On the interaction of parametrically excited spin waves with thermal spin waves

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It is shown with the aid of the diagram technique that the interaction between parametrically excited spin waves (PSW) and thermal spin waves (TSW) can be described using the kinetic equation. The damping constants in the equations for the normal (n) and anomalous (σ) correlators of the PSW are found to be identical. They can be calculated with the aid of the ordinary kinetic equation by substituting into it the TSW occupation numbers, which, because of the effect of the PSW, generally differ from the thermodynamic-equilibrium values. The relative contributions of the various mechanisms underlying this effect to the damping are thoroughly analyzed with allowance made for the real form of the dispersion law and the matrix elements of the interaction Hamiltonian for spin waves in cubic ferromagnets. The nonlinear PSW-damping constant is computed up to terms of first order in the PSW occupation numbers $n_{\mathbf{k}}$: $\gamma_{\mathbf{k}}=\gamma_{\mathbf{k}}^0+\int\eta_{\mathbf{kk}'}n_{\mathbf{k}}d\mathbf{k}'$. It is shown within the framework of the kinetic equation that, in the case of the dominant nonlinear-damping mechanisms, the coefficients $\eta_{\mathbf{kk}'}$ are singular: $\eta_{\mathbf{kk}'}\to\infty$ as $\mathbf{k}'\to\mathbf{k}$. The effect of the nonlinear damping on the form of the distribution function and on other characteristics of the PSW beyond the excitation threshold is studied with allowance made for this circumstance. It is shown, in particular, that positive nonlinear damping leads to the broadening of the angular distribution function of the PSW, while negative nonlinear damping leads to the narrowing of this function.

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Thus far, the main attention in investigations of parametrically excited spin waves has been given to the study of the properties of the narrow packet of parametric spin waves (PSW) whose frequencies lie in the region of parametric instability.[1] The rest of the spin waves are usually assumed to be close to thermodynamic equilibrium, and serve then only as a thermostat guaranteeing the PSW damping, which can be computed with the aid of the ordinary kinetic equation for spin waves after substituting into it the equilibrium distribution of the "thermal" spin waves (TSW):

$$n_{\mathbf{k}}=n_{\mathbf{k}}^0=[\exp(\hbar\omega_{\mathbf{k}}/T)-1]^{-1}. \qquad (1)$$

The assumption (1) that the TSW have an equilibrium distribution is based on the fact that the number N of PSW at low supercriticalities is significantly smaller than the total number N_t of TSW. But, as we shall now show, only a small part of the aggregate TSW reservoir participates effectively in the damping of the PSW. Therefore, even at low supercriticalities the energy dissipated by the PSW can cause the TSW occupation numbers in this region to significantly deviate from their equilibrium values. Indeed, in a typical experimental situation,[1] e.g., in the parametric excitation of spin waves in cubic ferromagnets [usually in yttrium iron garnet (YIG)], sufficiently long spin waves with wave vectors $k\lesssim 5\times10^5$ cm^{-1} are excited at a frequency of $\omega_p=2\pi\times10^{10}$ sec^{-1}. For these spin waves the damping constant $\gamma_{\mathbf{k}}$ is determined largely by the processes of wave coalescence

$$\omega_{\mathbf{k}}+\omega_{\mathbf{k}'-\mathbf{k}}=\omega_{\mathbf{k}'} \qquad (2)$$

and decay

$$\omega_{\mathbf{k}}=\omega_{\mathbf{k}'}+\omega_{\mathbf{k}-\mathbf{k}'}. \qquad (3)$$

As is well known,[2,3] in the wave-coalescence processes (2)

$$\gamma_c(\mathbf{k})=\pi\int d\mathbf{k}_1d\mathbf{k}_2|V_{1,k2}|^2(n_2-n_1)\delta(\omega_1-\omega_k-\omega_2)\delta(\mathbf{k}_1-\mathbf{k}-\mathbf{k}_2), \qquad (4)$$

while in the decay processes (3)

$$\gamma_{sp}(\mathbf{k})=\frac{\pi}{2}\int d\mathbf{k}_1d\mathbf{k}_2|V_{k,12}|^2(n_1+n_2)\delta(\omega_k-\omega_1-\omega_2)\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2). \qquad (5)$$

Here the $V_{1,23}\equiv V(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$ are the matrix elements of the three-wave interaction Hamiltonian

$$\mathcal{H}^{(3)}=\frac{1}{2}\int(V_{1,23}^{*}a_1^{+}a_2a_3+\text{H.c.})\delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3 \qquad (6)$$

and the abridged notation $n_1=n(\mathbf{k}_1)$, $\omega_1=\omega(\mathbf{k}_1)$, etc., has been adopted. The Planck constant $\hbar$ is "assumed to be equal to unity."

Spin-wave damping in the four-magnon scattering processes

$$\omega_k+\omega_1=\omega_2+\omega_3,\qquad \mathbf{k}+\mathbf{k}_1=\mathbf{k}_2+\mathbf{k}_3. \qquad (7)$$

may also prove to be important. In that case

$$\gamma_{sc}(\mathbf{k})=\frac{\pi}{2}\int d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3|T_{k1,23}|^2[n_1(n_2+n_3)-n_2n_3]\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\delta(\omega_k+\omega_1-\omega_2-\omega_3). \qquad (8)$$

Here $T_{12,34}$ is the matrix element in the four-magnon interaction Hamiltonian $\mathcal{H}^{(4)}$:

$$\mathcal{H}^{(4)}=\int T_{12,34}a_1^{+}a_2^{+}a_3a_4\delta(1+2-3-4)\,d1\,d2\,d3\,d4. \qquad (9)$$

It is clear that the TSW with frequencies $\omega_{\mathbf{k}'}$ lying within the interval $(\omega_{\mathbf{k}},\omega_{\mathbf{k}}-\omega_0)$, where $\omega_{\mathbf{k}}=\omega_p/2$ can participate in the decay processes (3). Consequently, ΔN_{sp} is smaller than the number of TSW in a sphere of radius k, i.e.,

$$\Delta N_{sp}\lesssim {}^4/_3\pi k^3n_k^0V/(2\pi)^3.$$

At a supercriticality of 6 dB, when the amplitude h of the parametric pump is higher than the threshold value by a factor of two, the total number of PSW is[1] $N\approx\gamma/|S|$, where $S\approx 2\pi g^2$ is the four-wave matrix element describing the interaction of the PSW with each other within the framework of the Hamiltonian (9) ($g=2.8$ GHz/kOe is the gyromagnetic ratio). Let us estimate the ratio

$$\xi=\Delta N_{sp}/N(6\text{ dB})\approx ST\Delta\omega_k/\gamma\omega_{ex}^{3/2} a^3,$$

where

$$\Delta\omega_k=\omega_k-\omega_0=\omega_k-gH-\omega_m/3, \qquad \omega_m=4\pi gM$$

(M is the magnetization), ω_{ex} is the exchange frequency, and a is the lattice constant. We shall make all the numerical estimates with the experiments with YIG at room temperature in mind:

$$\omega_m=4.9\cdot10^9 \text{ sec}^{-1}, \quad \omega_p=9.8\cdot10^9 \text{ sec}^{-1},$$

$$\gamma_k=2-10\cdot10^5 \text{ sec}^{-1}, \ T=300\text{ K}, \quad \omega_{ex}a^2\approx0.1 \text{ cm}^2\cdot\text{sec}^{-1}. \tag{10}$$

The quantity $\Delta\omega_k$ is determined by the applied magnetic field, and varies in experiments from 10^8 to 5×10^9 sec^{-1}. Under these conditions $\xi\approx10^{-8}\Delta\omega\approx1$, i.e., the number N of PSW and the number ΔN_{sp} of thermal spin waves that make the dominant contribution to the "decay" damping constant $\gamma_{sp}(k)$ of the PSW are indeed of the same order of magnitude. It is essential that the group of ΔN_{sp} spin waves have sufficiently long wavelengths, so that the scattering processes (7) do not have time to scatter the nonequilibrium part of its energy throughout the reservoir of thermal spin waves. It is therefore clear that the nonlinear PSW damping that arises as a result of the superheating of the group of ΔN_{sp} spin waves in the decay processes (3) can be substantial.

Similar arguments lead to the idea that a nonlinear dependence of γ_k on N can also arise in the coalescence processes (2). To be sure, the fact that the number ΔN_c of spin waves participating in the processes (2) is sufficiently small is not so obvious here; in fact, it does not follow from the conservation law (2) that the spin-wave vectors $|\mathbf{k}'|$ and $|\mathbf{k}-\mathbf{k}'|=k''$ are bounded from above. But analysis of the integral (4) (see §2 below for details) shows that the dominant contribution to it is made by the spin waves with k' and k'' of the order of k, and, consequently, $\Delta N_c\approx\Delta N_{sp}$.

