New classical inversion formulas for centrosymmetric electric and magnetic fields; focusing potentials

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New inversion formulas are obtained for the classical scattering of a charged particle by a spherical or axisymmetric electric or magnetic field at a fixed impact parameter or angular momentum. For different cases, focusing fields are obtained similar to those previously considered for scattering by an electric field at a given energy, viz., of the backscattering (cat's eye), Maxwell fish eye, or Luneberg lens type. A magnetoelectric analogy is formulated, namely the existence of equivalent axisymmetric electric and magnetic fields that scatter charged particles in identical fashion.

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1. INTRODUCTION

The trajectory of a particle scattered by a spherically symmetric potential $V(r)$ is determined in the classical approximation by two parameters, e.g., the particle velocity v_∞ at infinity and the impact parameter b, or else the total energy E and the angular momentum l. The deflection angle χ is a function of these two parameters, and when solving the inverse problem we must reconstruct $V(r)$, given χ and certain values of these parameters.

There are at present two quadrature solutions of the inversion problem: the Firsov algorithm,[1] in which the angle χ is specified as a function of the impact parameter b while the energy E is fixed, and the Hoyt algorithm,[2] in which l is fixed and the deflection angle is specified as a function of energy.

We shall show here there exists also a third algorithm with an explicit solution, with the impact parameter fixed and the deflection angle also specified as a function of energy.

The Firsov algorithm is most useful for scattering by a microtarget, for in this case the only quantity that we can fix is the energy. Next, knowing the intensity of scattering of a wide beam (compared with the target) of incident particles through various angles we can reconstruct the function $\chi(b)$ (if certain uniqueness conditions are satisfied).

When a macroscopic spherically or axially symmetric force field is probed, the value of $\chi(b)$ can be obtained by scanning the field with a particle beam that is narrow compared with the field dimensions. This measurement calls for the source to be moved. Another possibility is to fix the source position (and accordingly fix the impact parameter) and vary the particle energy, with $\chi(E)$ measured at constant b. This method may prove to be more convenient in some cases.

It appears that there exist also inverse-problem formulations other than the three mentioned above. The deflection angle can in the general case be specified on some line located in the plane of the parameters E and l or v_∞ and b. The question is: what conditions must this line satisfy to make a solution of the inverse problem possible?

For scattering of a particle by a magnetic field, the analog of a spherically symmetrical potential is a field $\mathbf{H}$ directed along the z axis and symmetrical with respect to rotations about this axis, while the trajectories of the incident charged particles are assumed to lie in the (x, y) plane.

An inversion formula for this problem could be obtained only for the case of a fixed generalized momentum $p_\varphi \equiv l$.

Inversion formulas can be used to obtain fields having various focusing properties.[3] For the Hoyt case, in electric and magnetic fields, lenses were obtained of the Luneberg type (which focuses a parallel particle beam onto the edge of the lens), of the Maxwell "fish-eye" type (in which the source and the focus are at diametrically opposite points), and of the "cat's-eye" type (in which the particles are scattered strictly backward). The possession of a particular focusing property by a potential is undoubtedly connected with the internal symmetry in the corresponding classical or quantum-mechanical problem.[4]

In Sec. 3 we obtain the focusing potentials for this case. In Sec. 4 we derive an inversion formula for a fixed impact parameter. Section 5 deals with the inversion problem for an axisymmetric magnetic field at a fixed angular momentum, and the magneto-electric analogy. The focusing magnetic fields for this case are obtained in Sec. 6.

2. HOYT'S FORMULA

We consider in the classical approximation three types of inversion formulas that reconstruct a centrosymmetric field $V(r)$ from the deflection angle χ of the particle: at a given particle energy E (Ref. 1), at a given angular momentum l (Ref. 2) and at a given impact parameter b. We assume hereafter $V(r)$ is a repulsive field, i.e., $V>0$, and that $V(\infty)=0$, although many results hold also in a more general case.

We consider first the second inversion-formula type, i.e., at a given angular momentum l. For the polar angle φ_0 in the field $V(r)$ we have (see, e.g., Ref. 5) the expression

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$$\varphi_0(E)=\int_{r_m}^{\infty}\frac{l\,dr/r^2}{[2m(E-V)-l^2/r^2]^{1/2}}=\lambda\int_{r_m}^{\infty}\frac{dr/r^2}{[E-V-\lambda^2/r^2]^{1/2}},$$

where $\lambda\equiv l/\sqrt{2m}$ is fixed.

