceptional cases solve the stability problem exactly; one is as a rule obliged to turn to the usual perturbation theory,[4] which obviously does not solve the entire problem. When the soliton possesses nontrivial topological properties its stability can be proved by using topological considerations.[5] There is, however, another exact and rather clear method by means of which the stability of three-dimensional solitons has been proved for a number of models.[6–8] This method is based upon a theorem of Lyapunov's according to which there is in the system at least one stable trajectory, provided some integral is bounded from above or below.

We apply this method in the present paper to the problem of the stability of two- and three-dimensional sound solitons in a weakly dispersive medium. We shall describe the propagation of the solitons by means of the Kadomtsev-Petviashvili (KP) equation[9]

$$\frac{\partial}{\partial x}(u_t+6uu_x+u_{xxx})=3\varkappa\Delta_\perp u. \tag{1}$$

This equation is written down in a system of coordinates moving with the sound velocity c_s. We have retained in it, after elimination of the velocity c_s, the main terms responsible for the weak dispersion ($\propto U_{xxx}$), for the small nonlinearity ($\propto uu_x$), and for diffractive divergence ($\propto\Delta_\perp u$). The KP equation has thus the same degree of universality as the known KdV equation, which is obtained from (1) when $\Delta_\perp u=0$. The sign of $\varkappa$ corresponds here to positive or negative dispersion.

It is well known that the dynamics of a system of sound waves is in some sense trivial when the wave dispersion is negative (examples are long-wavelength gravitational waves on the surface of a liquid, ion-sound waves in a plasma, sound in solids, and so on). The fact is that in that case one-dimensional solutions—solitons,[9,10] periodic stationary waves[11]—are stable against transverse perturbations. A nontrivial picture arises when the wave dispersion is positive (examples are long-wavelength gravitational-capillary waves, magnetosonic waves in a plasma, phonons in liquid helium under certain conditions,[12] and so on). In this case there occurs a decay instability even for small wave amplitudes. It is just only when positive dispersion occurs that there exist two-dimensional solitons, which can be found explicitly using the inverse scattering method,[2] and also three-dimensional solitons, which were observed by Petviashvili[13] in a numerical simulation.

In the present paper we show that two-dimensional solitons are stable against two-dimensional perturbations, while three-dimensional ones are unstable. This is shown in the second section of the paper. This is preceded by §1 in which we consider some properties of the KP equation and its stationary solutions. At the end of the paper we consider the problem of the stability of the two-dimensional soliton against three-dimensional perturbations.

§1. STATIONARY SOLUTIONS AND THEIR PROPERTIES

Before we consider the problem of the stationary solutions of the KP equation, we give the properties of (1) which are necessary for what follows, restricting ourselves to the case of only positive dispersion.

It is known that the KP equation is Hamiltonian:

$$u_t=\frac{\partial}{\partial x}\frac{\delta H}{\delta u} \tag{2}$$

with a Hamiltonian

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$$H=\int\left[\frac{u_x^2}{2}+\frac{3(\nabla_\perp w)^2}{2}-u^3\right]d\mathbf{r},$$

where $w_x=u$, $\varkappa=1$. Together with H, Eq. (1) conserves in the three-dimensional case the momentum

$$P=\int u^2 d\mathbf{r}$$

and the x-component of the angular momentum. For the two-dimensional KP equation one can find apart from H and P an infinite series of integrals of motion.[1,14] The reason for the existence of such a set when the dispersion is positive is, according to Ref. 14, the degeneracy of the dispersion law ω_k for the linear part of the KP equation, i.e., when one can find on the decay surface

$$\omega_k=\omega_{k_1}+\omega_{k_2}, \quad \mathbf{k}=\mathbf{k}_1+\mathbf{k}_2 \tag{3}$$

a nontrivial function f_k for which

$$f_k=f_{k_1}+f_{k_2}.$$

One should note that the dimensionality of the degenerate submanifold (3) equals four, i.e., less than the dimensionality of (3). There are therefore all grounds to think that in three-dimensional geometry there are no other integrals than the ones given above.

We now consider stationary solutions of Eq. (1) of the form $u=u(x-vt\cdot\mathbf{r}_\perp)$ which decrease in all directions. We shall call such solutions solitons; their form is determined from the equation

$$Lu=\left[v\frac{\partial^2}{\partial x^2}+3\Delta_\perp-\frac{\partial^4}{\partial x^4}\right]u=3\frac{\partial^2}{\partial x^2}u^2. \tag{4}$$

The operator L here is elliptic when $v>0$, guaranteeing the existence of a decreasing solution.[13] When the dispersion is negative, regardless of the sign of v, L is hyperbolic.