Similar qualitative arguments were adduced long ago by Schlömann,[4] LeGall *et al.*,[5] Melkov,[6] and others to show the nonlinearity of the damping in the three-wave processes (2) and (3). In a preprint of one of the authors of the present paper (V.L.),[7] these effects are quantitatively analyzed in the simplest possible model that still preserves the main characteristics of the phenomenon: the spin-wave spectrum is assumed to be isotropic, the dependence of the three-wave interaction constants $V_{1,23}$ on the angles is neglected, and the PSW-pump interaction constant V_k and the PSW-damping constant γ_k are also assumed to be isotropic. Fairly simple dependences of γ_k on N are obtained which qualitatively agree with experiment.

In the present paper we study the interaction between the PSW and the thermal spin waves without making model assumptions about the spin-wave dispersion law ω_k and the matrix elements of the interaction Hamiltonian. To begin with, in §1, we discuss the question whether the kinetic equation can be used to compute the damping of a spectrally narrow PSW packet. In spite of the fact that this question has an almost obvious answer within the framework of Wyld's diagram technique, it is still discussed in the literature.

In §2 we compute the nonlinear-damping constants $\eta_{\mathbf{k}\mathbf{k}'}$ for relatively small numbers N_k of PSW, when

$$\gamma_k(N)=\gamma_k^0+\pi^2\int\eta_{\mathbf{k}\mathbf{k}'}N_{\mathbf{k}'}\,d\mathbf{k}'. \tag{11}$$

Further, we compute the $\eta_{\mathbf{k}\mathbf{k}'}$ for the nonlinear damping due to the processes (2) of coalescence of two PSW. This mechanism was first proposed by Gottlib and Sühl.[8] We also carry out in this section a consistent comparison of the various mechanisms of nonlinear damping, and derive formulas giving the $\eta_{\mathbf{k}\mathbf{k}'}$ in different ranges of the external-magnetic-field strength. The contributions of the various three-wave processes to the nonlinear damping are, according to rough estimates, of the same order of magnitude. To ascertain the sign of the $\eta_{\mathbf{k}\mathbf{k}'}$ and the role of the various mechanisms, we must carry out a thorough quantitative analysis that takes account of the specific dispersion law and the form of the matrix elements. We are also careful to retain the numerical coefficients of the type 2π that have a tendency to enter in high powers into a final answer and make one contribution numerically small compared to another. The main qualitative result obtained in §2 should be considered to be the proof that $\eta_{\mathbf{k}\mathbf{k}'}$ is singular at $\mathbf{k}'=\mathbf{k}$: $\eta_{\mathbf{k}\mathbf{k}'}\propto|\mathbf{k}-\mathbf{k}'|^{-1}$. This means, in particular, that the nonlinear contribution to the damping diverges on the singular spectra predicted by the S theory (see Ref. 1). This divergence occurs as a result of the use of the kinetic equation to describe the narrow wave packet. The minimum packet width Δk for which the results presented in the present paper are valid can be estimated by comparing the $\eta_{\mathbf{k}\mathbf{k}'}$, (2.6), computed with the aid of the kinetic equation with the nonlinear-damping constant for a monochromatic wave[9]:

$$\Delta k/k\geqslant\gamma/\omega_k. \tag{12}$$

As is well known (see §1), this is the condition of applicability of the kinetic equation. As far as we know (see, for example, Ref. 10), the condition (12) is always fulfilled in experiments on the parametric excitation of spin waves.

Section 3 is devoted to the consideration of the effect of the nonlinear damping, in particular, its singular character, on the properties on the PSW. Here it is shown that positive nonlinear damping is an isotropicizing factor (increases the size of the packet in k space), while negative damping is an anisotropicizing factor. Owing to the latter circumstance, the S-theory equations with a negative nonlinear damping constant of the form (11) do not have a steady-state solution at all. It is shown that a small quantity of defects (the scattering of the spin waves by which is an isotropizing factor) leads to the existence of a steady-state solution with a finite packet width in k space. Besides the PSW distribution, the following integrated characteristics are found: the number of spin waves and the nonlinear susceptibilities.

§1. SYSTEM OF BASIC EQUATIONS

1. *The diagram technique.* We shall, in describing the spin waves, proceed from the total Hamiltonian of

the problem:

$$\mathcal{H}=\mathcal{H}^{(2)}+\mathcal{H}^{(3)}+\mathcal{H}^{(4)}+\mathcal{H}^{(p)}. \tag{1.1}$$

Here $\mathcal{H}^{(2)}$ is the Hamiltonian of the noninteracting spin waves:

$$\mathcal{H}^{(2)}=\int \omega_{\mathbf{k}} a_{\mathbf{k}}^{+} a_{\mathbf{k}}\, d\mathbf{k}, \tag{1.2}$$

$\mathcal{H}^{(3)}$ is the three-magnon interaction Hamiltonian (6) and $\mathcal{H}^{(4)}$ is the four-magnon Hamiltonian (9), in which we have, for simplicity, retained only the part describing the scattering processes (7). Finally, $\mathcal{H}^{(p)}$ is the spin wave–microwave pump interaction Hamiltonian:

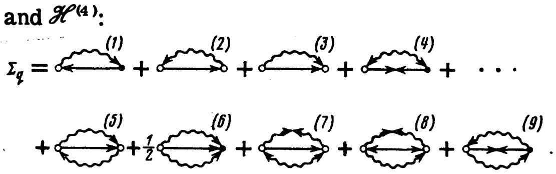
$$\mathcal{H}^{(p)}=\frac{1}{2}\int [h\exp(i\omega_p t)\, V_{\mathbf{k}} a_{\mathbf{k}}^{+} a_{-\mathbf{k}}^{+}+\text{H.c.}]\, d\mathbf{k}. \tag{1.3}$$

Since in the majority of experiments on the parametric excitation of spin waves in ferromagnets the temperature $T\approx 300$ K and is significantly higher than the pump frequency $\hbar\omega_p\approx 1$ K (in the three-centimeter region), we go over to the classical description of the spin waves by replacing the operators $a_{\mathbf{k}}^{+}$ and $a_{\mathbf{k}}$ by the corresponding c numbers, i.e., by the canonical variables $a_{\mathbf{k}}^{*}$ and $a_{\mathbf{k}}$. This allows the use of the canonical diagrammatic technique[11] for the statistical description of the PSW.

As shown in Ref. 12, a system of Dyson equations for the normal (G_q) and anomalous (L_q) Green functions arises upon the summation of the weakly coupled diagrams. These equations can be represented in the form

$$\begin{gathered} G_q=(\omega_p-\omega-\tilde{\omega}_{\mathbf{k}}-i\Gamma_q)\Delta_q^{-1}, \quad L_q=\tilde{P}_q\Delta_q^{-1}, \quad q=\mathbf{k},\ \omega; \\ \Delta_q=(\omega_p-\omega-\omega_{\mathbf{k}}-i\Gamma_q)(\omega-\tilde{\omega}_{\mathbf{k}}+i\Gamma_q)-|\tilde{P}_q|^2. \end{gathered} \tag{1.4}$$

The interaction-renormalized spin-wave frequency $\tilde{\omega}_{\mathbf{k}}$, the spin-wave damping constant Γ_q, and the renormalized pump power $\tilde{P}_q$ can be expressed in terms of the normal (Σ_q) and anomalous (Π_q) compact diagrams. For example, $\Gamma_{\mathbf{k}}=\operatorname{Im}\Sigma_{\mathbf{k}\omega_{\mathbf{k}}}$. Below we present the Σ diagrams that are of second order in the vertices in $\mathcal{H}^{(3)}$ and $\mathcal{H}^{(4)}$:

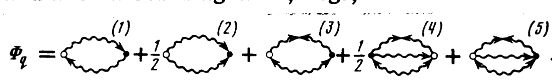


(1.5)

Here we have used the normal diagram notation: the straight lines represent the Green functions G_q and L_q; the wavy lines, the normal (n_q) and anomalous (σ_q) pair averages.[12] By summing the weakly coupled diagrams, we can obtain for n_q and σ_q equations that generalize the S-theory equations:

$$\begin{gathered} n_q=G_qA_q+L_qB_{\bar{q}}^{*}, \qquad \sigma_q=G_qA_{\bar{q}}^{*}+L_qB_q, \\ A_q=\Phi_qG_q^{*}+\Psi_qL_{\bar{q}}^{*}, \quad B_q=\Psi_q^{*}G_{\bar{q}}+\Phi_qL_q, \quad \bar{q}=-\mathbf{k},\ \omega_p-\omega. \end{gathered} \tag{1.6}$$

Here Φ_q and Ψ_q are the sums of the compact, normal, and anomalous diagrams, e.g.,



(1.7)

They differ from the diagrams for Σ and Π in that they can be cut into two parts only along n and σ lines.