Introducing now a new variable $U\equiv V+\lambda^2/r^2$ and changing from r to U, we obtain

$$\varphi_0(E)=-\lambda\int_{\infty}^{r_m}\frac{dr/r^2}{[E-U(r)]^{1/2}}=\lambda\int_0^E\frac{1}{[E-U]^{1/2}}\frac{d}{dU}\frac{1}{r(U)}dU, \tag{1}$$

where $r(U)$ is the function inverse to $U(r)$.

Formula (1) should be regarded as an integral equation for the function $1/r(U)$ when the left-hand side is given. This is an Abel equation and can be solved analytically[6]:

$$\frac{1}{r(U)}=\frac{1}{\pi\lambda}\int_0^U\frac{\varphi_0(E)\,dE}{(U-E)^{1/2}},\quad U(r)=V(r)+\lambda^2/r^2. \tag{2}$$

An inversion formula at fixed λ in the form (2) was first obtained by Hoyt.[2]

Using now the formulas $l=mv_\infty b$ and $E=mv_\infty^2/2$, we introduce the impact parameter b and note that under the conditions of our problem, when λ is fixed and E is varied, the impact distance is a variable quantity and $E=\lambda^2/b^2$ or $b=\lambda/E^{1/2}$. The Hoyt formula (2) can then be rewritten by introducing it in the impact parameter b explicity. Changing in (2) from E to b and introducing the particle deflection angle $\chi=2\varphi_0-\pi$, we get ultimately

$$\frac{\lambda}{r}=U^{1/2}+\frac{\lambda^2}{\pi U^{1/2}}\int_{\lambda/U^{1/2}}^{\infty}\frac{\chi(b)\,db}{b^2(b^2-\lambda^2/U)^{1/2}},\qquad U(r)=V(r)+\lambda^2/r^2. \tag{3}$$

Equations (3) constitute in fact the solution of the classical inverse scattering problem if $l=\lambda\sqrt{2m}$ is fixed and the deflection angle is specified as a function of the impact parameter.

3. FOCUSING POTENTIALS

We construct now with the aid of the inversion formulas (3) spherically symmetric systems that have a radius R and focus particles with a specified angular momentum, in the same manner as Firsov's formula 1 was used in Ref. 3 to construct systems having spherical symmetry and focusing particles with specified energy.

The focusing condition (see Ref. 3) calls here for all the particles with specified λ, emerging from a certain point on the symmetry axis at a distance R_1 from the force center, to land at another point of the same axis and located a distance R_2 from the center.

In this case, as shown in Ref. 3,

$$\chi(b)=\begin{cases}\arcsin(b/R_1)+\arcsin(b/R_2), & b<R\\ 0, & b>R.\end{cases} \tag{4}$$

We consider two interesting particular cases of (4), in which all the calculations can be carried through to conclusion.

1) Let $R_1=R_2=R$, i.e., the "source" and "sink" of the particles are at diametrically opposite points on a sphere of radius R. This focusing condition corresponds to the so-called Maxwell's fish eye (see Ref. 3). Here

$$\chi(b)=\begin{cases}2\arcsin(b/R), & b<R\\ 0, & b>R\end{cases}. \tag{4a}$$

Using the tabulated formulas[7]

$$\int_0^{\alpha}\frac{x\sin x\,dx}{\cos^2 x(\cos^2 x-\cos^2\alpha)^{1/2}}=\frac{\pi}{2}\frac{1-\cos\alpha}{\cos^2\alpha},$$

$$\int_a^1\frac{d\xi}{\xi^2(\xi^2-a^2)^{1/2}}=\frac{(1-a^2)^{1/2}}{a^2}$$

and putting $\xi\equiv b/R\equiv\cos x$ and $a\equiv\lambda/RU^{1/2}\equiv\cos\alpha$, we obtain from the inversion formula (3)