One must add that in the two-dimensional case the solution of Eq. (1) was found by using the inverse scattering method:[2]

$$u(x-vt,y)=4v'\cdot\frac{1+v'^2y^2-v'(x-3v't)^2}{[1+v'^2y^2+v'(x-3v't)^2]^2} \quad (v'=v/3). \tag{5}$$

It has the form of a two-dimensional soliton, decreasing like $1/r^2$ as $|r|\to\infty$. For this solution the momentum and the Hamiltonian have, respectively, the values

$$P=8\pi(v/3)^{1/2}, \quad H=-4\pi(v/3)^{3/2}.$$

The problem of the stability of the two-dimensional solitons has not been discussed earlier in the literature. Below we give a simple solution of this problem both for two- and for three-dimensional solitons without resorting to the inverse scattering method.

§2. STABILITY OF THE SOLITONS

We consider Eq. (4) which we rewrite, using (2) in the form

$$\delta(H+{}^1/_2vP)=0. \tag{6}$$

This form means that all finite solutions of Eq. (4) are stationary points of the Hamiltonian H for fixed P. In that case the velocity v has the meaning of a Lagrangian multiplier.

We now turn to the stability problem. To solve it we use directly the Lyapunov theorem according to which there exists in a dynamic system at least one stable trajectory if some integral, for instance, the Hamiltonian, is bounded from above or from below. The opposite is also true. If this integral is not bounded, there are no absolutely stable solutions. This does not, of course, exclude the existence of locally stable solutions. It is clear that such solutions will not be present if the given integral depends monotonically on its variables.

We must thus prove, in accordance with (6) that H is bounded (in this case, clearly, from below) for fixed P. We consider to begin with the simplest scale transformations

$$u(x,\mathbf{r}_\perp)\to\alpha^{-1/2}\beta^{(1-d)/2}u(x/\alpha,\mathbf{r}_\perp/\beta),$$

which conserve P (d is the spatial dimensionality).

In this case H will depend on the two parameters, α and β:

$$H=\frac{1}{2\alpha^2}I_1+\frac{3\alpha^2}{2\beta^2}I_2-\alpha^{-1/2}\beta^{(1-d)/2}I_3,$$

where

$$I_1=\int u_x^2 dr,\quad I_2=\int(\nabla_\perp w)^2 dr,\quad I_3=\int u^3 dr.$$

When $d=2$ a simple analysis shows that H as a function of the two parameters is bounded from below, while direct substitution shows that the minimum is realized by the soliton solution (5). In the three-dimensional case the situation is the opposite: there is no minimum, the focus is replaced by a saddle-point. One checks this easily by considering the lines $\alpha^2=c\beta$ on which H changes monotonically, which guarantees the absence of locally stable solutions. Moreover, the absence of additional integrals of motion, mentioned earlier, finally leads to the conclusion that the three-dimensional soliton is unstable. As for the two-dimensional case, scale transformations clearly do not exclude all possible deformations. Dimensionality estimates only indicate that H is bounded. We give a rigorous proof of this fact. To do this we set an upper bound on I_3 in terms of I_1, I_2, and P.

First of all we have from the Hölder inequality

$$\int u^3dxdy\le\left(\int u^2dxdy\right)^{1/2}\left(\int u^4dxdy\right)^{1/2}.$$

We estimate next $\int u^4dxdy$:

$$\int u^4dxdy\le\int_{-\infty}^{\infty}\max_x u^2dy\int_{-\infty}^{\infty}u^2dx=2\int_{-\infty}^{\infty}\max_x u^2dy\int_{-\infty}^{\infty}dx\int_{-\infty}^{y}uu_{y'}dy'$$

(in the last integral we interchange the order of integration over x and y' and then integrate by parts)

$$=2\int_{-\infty}^{\infty}dy\max_x u^2\int_{-\infty}^{y}dy'\int_{-\infty}^{\infty}uu_{y'}dx=-2\int_{-\infty}^{\infty}dy\max_x u^2\int_{-\infty}^{y}dy'\int_{-\infty}^{\infty}u_xw_{y'}dx.$$

Further using the obvious inequality

$$\max_x u^2\le2\int_{-\infty}^{\infty}|uu_x|dx,$$

we get for the integral $\int u^4dxdy$ the following upper limit estimate:

$$\int u^4 dxdy \leq 4 \int dxdy|uu_x| \int dxdy|u_x w_y|$$
$$\leq 4\left(\int u^2 dxdy\right)^{1/2} \int u_x^2 dxdy \left(\int w_y^2 dxdy\right)^{1/2}.$$

Hence the required estimate for I_3 has the form

$$I_3 \leq 2P^{3/4} I_1^{3/4} I_2^{1/4}.$$

Substituting this inequality into the Hamiltonian gives a lower limit of H:

$$H \geq {}^1/_2 I_1 + {}^3/_2 I_2 - 2P^{3/4} I_1^{3/4} I_2^{1/4} \geq -{}^2/_3 P^3.$$

Thus, by virtue of the boundedness of the Hamiltonian its lower limit corresponds to some stationary point—to a two-dimensional soliton. One then sees easily that H for fixed P has a single minimum (in the opposite case H as function of α and β would have at least one more minimum under scale transformations). Hence it follows necessarily that the Hamiltonian reaches its smallest value for the soliton solution (5). This proves indeed the stability of the two-dimensional soliton against two-dimensional perturbations. We study the stability against three-dimensional perturbations in the next section.