2. *Transition to the kinetic equation in the absence of a pump.* If $hV_{\mathbf{k}}=0$, there are no anomalous correlators, $L_q=0$ and $\sigma_q=0$ in the equations (1.6), and expression (1.5) for the normal Green function assumes the simpler standard form

$$G_q=(\omega-\omega_{\mathbf{k}}-\Sigma_{\mathbf{k}\omega})^{-1}=(\omega-\tilde{\omega}_{\mathbf{k}}+i\Gamma_{\mathbf{k}})^{-1}. \tag{1.8}$$

Then, instead of (1.6), we obtain

$$n_q=|G_q|^2\Phi_q=\Phi_{\mathbf{k}\omega_{\mathbf{k}}}[(\omega-\tilde{\omega}_{\mathbf{k}})^2+\Gamma_{\mathbf{k}}^2]^{-1}. \tag{1.9}$$

The quantity $\Phi_{\mathbf{k}\omega}$ is a smooth function of $\mathbf{k}$ and ω, and, consequently, the dependence of $n_{\mathbf{k}\omega}$ on ω is determined by the square of the modulus of the Green function. Integrating (1.9) over ω, we have

$$\Gamma_{\mathbf{k}}n_{\mathbf{k}}=\pi\Phi_{\mathbf{k}\omega_{\mathbf{k}}}, \quad n_{\mathbf{k}}=\int d\omega\, n_{\mathbf{k}\omega}. \tag{1.10}$$

For wave packets that are spectrally broad enough (i.e., whose spectral width $\Delta\omega_{\mathbf{k}}\gg\Gamma_{\mathbf{k}}$), we can, by setting $n_{\mathbf{k}\omega}=n_{\mathbf{k}}\delta(\omega-\omega_{\mathbf{k}})$, perform the integrations in the expressions for $\Gamma_{\mathbf{k}}$ and Φ_q over all the internal frequencies. As a result, the diagrams 1 and 2 in (1.5) give the standard expression (4) for the spin-wave damping constant due to the coalescence processes (2), the diagram 3 gives the expression (5) for the damping constant due to the decay processes (3), and the diagrams 5 and 6 give the expression (8) for the damping constant due to the scattering processes (7). The diagrams 1, 2, and 4 in (1.7) for Φ_q give the sell-known expressions for the arrival terms in the above processes. Thus, we indeed arrive at the steady-state kinetic equation. Let us recall that we do not, in deriving it, take the diagrams with three or more vertices into consideration. It can be shown[11] that these diagrams can indeed be neglected in the case of sufficiently broad wave packets, i.e., when

$$\Gamma_{\mathbf{k}}\ll\Delta k\frac{\partial\omega_{\mathbf{k}}}{\partial k}, \quad \Gamma_{\mathbf{k}}\ll\frac{\partial^2\omega}{\partial k^2}(\Delta k)^2. \tag{1.11}$$

As is well known, the solution to the kinetic equation is the Rayleigh–Jeans distribution, which is also established in the absence of a pump. By substituting $n_{\mathbf{k}}^0$ into (4), (5), and (8), we can compute the spin-wave damping constant in the near-equilibrium state:

$$\gamma_{\mathbf{k}}^0=\gamma_{sp}^0+\gamma_c^0+\gamma_{sc}^0.$$

The contributions from other processes, e.g., the Kasuya–LeCroy process,[3] should be added to $\gamma_{\mathbf{k}}^0$ as the need arises.

3. *Procedure for dividing the spin waves into parametric and thermal spin waves in the presence of a pump.* A pump changes the occupation numbers $n_{\mathbf{k}}$ not just in the parametric-resonance region, but in the entire $\mathbf{k}$ space. Therefore, the question arises: "Which waves should be considered to have been parametrically excited, and which ones should, as before, be classified as thermal waves?" A qualitative answer is this: "The parametric spin waves (PSW) are those pairs of spin waves whose phases are correlated with the phase of the pump; the remaining spin waves can be called thermal spin waves if their occupation numbers $n_{\mathbf{k}}$ do not differ too much from the equilibrium level." As shown in our preprint,[13] it is reasonable to define the PSW distribution function $n_p(\mathbf{k},\omega)$ by the formula

$$n_p(\mathbf{k},\ \omega)=n(\mathbf{k},\ \omega)-n_t(\mathbf{k},\ \omega), \tag{1.12}$$

where

$$n_t(\mathbf{k},\omega)=\frac{\Phi_{\mathbf{k}\omega}}{(\omega-\tilde{\omega}_{\mathbf{k}})^2+\Gamma_{\mathbf{k}}^2} \tag{1.13}$$

is defined by analogy with (1.9), with the difference that here the quantity $\Phi_{\mathbf{k}\omega}$ should be computed not for the thermodynamic-equilibrium distribution $n_{\mathbf{k}}^0$, but for the real spectrum, $n_t(\mathbf{k},\omega)$, $n_p(\mathbf{k},\omega)$, and $\sigma(\mathbf{k},\omega)$, computed in the presence of a pump. For such a definition, $n_t(\mathbf{k},\omega)$ is everywhere a smooth function of $\mathbf{k}$, and asymptotically tends to the equilibrium spectrum as we move far away from the resonance surface, i.e., as $\omega_{\mathbf{k}} \gg \omega_p/2$ increases. As for the quantities

$$n_p(\mathbf{k})=\int n_p(\mathbf{k},\omega)\,d\omega$$

and $\sigma(\mathbf{k})$ they fall off rapidly with distance from the resonance surface. It can be shown that

$$n_p(\mathbf{k}),\ \sigma(\mathbf{k})\sim(\tilde{\omega}_{\mathbf{k}}-\omega_p/2)^{-2}.$$

4. *The damping of the spin waves in the presence of a pump.* An essential feature of the basic equations (1.6) is that they contain only one quantity (Γ_q) that characterizes the damping of the waves. This means that, in the S-theory equations and any of their generalizations, the damping constants for the normal ($n_{\mathbf{k}}$) and anomalous ($\sigma_{\mathbf{k}}$) correlators have the same value $\Gamma_{\mathbf{k},\omega_p/2}$. At the same time, in some experiments[14] different damping times τ_N and τ_Σ are observed for the integrated quantities

$$N=\int n_{\mathbf{k}}\,d\mathbf{k},\quad \Sigma=\int \sigma_{\mathbf{k}}\,d\mathbf{k}.$$

Indeed, a question that has still not been fully answered is: "How are the times τ_N and τ_Σ related in the various experiments with the quantity $\Gamma_{\mathbf{k}}$ and the other characteristics of the PSW—the two-magnon damping constant γ_{def}, the PSW packet width $\Delta\omega_{\mathbf{k}}$ with respect to the modulus of $\mathbf{k}$, etc.? Let us set this question aside, and compute the quantity $\Gamma_{\mathbf{k}}$. Generally speaking, this should be done by substituting the quantities $\sigma(\mathbf{k},\omega)$, $n(\mathbf{k},\omega)=n_p+n_t$, and the exact expressions for the Green functions into the diagrammatic expressions (1.5) for Σ_q. Then $\Gamma_{\mathbf{k}}=\operatorname{Im}\Sigma_{\mathbf{k},\omega_{\mathbf{k}}}$ can be split up into the terms

$$\Gamma_{\mathbf{k}}=\gamma_{\mathbf{k}}^0+\gamma^{NL}=\gamma_{\mathbf{k}}^0+\gamma_1(\mathbf{k})+\gamma_2(\mathbf{k}), \tag{1.14}$$

the first $\gamma_{\mathbf{k}}^0$ of which depends only on n_t, while the second and third, $\gamma_1(\mathbf{k})$ and $\gamma_2(\mathbf{k})$, contain the first and second powers, respectively, of n_p and σ. The computation of $\gamma^0\{n_t\}$ is, in principle, simplified by the fact that $n_t(\mathbf{k},\omega)$ is a smooth function of $\mathbf{k}$, and the dominant contribution to the integrals is made by the region where the wave vector of the Green function does not lie on the resonance surface, with the result that the approximation (1.8) for the Green function can be used. As a result, on performing the integration over the internal frequencies, we again arrive at the well-known expressions (4), (5), and (8) with the thermodynamic-equilibrium spectrum $n_{\mathbf{k}}^0$ replaced by the spectrum $n_t(\mathbf{k})$, which differs from $n_{\mathbf{k}}^0$ because of the interaction with the PSW:

$$\gamma_{sp,1}=\pi\int|V_{k,12}|^2 n_p(\mathbf{k}_1)\,\delta(\omega_k-\omega_1-\omega_2)\,\delta(\mathbf{k}-1-2)\,d1d2. \tag{1.15}$$