$$\lambda/r=U^{1/2}+(\lambda^2/R^2a^2U^{1/2})\left((1-a^2)^{1/2}+a-1\right).$$

From this we get the potential

$$V(r)=\frac{2\lambda^2}{R^2}\left(1-\frac{R}{r}\right),\quad r<R. \tag{5}$$

Thus, the Coulomb potential (5), cut off at $r=R$, focuses particles leaving an arbitrary point on a sphere of radius R unto the diametrically opposite point of this sphere, if the particle orbital momentum is $l=\lambda\sqrt{2m}$. Just as in the case of the "fish-eye" potential, we can continue this potential analytically farther than the radius R; the focusing property will be preserved also for trajectories that emerge outside the sphere $r=R$. This focusing property of a Columb field reflects the known fact[5] that a bundle of Coulomb trajectories passing through two points diametrically opposite relative to the force center corresponds to a constant angular momentum.

2) Let $R_1=\infty$ and $R_2=R$, i.e., a plane-parallel beam is focused onto the surface of a sphere of radius R. Such a focusing system is usually called a Luneburg lens.[3] In this case

$$\chi(b)=\begin{cases}\arcsin(b/R), & b<R\\ 0, & b>R\end{cases}. \tag{5a}$$

Using the cited tabulated values of the integrals, we obtain from (3)

$$2\lambda/r=U^{1/2}+(U-\lambda^2/R^2)^{1/2}+\lambda/R,$$

from which we determine the potential. If a centrifugal potential λ^2/r^2 is added to the potential V and is analytically continued to the singularity $r=2R$, the potential becomes symmetrical about the point $r=R$. This yields, e.g., an additional symmetry property for bound states of the corresponding quantum problem. We have

$$V(r)=\frac{\lambda^2}{4R^2}\left[3-\frac{4R}{r}+\frac{r^2}{(2R-r)^2}\right],\quad r<R. \tag{6}$$

Equation (6) specifies in fact the potential that realizes a Luneburg lens for particles with a given angular momentum.

The inversion problem has also an exact solution in the case of scattering at a constant angle, i.e., $\chi=c\pi$, where c is an arbitrary real positive quantity.

The inversion algorithm (3) yields for this case

$$\lambda/r=U^{1/2}+c(U-\lambda^2/R^2)^{1/2}. \tag{7}$$

If the field $V(r)$ is specified in all of space, R must be extended to infinity. The potential is then

$$V(r)=-c(c+2)\lambda^2/(c+1)^2r^2. \tag{8}$$

At $c=1$ we obtain the so-called cat's eye (backscattering). The corresponding potential is

$$V(r)=-3\lambda^2/4r^2. \tag{9}$$

For a field cut off at $r=R$ it is easy to obtain from (7) for V a quadratic equation that becomes linear at $c=1$ and yields for the potential the expression

$$V(r)=-\frac{3}{4}\frac{\lambda^2}{r^2}+\frac{\lambda^2}{2R^2}\left(1+\frac{r^2}{2R^2}\right),\quad r<R. \tag{10}$$

The cut-off field (10) is thus capable of reflecting particles incident on it with an angular momentum $l=\lambda\sqrt{2m}$.

4. INVERSION FORMULA FOR A GIVEN IMPACT PARAMETER

We obtain now a simple inversion formula at a fixed particle impact parameter b. We assume that the deflection angle χ is given as a function of the energy E. From Ref. 5 we have

$$\varphi_0(E)=\int_{r_m}^{\infty}\frac{bdr}{r^2[1-V/E-b^2/r^2]^{1/2}},$$

or

$$\varphi_0/E^{1/2}=\int_{r_m}^{\infty}\frac{bdr}{r(r^2-b^2)^{1/2}[E-r^2V/(r^2-b^2)]^{1/2}}.$$

We introduce now $U(r)\equiv r^2V(r)/(r^2-b^2)$ and change from r to U in the integral. Then

$$\varphi_0/E^{1/2}=\int_{\infty}^{r_m}\frac{-bdr}{r(r^2-b^2)^{1/2}[E-U(r)]^{1/2}}=\int_0^E\frac{dU}{(E-U)^{1/2}}\frac{d}{dU}\arcsin\frac{b}{r(U)}. \tag{11}$$

This is an Abel integral equation for the function $\arcsin(b/r(U))$ at a given angle $\varphi_0(E)$. Introducing the particle deflection angle $\chi(E)$ and using Eq. (6), we obtain

$$\arcsin\frac{b}{r(U)}=\frac{1}{2\pi}\int_0^U\frac{\pi-\chi(E)}{[UE-E^2]^{1/2}}dE.$$

From this we get ultimately

$$r(U)=b\sec\frac{1}{2\pi}\int_0^U\frac{\chi(E)dE}{[UE-E^2]^{1/2}},\quad U(r)\equiv r^2V(r)/(r^2-b^2). \tag{12}$$

The inversion formulas (12) solve in fact the foregoing inverse problem.