§3. BENDING INSTABILITY OF THE TWO-DIMENSIONAL SOLITON

We consider the three-dimensional Eq. (1) linearized with respect to the solution (5):

$$A\psi - i\omega \frac{\partial \psi}{\partial x} = -3k^2\psi, \tag{7}$$

where the perturbation δu is chosen in the form

$$\delta u = \psi(x - vt, y) e^{-i\omega t + ikz},$$

while the operator

$$A = \frac{\partial^2}{\partial x^2}\left[-v + 6u_0 + \frac{\partial^2}{\partial x^2}\right] - 3\frac{\partial^2}{\partial y^2}.$$

In the general form it is difficult to solve this spectral problem and we therefore turn to a determination of the spectrum $\omega(k)$ in the long-wavelength limit. To do this we seek the eigenfunction in the form of a series

$$\psi = \psi_0 + \psi_1 + \psi_2 + \ldots.$$

Here ψ_0 is determined by the equation

$$A\psi_0 = 0$$

and is an indifferent-equilibrium perturbation. Due to the translational invariance of (1), ψ_0 can be chosen in the form

$$\psi_0 = c_1 \frac{\partial u_0}{\partial x} + c_2 \frac{\partial u_0}{\partial y},$$

which is a small shift of the soliton as a whole; the first function represents a shift along x and the second along y. It is clear that these perturbations are independent and can therefore be considered separately.

The next approximation in (7) for the first type of perturbation gives

$$A\psi_1 - i\omega \frac{\partial}{\partial x} u_{0x} = 0. \tag{8}$$

One can give the solution of this equation explicitly. To do this we differentiate Eq. (4) with respect to the velocity v and compare it with (8). As a result we get

$$\psi_1 = i\omega \partial u_0 / \partial v.$$

In the second approximation we have

$$A\psi_2 + \omega^2 \frac{\partial^2}{\partial x \partial v} u_0 = -\frac{k^2}{3}\frac{\partial u_0}{\partial x}. \tag{9}$$

We obtain the spectrum $\omega(k)$ from this as the condition for solubility. For this it is necessary to multiply Eq. (9) scalarly with the eigenfunction φ_0 of the zero-eigenvalue operator which is the adjoint of A. Because of (2), $\varphi_0 = w_0$ and thus

$$\omega^2 \int w_0 \frac{\partial^2}{\partial x \partial v} u_0 dxdy = -\frac{k^2}{3}\int w_0 \frac{\partial u_0}{\partial x} dxdy.$$

Integrating by parts we get the expression

$$\frac{\omega^2}{2}\frac{\partial P}{\partial v} = -\frac{k^2}{3} P. \tag{10}$$

Hence it follows that the two-dimensional soliton is unstable against bending of the entire front:

$$\omega^2 = -{}^4/_3 k^2 v < 0.$$

The qualitative reasons for the instability are here the same as for the instability of the one-dimensional soliton[9] where the result which was obtained earlier by a somewhat different method by Kadomtsev and Petviashvili[9,15] follows directly from Eq. (10) for the case of a one-dimensional soliton. As for the other perturbation, it is stable in the long-wavelength limit. The expression for the square of the frequency is obtained in a similar manner:

$$\omega^2 = k^2 P^{-1} \int w_{0y}^2 \, dx \, dy > 0.$$

CONCLUSION

Notwithstanding the fact that two-dimensional solitons are stable when $d = 2$, they turn out to be unstable in the three-dimensional case. This instability is, as in the one-dimensional case (cf. Ref. 15), of the self-focussing type and its development must lead to a division of the front of the two-dimensional soliton into three-dimensional localized clusters. The subsequent behavior of the system will be determined by the evolution of each of these clusters.

It is well known that in the one-dimensional case the evolution of any localized perturbation leads to the separation of the solitons from the non-soliton part that approaches asymptotically a self-similar solution as $t \to \infty$.[16] An analogous situation occurs when $d = 2$ when the non-soliton part also reaches the self-similar regime as $t \to \infty$.[17] In the three-dimensional case the situation must change radically. Here there are no stable solitons and therefore in the process of the evolution of any initial condition there does not occur a splitting off of solitons as occurred in the one- and two-dimensional cases. On the other hand, one should expect that the solution at $d = 3$ must also reach the self-similar regime. This is, in particular, indicated by the estimates of §2, from which it is clear that H as a function of α and β decreases fastest on the lines $\alpha^2 = c\beta$, which corresponds to the self-similar behavior. In contrast to the one- and two-dimensional cases the attainment of the self-similar regime must occur for $d=3$ after a finite time which, in turn, leads to the appearance of a singularity—a phenomenon such as a collapse. At the present it is as yet unclear what is the nature of the singularity, what are the integral criteria for the formation of singularities, or whether they can be described as self-focusing. To a large extent numerical experiments must give the answers to these problems.

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