The part of γ of first order in $n_p(\mathbf{k},\omega)$ is just as easy to compute: we should integrate first over the PSW frequencies, setting

$$n_p(\mathbf{k},\omega)=n_p(\mathbf{k})\,\delta(\omega-\omega_p/2),$$

and then over the remaining internal frequencies, replacing $\operatorname{Im}G_{\mathbf{k}\omega}$ by $\pi\delta(\omega-\omega_{\mathbf{k}})$. This can be done if the range of integration over at least one of the internal $\mathbf{k}'$ is broad, so that $\omega_{\mathbf{k}'}$ changes by an amount significantly greater than the greatest of the damping constants for the spin waves participating in the process. Thus, for example,

$$\gamma_{c,1}=\pi\int|V_{1,\mathbf{k}2}|^2\,[n_p(1)\,\delta(\omega_{\mathbf{k}}+\omega_2-\omega_p/2)-n_p(2)\,\delta(\omega_{\mathbf{k}}+\omega_p/2-\omega_1)]\times\delta(\mathbf{k}+2-1)\,d1\,d2. \tag{1.16}$$

By comparing this expression with (4), we can easily verify that $\gamma_{c,1}$ can be obtained from the kinetic equation [i.e., from (4)] by substituting into it the PSW distribution $n_p(\mathbf{k})$ with allowance made for the fact that $\omega_{\mathbf{k}}=\omega_p/2$. The expressions for $\gamma_{sp,1}$ and $\gamma_{sc,1}$ can be obtained from (5) and (8) in similar fashion. The second-order terms $\gamma_{sc,2}$ can also be computed in much the same way. It is only necessary to take also into account the contribution of the diagrams 7 and 8 that arises because of the presence of the anomalous correlators.

All the preceding expressions for the damping constants are valid for any $\mathbf{k}$, including $\mathbf{k}$ lying on the resonance surface. But if we set $\omega_{\mathbf{k}}=\omega_p/2$ in the analytic expressions for the diagrams 6–9 in (1.5), then it turns out that the frequency figuring in the Green functions also falls on the resonance surface. The approximations (1.8) used by us for G and n_q are then no longer valid, and, instead of the simple formulas for $\gamma_{sc,2}$ that follow from the kinetic equation, we obtain an entirely different expression: see the formulas (3) and (4) in Refs. 9 and 17.

The arrival terms Φ_q of the kinetic equation are computed in the same scheme as the damping constant, and split up in exactly the same way into terms of zeroth, first, etc., orders in n_p and σ (see our preprint[13]).

Summarizing the results obtained in this section, we can say that the kinetic equation can be used to study the interaction between the parametrically excited waves and the thermal waves if the wave vectors of the TSW lie far away from the resonance surface $\omega_{\mathbf{k}}=\omega_p/2$. In this case we must take into account the additional terms that arise as a result of the correlation of the phases in the PSW pairs, and we can set $\omega_{\mathbf{k}}=\omega_p/2$.

If, on the other hand, the wave vectors of the TSW lie close to the resonance surface (i.e., if $|\omega_{\mathbf{k}}-\omega_p/2|\lesssim\gamma$), then the description scheme becomes complicated: it becomes necessary to include more terms in the expressions for $\gamma_{\mathbf{k}}$ and $\Phi_{\mathbf{k}}$, but no fundamental difficulties arise here.

§2. NONLINEAR DAMPING IN THE CUBIC FERROMAGNETS

Many new and interesting effects connected with nonlinear damping occur even in the case of a small number of PSW, when the deviation of the state of the TSW from the equilibrium state is small, i.e., when $\delta n_{\mathbf{k}}\ll n_{\mathbf{k}}^0$. This case, which admits of a detailed analytical investigation, constitutes the subject of the present paper. In this section, using the principles validated in §1, we compute that correction to the PSW-damping constant which is proportional to the first power of the number of PSW.

1. *Basic equations.* In the experimental situation (10) of interest to us, we can neglect the contribution of the four-wave processes to the nonlinear damping. The deviations $\delta n_{\mathbf{k}}$ of the TSW from equilibrium should be found from the kinetic equation for thermal magnons:

$$\gamma_{\mathbf{k}}^0 \delta n_{\mathbf{k}} = -\pi \int |V_{\mathbf{k}+\mathbf{k}'\mathbf{k}\mathbf{k}'}|^2 \delta(\omega_{\mathbf{k}+\mathbf{k}'} - \omega_{\mathbf{k}} - \omega_{\mathbf{k}'}) (n_{\mathbf{k}}^0 - n_{\mathbf{k}+\mathbf{k}'}^0) n_p(\mathbf{k}') d\mathbf{k}'$$
$$+\pi \int |V_{\mathbf{k}\mathbf{k}'\mathbf{k}-\mathbf{k}'}|^2 \delta(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} - \omega_{\mathbf{k}-\mathbf{k}'}) (n_{\mathbf{k}-\mathbf{k}'}^0 - n_{\mathbf{k}}^0) n_p(\mathbf{k}') d\mathbf{k}'$$
$$+\pi \int |V_{\mathbf{k}'\mathbf{k}\mathbf{k}'-\mathbf{k}}|^2 \delta(\omega_{\mathbf{k}'} - \omega_{\mathbf{k}} - \omega_{\mathbf{k}'-\mathbf{k}}) (n_{\mathbf{k}'-\mathbf{k}}^0 - n_{\mathbf{k}}^0) n_p(\mathbf{k}') d\mathbf{k}'. \quad (2.1)$$

We shall use for the analysis the actual dispersion law for spin waves in an isotropic ferromagnet:

$$\omega_{\mathbf{k}}^2 = (\omega_0 + \alpha k^2 + {}^1/_2 \omega_m \sin^2\theta) - {}^1/_4 \omega_m^2 \sin^4\theta, \quad (2.2)$$

where θ is the angle between the wave vector of the wave and the magnetic field,

$$\alpha = \omega_{ex} a^2, \quad \omega_0 = gH - {}^1/_3 \omega_m.$$

For a wave close to the equator of the resonance surface, the dispersion law (2.2) can be approximated by the formula

$$\omega_{\mathbf{k}} \approx \omega_1 + \alpha k^2 = \omega_0 + {}^1/_2 \omega_m + \alpha k^2, \quad (2.3)$$

which for (10) corresponds to a roughly 10% error.

In the case of the dispersion law (2.2) decays are allowed for waves with

$$\alpha k^2 > \alpha k_p^2 = 2(\omega_0^2 + \omega_0\omega_m \sin^2\theta)^{1/2}.$$

The frequency of the parametric waves is fixed ($\omega_{\mathbf{k}} = \omega_p/2$), and the variation of the wave vector is achieved in experiment through the variation of H (and, consequently, of ω_0). The decays become allowed when

$$\omega_0 \leqslant {}^1/_3 (\omega_p^2 + 4\omega_m^2 \sin^4\theta)^{1/2} - {}^1/_6 (\omega_p^2 + \omega_m^2 \sin^4\theta)^{1/2}.$$

2. *Negative nonlinear damping in the processes of coalescence of a parametric wave with a thermal wave.* Let us first consider the case in which the decay of a PSW is allowed. Then, as noted at the qualitative level in Ref. 1, the nonlinear damping of the PSW will be negative, i.e., $\gamma_{\mathbf{k}}(N)$ will decrease with increasing N. For a quantitative analysis, it is essential that we take into consideration the angular dependence of the three-wave interaction matrix element

$$V_{\mathbf{k}12} = V(\sin 2\theta_{1H} e^{-i\varphi_1} + \sin 2\theta_{2H} e^{-i\varphi_2}), \quad V = (8\pi^2 g^3 M)^{1/2}, \quad (2.4)$$

where the z axis has been oriented along the magnetic field and θ_{1H}, θ_{2H} and φ_1, φ_2 are respectively the polar and azimuthal angles of the vectors $\mathbf{k}_1$ and $\mathbf{k}_2$. Substituting $\delta n_{\mathbf{k}}$ from (2.1) into (4) and (5), we obtain in accordance with (11) the expression