It is thus known, for example (Ref. 5, p. 72) that for a field $V=\alpha/r^2$ with $\alpha>0$ the deflection angle is

$$\chi(E)=\pi(1-1/[1+\alpha/b^2E]^{1/2}). \tag{13}$$

We apply to this case the inversion algorithm (12). Substituting (13) in (12) and carrying out elementary integration, we obtain

$$U(r)=\alpha/(r^2-b^2),\quad V=\alpha/r^2,$$

as it should be.

5. INVERSION FORMULA FOR A MAGNETIC FIELD

Consider the scattering of a classical nonrelativistic particle of mass m and charge e in an axially symmetric field $\mathbf{H}$ directed along the z axis. The vector potential is

$$\mathbf{A}=(0,A_\varphi(\rho),0)\equiv(0,A(\rho),0).$$

The component of the vector $\mathbf{H}$ are

$$H_\rho=0,\ H_\varphi=0,\ H_z\equiv H=\mathrm{rot}_z\mathbf{A}=A(\rho)/\rho+\partial_\rho A(\rho).$$

The Coulomb gauge condition $\operatorname{div}\mathbf{A}=0$ is found to be automatically satisfied. We consider only planar (at $p_z=m\dot{z}=0$) motion of the particle, and replace ρ with r.

From the Lagrangian[8] of such a particle in a magnetic field of the indicated configuration ($c=1$)

$$L=\frac{m}{2}(\dot{r}^2+r^2\dot{\varphi}^2)+er\dot{\varphi}A(r)$$

we obtain two integrals of motion: the generalized angular momentum

$$l\equiv p_\varphi=mr^2\dot{\varphi}+erA(r)$$

and the energy

$$E=mv^2/2=m(\dot{r}^2+r^2\dot{\varphi}^2)/2.$$

Now, using these two conservation laws, we easy construct a quadrature expression for the particle trajectory and the deflection angle. We have $\dot{\varphi}=(l-erA)/mr^2$ and from the energy integral we get

$$dr/dt=[2E/m-(l-erA)^2/m^2r^2]^{1/2}.$$

Eliminating dt with the aid of the angular-momentum conservation law and integrating, we obtain ultimately

$$\varphi(r)=\int\frac{dr}{r^2[2mE/(l-erA)^2-1/r^2]^{1/2}}+\text{const}. \tag{14}$$

In analogy with the case of a central electric field, such a trajectory is symmetric about a straight line drawn from the symmetry center to the closest trajectory point. The polar angle is

$$\varphi_0=\int_{r_m}^{\infty}\frac{dr}{r^2[2mE/(l-erA)^2-1/r^2]^{1/2}}. \tag{15}$$

Using (15), we construct now an inversion algorithm that allows us to reconstruct the potential $A(r)$ given the function $\chi(E)$, where $\chi=2\varphi_0-\pi$, at a fixed particle angular momentum l. We assume hereafter that

$$rA(r)\to0,\quad r\to\infty.$$

We introduce in (15) new variables

$$s\equiv1/r,\quad x\equiv2mE/l^2,\quad g(s)\equiv l/(l-eA(s)/s). \tag{16}$$

With allowance for (16), we have

$$\varphi_0(x)=\int_0^{s_0}\frac{ds}{[xg^2(s)-s^2]^{1/2}},\quad xg^2(s_0)-s_0^2=0,\quad g(0)=1. \tag{17}$$

Expression (17) is a nonlinear integral equation with a boundary condition for the function $g(s)$ at a specified $\varphi_0(x)$ [or $\chi(x)$]. It can be solved by various means. One is to transform (17) into

$$\varphi_0(x)=\int_0^{s_0}\frac{ds/g(s)}{[x-f(s)]^{1/2}},$$

where $f(s)\equiv s^2/g^2(s)$, and changing from the variable s to f. The resultant Abel-type equation[6] in terms of the antiderivative of the function $1/g(s)$ is inverted, the result of the inversion is differentiated with respect to s, and a first-order differential equation in $g(s)$ is obtained and can be directly integrated.