$$\eta_{\mathbf{k}\mathbf{k}'} = \int d\mathbf{1}\, |V_{1,\mathbf{k}1-\mathbf{k}}|^2 \delta(\omega_1 - \omega_{\mathbf{k}} - \omega_{1-\mathbf{k}}) |V_{1+\mathbf{k}',1\mathbf{k}'}|^2 \delta(\omega_{1+\mathbf{k}'} - \omega_1$$
$$- \omega_{\mathbf{k}'}) \frac{n_1^0 - n_{1+\mathbf{k}'}^0}{\gamma_1^0} + \int d\mathbf{1}\, |V_{1+\mathbf{k},1\mathbf{k}}|^2 \delta(\omega_{1+\mathbf{k}} - \omega_1 - \omega_{\mathbf{k}}) |V_{1\mathbf{k}'1-\mathbf{k}'}|^2 \delta(\omega_1$$
$$- \omega_{\mathbf{k}'} - \omega_{1-\mathbf{k}'}) \frac{n_{1-\mathbf{k}'}^0 - n_1^0}{\gamma_1^0} - \int d\mathbf{1}\, |V_{1+\mathbf{k},1\mathbf{k}}|^2 \delta(\omega_{1+\mathbf{k}} - \omega_1 - \omega_{\mathbf{k}})$$
$$\times |V_{1+\mathbf{k}',1\mathbf{k}'}|^2 \delta(\omega_{1+\mathbf{k}'} - \omega_1 - \omega_{\mathbf{k}'}) \frac{n_1^0 - n_{1+\mathbf{k}'}}{\gamma_1^0} - \int d\mathbf{1}\, |V_{1,\mathbf{k}'1-\mathbf{k}'}|^2 |V_{1,\mathbf{k}1-\mathbf{k}}|^2$$
$$\times \delta(\omega_1 - \omega_{\mathbf{k}} - \omega_{1-\mathbf{k}}) \delta(\omega_1 - \omega_{\mathbf{k}'} - \omega_{1-\mathbf{k}'}) \frac{n_{1-\mathbf{k}'}^0 - n_1^0}{\gamma_1^0}. \quad (2.5)$$

The third and fourth terms in $\eta_{\mathbf{k}\mathbf{k}'}$ have a singularity at $\mathbf{k}' = \mathbf{k}$, which arises as a result of the coincidence of the arguments of the δ functions. We can, by performing the integration, easily verify that for $|\varkappa| \ll k\ \eta_{\mathbf{k}\mathbf{k}+\varkappa} \propto 1/|\varkappa|$. Let us analyze the expression (2.5) for $\mathbf{k}'$ close to $\mathbf{k}$. Assuming that $x_{\mathbf{k}}, x_{\mathbf{k}'} \ll 1$, and limiting ourselves to the consideration of only the singular term in (2.5), we obtain

$$\eta_{\mathbf{k}\mathbf{k}'} \approx -\frac{4T\omega_p V^4}{\alpha k\,[2\sin^2(\varphi/2) + (x_k - x_{k'})^2 + ((k-k')/k)^2]^{1/2}}$$
$$\times \Bigg[\frac{1}{\omega_1 + \alpha k^2} \int_{\tilde{k}_1}^{\infty} \frac{k_1\, dk_1\, (\tilde{k}_1/k_1)^5 (1 - \tilde{k}_1^2/k_1^2)^{3/2}}{\gamma_1^0 \omega_1 (\omega_1 - \omega_p/2)}$$
$$+ \frac{1}{\omega_1} \int_{\tilde{k}_2}^{\infty} \frac{k_1\, dk_1\, (\tilde{k}_2/k_1)^5 (1 - \tilde{k}_2^2/k_1^2)^{3/2}}{\gamma_1^0 \omega_1 (\omega_1 - \omega_p/2)}\Bigg]. \quad (2.6)$$

Here

$$\tilde{k}_1 = \frac{\omega_1 + 2\alpha k^2}{2\alpha k \cos(\varphi/2)}, \quad \tilde{k}_2 = \frac{\omega_1}{2\alpha k \cos(\varphi/2)}, \quad x_{\mathbf{k},\mathbf{k}'} = \frac{\pi}{2} - \theta_{\mathbf{k},\mathbf{k}'},$$

where φ is the difference between the azimuthal angles of the vectors $\mathbf{k}$ and $\mathbf{k}'$.

As is well known, when $\mathbf{k}$ goes over from the nondecay region into the decay region, the damping constant $\gamma_{\mathbf{k}}^0$ increases severalfold. It is easy to see that $\tilde{k}_1$ satisfies the inequality $\tilde{k}_1 > k_p \approx (2\omega_1/\alpha)^{\frac{1}{2}}$. Therefore, for $\tilde{k}_2 < k_p$, which is equivalent to

$$H < H_1 \approx g^{-1}({}^4/_9 \omega_p - {}^1/_6 \omega_m),$$

the dominant contribution in (2.6) is made by the second term. For field intensities higher than H_1, $\eta_{\mathbf{k}\mathbf{k}'}$ is several times smaller. To study the effect of the nonlinear damping on the PSW distribution in the case of axial symmetry, we need

$$\eta_{kk'} = \frac{1}{2\pi} \int \eta_{\mathbf{k}\mathbf{k}'}\, d\varphi.$$

Computing this quantity for $H < H_1$, we obtain

$$\eta_{kk'} \approx -\frac{T\omega_p V^4 \ln[(x_k - x_{k'})^2 + (k-k')^2/k^2]^{-1}}{20\pi(\omega_1 + \omega_p/2)\gamma^0(\omega_1/2\alpha k)\alpha^3 k^3}. \quad (2.7)$$

3. *Nonlinear damping due to the decay of the* PSW. Let us now consider the nonlinear damping of the parametric waves in the decay region of the spectrum. Limiting ourselves in the expression for the damping constant to the consideration of the term corresponding to the decay of the parametric waves, we write

$$\gamma_{\mathbf{k}}(N) = \gamma_{\mathbf{k}}^0 + \pi \int d\mathbf{k}_1 |V_{\mathbf{k},1\mathbf{k}-1}|^2 \delta(\omega_{\mathbf{k}} - \omega_1 - \omega_{\mathbf{k}-1}) \delta n_1.$$

Neglecting all the terms in (2.1) except the second, since they make a "nonresonance" contribution (the corresponding $\eta_{\mathbf{k}\mathbf{k}'}$ does not have a singularity at $\mathbf{k}' = \mathbf{k}$), we obtain

$$\eta_{\mathbf{k}\mathbf{k}'} = \int d\mathbf{k}_1 |V_{\mathbf{k},1\mathbf{k}-1}|^2 |V_{\mathbf{k}',1\mathbf{k}'-1}|^2 (\omega_{\mathbf{k}} - \omega_1 - \omega_{\mathbf{k}-1}) \delta(\omega_{\mathbf{k}'}$$
$$- \omega_1 - \omega_{\mathbf{k}'-1}) \frac{n_1^0 + n_{\mathbf{k}'-1}}{\gamma_1^0}. \quad (2.8)$$

For the quantitative calculations, let us again use the dispersion laws (2.3) and the matrix element (2.4). For the PSW close to the equator, we obtain, similarly to (2.6), the expression

$$\eta_{\mathbf{k}\mathbf{k}'} \approx \frac{8V^4 T\omega_k}{\alpha^2 k^2 \cos(\varphi/2)} \left[\sin^2\frac{\varphi}{2} + \left(\frac{x_{\mathbf{k}} - x_{\mathbf{k}'}}{2}\right)^2 + \left(\frac{k-k'}{2k}\right)^2\right]^{-1/2}$$
$$\times \int_{k_-}^{k_+} \frac{dk_1 \sin^4\theta_0 \cdot 2}{\cos\theta_0 \gamma_1^0 \omega_1 (\omega_p/2 - \omega_1)}.$$

Here

$$\sin\theta_0(k_1)=\frac{\omega_0+2\alpha k_1^2}{2\alpha kk_1\cos(\varphi/2)},\quad k_\pm=\frac{k}{2}\cos\frac{\varphi}{2}\pm\left(\frac{k^2}{4}\cos^2\frac{\varphi}{2}-\frac{\omega_1}{2\alpha}\right)^{1/2}.$$

In the computation of $\eta_{kk'}$ the dominant contribution is made by the region around $\varphi=0$:

$$\eta_{kk'}\approx\frac{4\omega_p V^4 T}{\pi\omega_{k/2}}\left[3-\frac{k_p^4}{k^4}-\frac{2k_p^2}{k^2}+x_k^2\right]\left[3-\frac{k_p^4}{k^4}-\frac{2k_p^2}{k^2}+x_{k'}^2\right]\left(1-\frac{k_p^2}{k^2}\right)^{1/2}$$
$$\times\left[\left(\frac{\omega_p}{2}-\omega_{k/2}\right)\gamma^0\left(\frac{k}{2}\right)\alpha^2 k\right]^{-1}\left[3-\frac{k_p^4}{k^4}-\frac{2k_p^2}{k^2}\right]^{-1/2}\ln\left[(x_k-x_{k'})^2+\left(\frac{k-k'}{k}\right)^2\right]^{-1}. \qquad (2.9)$$

Here $k_p^2=2\omega_1/\alpha$.

If the wave vector k of the PSW lies near the decay edge $k^2=k_p^2(1+\delta)$, $\delta\ll1$, then

$$\eta_{kk'}\approx\frac{2^6V^4T(\delta+x_k^2)(\delta+x_{k'}^2)}{\pi\omega_p\gamma(k/2)\alpha^2 k}\ln\left[(x_k-x_{k'})^2+\frac{k-k'}{k}{}^2\right]^{-1}. \qquad (2.10)$$

It can be seen that, for PSW at exactly the equator (i.e., for $x_k=x_{k'}=0$), $\eta_{kk'}\propto\delta^2\ll1$ because of the fact that the matrix element $V_{k,k'k-k'}$ vanishes for waves propagating in the plane perpendicular to the magnetization.

A comparison of (2.10) and (2.7) shows that the negative nonlinear damping is replaced by positive damping when $\delta>1/7$, i.e., the formula (2.9) or (1.10) can be used to describe the nonlinear damping in the decay region.

4. *Nonlinear damping due to the coalescence of two* PSW. In the case of the dispersion law (2.3) we obtain from (4) the expression

$$\eta_{kk'}=\int V^2|\sin 2\theta_k+\sin 2\theta_{k'}e^{-i\varphi}|^2\,\delta(\omega_{k+k'}-\omega_k-\omega_{k'})\,\frac{d\varphi}{2\pi}$$
$$=\frac{V^2}{\pi}\frac{\cos\varphi_0(\sin^2 2\theta_k+\sin^2 2\theta_{k'}+(2\omega_1/\alpha kk')\cos\theta_k\cos\theta_{k'})}{2\alpha kk'\sin\theta_k\sin\theta_{k'}\sin\varphi_0},$$

where $\cos\varphi_0=\omega_1/2\alpha kk'\sin\theta_k\sin\theta_{k'}$.

The coalescence process is allowed for

$$k,k'>k_{3\,\mathrm{mag}}=\left(\frac{\omega_p^2+\omega_m\sin^4\theta}{6\alpha^2}\right)^{1/4}\approx\left(\frac{\omega_p}{2\alpha}\right)^{1/2}\ \text{or}\ 2\alpha kk'>\omega_1.$$

If $kk'=(\omega_1/2\alpha)(1+\delta')$, $\delta'\ll1$, then for the PSW near the equator

$$\eta_{kk'}\approx\frac{V^2}{\pi\omega_1}\frac{(x_k+x_{k'})^2}{(\delta'-1/2(x_k^2+x_{k'}^2))^{1/2}}. \qquad (2.11)$$

Let us now compare the contributions of the various processes to the nonlinear damping. For this purpose, let us compute

$$\tilde\eta=\frac{1}{2\Delta}\int_{-\Delta}^{\Delta}\eta_{0x}\,dx$$

for $\eta_{xx'}$ given by the formulas (2.7), (2.10), and (2.11). This quantity is equal to the nonlinear damping constant for the PSW at the equator in the model with $N(x)$ chosen in the form of a rectangle of width 2Δ. To simplify the comparison, let us assume that the γ_k^0 entering into (2.7) and (2.10) is equal to $\gamma_k^0=\zeta\gamma_c=\zeta V^2T/4\pi\alpha^2k$, where, as experiment shows, $\zeta\approx1-5$. Then

$$\tilde\eta_{2c}\approx\frac{8V^2}{\pi\omega_1}\frac{\delta'}{\Delta}\left[\arcsin\frac{\Delta}{\delta'}-\frac{\Delta}{\delta'}\left(1-\left(\frac{\Delta}{\delta'}\right)^2\right)^{1/2}\right],\quad \tilde\eta_{2c,max}\approx\frac{2V^2\Delta}{\omega_1},$$
$$\tilde\eta_c\approx-\frac{V^2\omega_1\ln\Delta^{-1}}{5\zeta\alpha^2k^4},\quad \tilde\eta_{sp}\approx\frac{4V^2\delta^2\ln\Delta^{-1}}{\zeta\omega_{k/2}(1/2\,\omega_p-\omega_{k/2})}. \qquad (2.12)$$

It can be seen from (2.12) that $\tilde\eta_{2c}$ is substantial in a small region of width Δ around $H_{3\mathrm{mag}}$. A comparison of $\tilde\eta_c$ and $\tilde\eta_{2c}$ for $H=H_{3\mathrm{mag}}(1+\Delta)$ shows that

$$\tilde\eta_c/\tilde\eta_{2c,\,max}\approx2\ln\Delta^{-1}/5\zeta\Delta.$$

Thus, the contribution of the process of coalescence of two PSW increases with increasing angular width of the packet. Figure 1 shows typical behavior of the nonlinear damping constant for the case $\Delta=0.1$. This behavior agrees well with the experimental data reported in Ref. 15.

Summarizing, we can assert that, for $H<H_3\approx650$ Oe, the nonlinear damping constant $\eta_{kk'}$ for the waves with wave vectors lying close to the equator of the resonance surface is given by the formula (2.10). For $H_3<H<H_1$, $\eta_{kk'}$ is negative, and is given by the expression (2.7). At $H\approx H_{3\mathrm{mag}}\approx1.1$ kOe, $\eta_{kk'}$ has a narrow peak. In the region $H>H_1$, $|\eta|$ decreases severalfold.

§3. THE EFFECT OF THE NONLINEAR DAMPING ON THE STEADY STATE OF THE PARAMETRICALLY EXCITED WAVES

As is easy to see, the nonlinear contribution to the PSW damping constant (2.7), (2.10) diverges on the S-theory distributions that are singular in k space.[1] It is natural to assume that the S-theory equations with a nonlinear damping constant will have as a solution the PSW spectra having in k space a finite width with respect to both θ and k. This assumption is correct, but only in the case of positive nonlinear damping. Indeed, it is known from the S theory that the PSW distribution is stable against the production of new pairs if the surface of the renormalized pump

$$P_k=hV_k+\int S_{kk'}\sigma_{k'}\,d\mathbf{k}'$$

is enclosed by the γ_k surface and touches it at those points where $N_k\neq0$. Positive nonlinear damping increases the curvature of the γ_k surface, and, as a consequence, can be a factor increasing the angular width of the PSW packet, whereas negative nonlinear damping, on the other hand, is an anisotropicizing factor. It is easy to see (see below) that the S-theory equations with a nonlinear damping constant of the form (11) possess a solution with a finite width with respect to the angle θ and the modulus k when $\eta>0$, but possess no

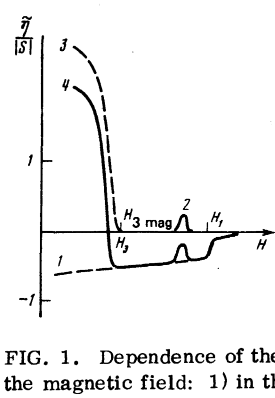


FIG. 1. Dependence of the nonlinear damping constants on the magnetic field: 1) in the processes involving the coalescence of a PSW and a TSW ($\tilde\eta_c$); 2) in the processes involving the coalescence of two PSW ($\tilde\eta_{2c}$); 3) in the decay processes. The curve 4) is the total curve.

solutions at all when $\eta<0$. Let us now proceed to perform analytic computations.

1. *The case of positive nonlinear damping.* Anticipating that the waves will be concentrated in a narrow region of k space (as we shall see below, this is guaranteed by the condition $\eta N \ll \gamma$), and assuming that the wave packet will be axially symmetric, we neglect the dependence of the coefficients $S_{\mathbf{kk}'}$ and $\gamma_{\mathbf{k}}$ on θ and k within the limits of the width of the packet. Then we can assume that

$$\int S_{\mathbf{kk}'} N_{\mathbf{k}'}\, d\mathbf{k}' = S(\theta_{\mathbf{k}}) N, \quad N = \int N_{\mathbf{k}'}\, d\mathbf{k}'. \tag{3.1}$$

The formula (3.1) expresses the fact that the integral of interest to us is proportional to the total number of PSW and at the same time serves as a definition of the coefficients $S(\theta_{\mathbf{k}})$.

In the narrow-packet approximation, the steady-state S-theory equation with a nonlinear damping constant assumes the form[1]

$$\int \eta_{\mathbf{kk}'} N_{\mathbf{k}'}\, d\mathbf{k}' = [h^2 V^2(\theta) - S^2(\theta) N^2 - (\tilde{\omega}_{\mathbf{k}} - \omega_p/2)^2]^{1/2}. \tag{3.2}$$

This equation is valid in the region where $N_{\mathbf{k}} \neq 0$. Taking account of the fact that $V(\theta) = V\cos^2\theta, S(\theta) = S\cos^2\theta$ (the angle θ is measured here from the equator of the resonance surface $\tilde{\omega}_{\mathbf{k}_0} = \omega_p/2$), and $\tilde{\omega}_{\mathbf{k}} = \tilde{\omega}_{\mathbf{k}0} + (\mathbf{k} - \mathbf{k}_0)\mathbf{v}$, where $\mathbf{v} = \partial\omega/\partial\mathbf{k}$ is the group velocity of the spin waves, we can represent Eq. (3.2) in the vicinity of the equator of the resonance surface in the form

$$-\eta \frac{k_0^2}{2} \int \ln\left[(\theta-\theta')^2 + \left(\frac{k-k'}{k}\right)^2\right] N(k', \theta')\, dk'\, d\theta' = P - \gamma - P\sin^2\theta - \frac{(k-k_0)^2 v^2}{2P}. \tag{3.3}$$

Here

$$P = (h^2V^2 - S^2N^2)^{1/2}. \tag{3.4}$$

Notice that, since $k_0 v_0/P \approx k_0 v_0/\gamma \gg 1$, the dimensionless PSW-packet width with respect to the modulus k is much smaller than the width with respect to the angle θ. Therefore, as a first approximation, let us consider (3.3) for $k = k_0$, setting $N(k,\theta) = N(\theta)\delta(k-k_0)$:

$$-\eta \int \ln(\theta-\theta') N(\theta')\, d\theta' = P\left(\frac{P-\gamma}{P} - \theta^2 - \sum_{n=2}^{\infty} \frac{\theta^{2n}}{(2n)!}\right). \tag{3.5}$$

Equation (3.5) possesses a solution that is nonzero in the interval $-\Delta < \theta < \Delta$ and equal to zero outside it. This solution can be expanded in a Taylor series in powers of θ^2. We can verify by direct substitution that, up to terms $\sim \ln^{-1}[P/(P-\gamma)]$ it is given in the entire interval by the formula

$$N(\theta) = 3N(\Delta^2 - \theta^2)/4\Delta^3, \tag{3.6}$$

where Δ is the PSW-packet width with respect to the polar angle and N is the total number of PSW:

$$\Delta^2 = 9\frac{P-\gamma}{P}\left(1 + \frac{1}{2\ln[P/(P-\gamma)]} + \ldots\right), \tag{3.7}$$

$$\eta N \ln[P/(P-\gamma)] = P - \gamma. \tag{3.8}$$

It is difficult to perform such a detailed analysis in the two-dimensional case; therefore, we shall, in computing the packet width, limit ourselves to an estimate. Assuming that $N(k,\theta)$, as a function of k, is a rectangle of width Δk, we find from (3.3) that

$$(\Delta k)^2 \approx (P-\gamma)P/v^2.$$

Equation (3.8) for the total number N of PSW will, of course, not change. Let us note that (3.8) differs only by the logarithm from the S-theory equation with a nonlinear damping constant of the form $\eta_{kk'} = \text{const}$, obtained in Ref. 7. Thus, the simple model used in Ref. 7 describes the integrated characteristics of the PSW fairly well. Indeed, neglecting the slowly varying logarithmic function in (3.8), we find that

$$\frac{SN}{\gamma} = \frac{-c + [\xi^2(c^2+1) - 1]^{1/2}}{c^2+1}, \tag{3.9}$$

where $\xi = hV/\gamma$ and $c = \eta/|S|$. The real and imaginary parts of the nonlinear susceptibility can be obtained in the usual manner (see Ref. 7):

$$\chi'' = \frac{2V^2}{|S|}\left\{\frac{c}{c^2+1} + \frac{(1-c^2)[\xi^2(c^2+1)-1]^{1/2} - 2c}{\xi^2(c^2+1)}\right\}, \tag{3.10}$$

$$\chi' = \frac{2V^2}{S}\left\{\frac{-c + [\xi^2(c^2+1)-1]^{1/2}}{\xi^2(c^2+1)}\right\}^2. \tag{3.11}$$

From (3.10) and (3.11) we can obtain the properties, discussed in Ref. 7, of the nonlinear susceptibility: the finite slope of $\chi''(\xi)$ and the zero slope of $\chi'(\xi)$ at the threshold point, the finite $\chi''(\infty)$ value, the displacement of the $\chi''(\xi)$ peak toward the region of high ξ values as c increases, etc. Let us only note that allowance for the logarithm in (3.8) leads to a situation in which in the vicinity of the threshold, when $\xi - 1 \ll (\eta\ln(\xi-1)/S)^2$,

$$\eta N = \frac{\xi-1}{2}\ln^2(\xi-1),$$

i.e., has a zero derivative at $\xi = 1$.

A qualitatively new phenomenon, which is connected with the singular character of $\eta_{kk'}$, is, as we can see, the possession of a finite width by the packet. Knowing the function $N(\xi)$, we can determine this width from (3.7):

$$\Delta^2 = 9\frac{\eta N}{\gamma} \approx 9\frac{-c + [\xi^2(c^2+1)-1]^{1/2}}{c + 1/c}. \tag{3.12}$$

As we saw in §2, by varying the magnetic field, we can vary the quantity $c = \eta/|S|$ within wide limits. For $c^2/(c^2+1) \ll \xi^2 - 1$,

$$\Delta^2 \approx 9(\xi^2-1)/2(c^2+1)^{1/2}. \tag{3.13}$$

In the opposite limiting case, when $1/c \gg \xi^2 - 1 \gg c^2/(1+c^2)$,

$$\Delta^2 \approx 9c(\xi^2-1)^{1/2}. \tag{3.14}$$

It can be seen that, for $c \approx 1$, the PSW packet is greatly broadened with respect to the angle even at supercriticalities ≈1.5 dB.

Let us recall that all the above-presented formulas have been obtained under the assumption that $\eta N \ll \gamma$. At the applicability limit $\Delta \approx 1$,

$$\Delta k/k \approx (\gamma/\omega)^{1/2} \ll 1.$$

On the other hand, it follows from (12) that, for the results obtained to be applicable, it is necessary that Δ

$>\gamma/\omega$, which, together with (3.12), leads to the condition

$$\gamma(\gamma/\omega)^2 \ll \eta N \ll \gamma. \tag{3.15}$$

Besides the positive nonlinear damping, the presence of thermal noise, the inter-PSW interaction, which is described by the off-diagonal terms of the Hamiltonian, and the two-magnon scattering of the PSW by the static inhomogeneities lead to the broadening of the spectrum. But a comparison of (3.8) with the results of Ref. 10 shows that, for $\xi \gg \beta\gamma_{def}/\gamma$, where $\beta = (ak)T/T_c \approx 10^{-2} - 10^{-3}$ and γ_{def} is the two-magnon damping constant in the absence of a pump, the broadening of the packet in k space is due largely to the nonlinear damping, i.e., to the interaction between the parametric and thermal waves.

2. *The case of negative nonlinear damping.* As has already been noted, negative nonlinear damping decreases the PSW-packet width, which, in the S-theory approximation, is equal to zero anyway. Therefore, it is absolutely necessary in the consistent theory that we take into account, besides the negative nonlinear damping, the interactions leading to the smearing of the singular distribution of the PSW. At not too high supercriticalities, the dominant interaction is usually the two-magnon scattering.

Let us first consider the case in which the two-magnon damping constant γ_{def} is much smaller than the total damping constant: $\gamma_{def} \ll \gamma$. In this situation we can neglect the effect of the scattering by the defects on the interaction between the parametric and thermal waves, and use the nonlinear damping constants $\eta_{kk'}$ computed in §2. As in Ref. 16, the equations describing the angular distribution of the PSW can be derived with the aid of the diagram technique. In the axially symmetric case they have the form

$$2[\Gamma_x^2 - P_x^2]^{3/2} N_x = \gamma_{def}\Gamma_x^2 N, \quad \Gamma_x = \Gamma - \gamma^{NL}(x),$$
$$\Gamma = \gamma + \gamma_{def}\Gamma \int_{-1}^{1} \frac{dx}{(\Gamma_x^2 - P_x^2)^{1/2}} \tag{3.16}$$
$$\gamma^{NL}(x) = \int_{-1}^{1} \eta_{xx'} N_{x'}\, dx', \quad P_x = hV_x - i\int_{-1}^{1} \frac{S_{xx'}P_{x'}N_{x'}}{\Gamma_{x'}}\, dx'.$$

Here $x = \cos\theta$; Γ_x and P_x are respectively the renormalized damping constant and pump power.

Naturally, it is difficult to obtain the exact solution to this complex system of nonlinear integral equations. Even the smallness of the quantity $\gamma_{def}/\gamma \ll 1$, which leads to a situation in which the distribution function N_x of the PSW should have at $x = 0$ a sharp peak with characteristic width $\Delta \ll 1$, is of no help. The point is that, because of the singularity of the kernel $\eta_{xx'}$, the form of the function significantly depends on the shape of the packet N_x. Therefore, we shall just estimate the angular width Δ without computing N_x. For this purpose, we represent N_x in the form of a rectangle of width 2Δ, and use the fact that N_x is a narrow packet. Then we can, after approximately evaluating the integrals in (3.16), reduce this system of integral equations to the following system of algebraic equations:

$$\tilde{P}\tilde{\nu}^2 = \Gamma^2\gamma_{def}, \quad \tilde{P}^2 = P^2 + \frac{\Gamma\eta N}{2\Delta^2}, \quad \Delta\tilde{P} = \tilde{\nu},$$
$$\tilde{\nu}^2 = \Gamma^2 - P^2 - 2\Gamma\eta N \ln\Delta^{-1}, \quad (hV)^2 = P^2[1 + (SN)^2/(\Gamma - \eta N \ln\Delta^{-1})^2]. \tag{3.17}$$

Let us give its solution in two limiting cases: if

$$\frac{\eta^2}{S^2}\ln^2\frac{\gamma}{\gamma_d} \ll \frac{\gamma_d}{\gamma}, \tag{3.18}$$

then at low supercriticalities, when $h - h_c \ll h_c$,

$$N(h) = N_+/2 + [N_+^2/4 + V^2(h^2 - h_c^2)/S^2]^{1/2}, \tag{3.19}$$

where N_+ is the magnitude of the "forward" jump in N at the threshold:

$$SN_+ = \frac{\gamma\eta}{S}\ln\frac{\gamma}{\gamma_d}. \tag{3.20}$$

The width Δ of the packet is then determined by the two-magnon scattering, and practically does not depend on the nonlinear damping:

$$\Delta^2 \approx \frac{\gamma_d}{\gamma} - \frac{\eta^2}{S^2}\ln^2\frac{\gamma}{\gamma_d} \approx \frac{\gamma_d}{\gamma}. \tag{3.21}$$

The expressions for the jumps in the susceptibilities χ'' and χ' and the jump in the phase of the PSW have the form

$$\Delta\chi'' = \frac{V^2}{S}\frac{\eta}{S}\ln\frac{\gamma}{\gamma_d},$$
$$\Delta\chi' = \frac{V^2}{S}\frac{\eta^2}{S^2}\ln^2\frac{\gamma}{\gamma_d}, \quad \Delta\psi = \frac{\eta}{S}\ln\frac{\gamma}{\gamma_d}. \tag{3.22}$$

The results of the theory in this limiting case differ only by the logarithmic factor attached to η from the results of the simple theory presented in Ref. 7, which assumes $\eta_{kk'} = \text{const}$.

If the coefficient η is large, and, instead of (3.19), the inverse relation is fulfilled, then the analysis of the equations (3.17) is significantly less trivial. Here we give only the expressions for the N, χ', and χ'' jumps occurring at the threshold. Instead of (3.20), the magnitude of the "forward" N jump is now given by a formula that does not contain S:

$$\eta N_+ \ln\frac{\gamma}{\gamma_d} = \frac{1}{2}\gamma_d. \tag{3.23}$$

It can be seen that the magnitude of the jump N_+ is set such that γ^{NL} does not exceed the two-magnon damping constant γ_{def}. The width Δ of the packet is then smaller than the width given by (3.21):

$$\Delta^2 = \frac{\gamma_d}{\gamma}\frac{1}{\ln^3(\gamma/\gamma_d)}. \tag{3.24}$$

It can be seen from the formulas (3.21) and (3.24) that negative nonlinear damping indeed decreases the width of the PSW packet. In this limiting case the χ'', χ', and ψ jumps are given by the expressions

$$\Delta\chi'' = \frac{V^2}{S}\frac{S}{\eta}\frac{\gamma_d}{\gamma}\frac{1}{\ln(\gamma/\gamma_d)},$$
$$\Delta\chi' = \frac{V^2}{S}\frac{\gamma_d}{\gamma}, \quad \Delta\psi = \left(\frac{\gamma_d}{\gamma}\right)^{1/2} \tag{3.25}$$

instead of (3.22). Comparing the results of the calculations in the two cases, we see that the behavior of the PSW (e.g., the jumps in the phase, $\chi', \chi'', \ldots$) is determined by the smaller of the two quantities $(\eta/S)\ln(\gamma/\gamma_d)$ and $(\gamma_d/\gamma)^{\frac{1}{2}}$.

Notice that we have consistently neglected the four-wave processes. At the same time, they make to the nonlinear damping a positive contribution that is, in particular, proportional to the square of the number of PSW:

$$\gamma=\gamma^0+\int\eta_{\mathbf{kk}'}N_{\mathbf{k}'}d\mathbf{k}'+\alpha N^2.$$

The nonlinear damping constant due to the four-wave processes naturally does not have singularities, and is therefore determined by the total number of PSW. The coefficient α can be roughly estimated for the process involving the scattering of two PSW: $\alpha \approx T^2/\omega_0$. The formulas (3.19)–(3.25) are valid when $\alpha N_+ < \eta \ln\Delta^{-1}$. In the opposite case the jump is limited by the positive contribution to the nonlinear damping. Evidently, the quadratic—in N—contribution to the nonlinear damping constant explains the change, observed by Melkov and Krutsenko,[15] from negative to positive nonlinear damping as the supercriticality is increased.

The case of a large number of impurities, i.e., the case in which $\gamma_{\mathrm{def}} \gg \gamma$, can be considered in similar fashion. In this situation, because of the two-magnon scattering, the singularity in $\eta_{\mathbf{kk}'}$ is smeared, and the distribution of the PSW over the $\tilde{\omega}_{\mathbf{k}} = \omega_p/2$ surface is isotropic.

In conclusion, let us note that the singular character of $\eta_{\mathbf{kk}'}$ was neglected in earlier[6,10] interpretations of the experiments on nonlinear damping. Therefore, for any detailed comparison of the results of the present paper with the results of measurements to be possible, we need new purposeful experiments.

[1] V. E. Zakharov, V. S. L'vov, and S. S. Starobinets, Usp. Fiz. Nauk **114**, 609 (1974) [Sov. Phys. Usp. **17**, 896 (1975)].

[2] A. I. Akhiezer, V. G. Bar'yakhtar, and S. P. Peletminskiĭ, Spinovye volny (Spin Waves), Nauka, Moscow, 1967 [Eng. Transl., Wiley, New York, 1968].

[3] A. G. Gurevich, Magnitnyĭ resonans v ferritakh i antiferromagnetikakh (Magnetic Resonance in the Ferrites and Antiferromagnets), Nauka, Moscow, 1973.

[4] E. Schlömann, J. Appl. Phys. **33**, 527 (1962).

[5] H. LeGall, B. Lemaire, and D. Sere, Solid State Commun. **5**, 919 (1967).

[6] G. A. Melkov, Zh. Eksp. Teor. Fiz. **61**, 373 (1971) [Sov. Phys. JETP **34**, 198 (1972)].

[7] V. S. L'vov, Preprint 69-72, Inst. Nucl. Phys., Novosibirsk, 1972.

[8] P. Gottlib and H. Sühl, J. Appl. Phys. **33**, 1508 (1962).

[9] V. S. L'vov, Zh. Eksp. Teor. Fiz. **68**, 308 (1975) [Sov. Phys. JETP **41**, 150 (1975)].

[10] I. V. Krutsenko, V. S. L'vov, and G. A. Melkov, Zh. Eksp. Teor. Fiz. **75**, 1114 (1978) [Sov. Phys. JETP **48**, 561 (1978)].

[11] V. E. Zakharov and V. S. L'vov, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **18**, 1470 (1975).

[12] V. E. Zakharov and V. S. L'vov, Zh. Eksp. Teor. Fiz. **59**, 1200 (1970) [Sov. Phys. JETP **32**, 656 (1971)].

[13] V. S. L'vov and G. E. Fal'kovich, Preprint No. 161, Inst. Automat. Electrom., 1981.

[14] V. S. Zhitnyuk and G. A. Melkov, Zh. Eksp. Teor. Fiz. **75**, 1755 (1978) [Sov. Phys. JETP **48**, 884 (1978)].

[15] G. A. Melkov and I. V. Krutsenko, Zh. Eksp. Teor. Fiz. **72**, 564 (1977) [Sov. Phys. JETP **45**, 295 (1977)].

[16] V. E. Zakharov and V. S. L'vov, Fiz. Tverd. Tela (Leningrad) **14**, 2913 (1972) [Sov. Phys. Solid State **14**, 2513 (1973)].

[17] V. S. L'vov and V. B. Cherepanov, Zh. Eksp. Teor. Fiz. **76**, 2266 (1979) [Sov. Phys. JETP **49**, 1145 (1979)].

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