Another possible solution procedure is considered in Ref. 5. The final solution of the problem (17) is

$$g(r)=\exp\frac{1}{\pi}\int_0^{l^2/2mr^2g^2}\chi'(E)\operatorname{arch}(l/rg\sqrt{2mE})\,dE, \tag{18}$$

$$g(r)\equiv l/(l-erA(r))=bv/rv_\varphi,$$

where v is the particle velocity.

We note now that under our conditions, when the angular momentum of the particle is specified, the impact distance b is the varied quantity and $E=\lambda^2/b^2$, where $\lambda^2\equiv l^2/2m$.

We shall need below an inversion formula that contains the impact parameter explicitly. Changing in (18) from E to b and integrating by parts, we obtain the variant of interest to us

$$g(r)=\exp\frac{1}{\pi}\int_{rg}^{\infty}\frac{\chi(b)\,db}{[b^2-r^2g^2]^{1/2}},\quad g(r)\equiv l/(l-erA(r)). \tag{19}$$

This equation is mathematically equivalent to the known Firsov formula,[1] since it coincides with the latter when $g(r)$ is replaced by $n(r)=[1-e\Phi(r)/E]^{1/2}$, where $\Phi(r)$ is the centrosymmetric scattering field.

From the physical point of view the equivalence of these two formulas lies in the fact that if charged particles with specified energy E move in an electric field $\Phi(r)$ along certain trajectories, then the same particles with specified angular momentum l will move in the magnetic field

$$A(r)=\frac{l}{er}(1-1/[1-e\Phi(r)/E]^{1/2})$$

along the same trajectories. This statement can be called the magneto-electric analogy.

The weak-field approximation can be considered in (19) when $erA/l\ll 1$ and $\chi\ll 1$ (in analogy with the Firsov problem). In fact, expanding the exponential in a series, we have

$$g\simeq 1+\frac{1}{\pi}\int_r^{\infty}\frac{\chi(b)\,db}{(b^2-r^2)^{1/2}},$$

where g is replaced by unity in the integrand and at the limits. Next, in the left-hand side $g=(1-erA/l)^{-1}\approx 1+erA/l$, and we ultimately obtain a simple explicit formula in the weak-field approximation:

$$A(r)\approx\frac{l}{e\pi r}\int_r^{\infty}\frac{\chi(b)\,db}{(b^2-r^2)^{1/2}}. \tag{20}$$

6. FOCUSING MAGNETIC FIELDS

We shall use the inversion algorithm (19) for the magnetic field to construct planar focusing systems of radius R, of the type already considered in Sec. 3 of this paper and in Ref. 3 for the case of a scalar field $V(r)$.

Corresponding to one such exactly solvable systems is the focusing condition (4a), where the "source" and "sink" of particles of charge e lie on a circle of radius R. This is a system of the Maxwell "fish-eye" type.

Substituting (4a) in (19) we obtain (see Ref. 3)

$$g=2/(1+r^2/R^2),$$

so that the magnetic potential is

$$A(r)=\frac{l}{2er}\left(1-\frac{r^2}{R^2}\right)\ \text{for}\ r<R. \tag{21}$$

It is easy to verify that this vector potential corresponds to a homogeneous magnetic field $H=-l/eR^2$, and all the trajectories are thus circles in this case.

We now extend the homogeneous magnetic field $H=H_z$ over all of space. The charged particles move in such a field in circles at constant velocities proportional to the radii of these circle. On the basis of the magneto-electric analogy obtained in the preceding section we conclude that this case corresponds to the Maxwell "fish eye" problem considered in Ref. 3. Classical particles of energy, in a "fish-eye" field

$$V(r)=E\{1-4R^4/(R^2+r^2)^2\},$$

specified in all of space, have circular trajectories.[4] If these circles pass through a point $\mathbf{r}$, they pass also through the point $-R^2\mathbf{r}/r^2$ obtained from the initial point as a result of inversion in a sphere having a radius R and a center at zero, as well as reflection at the origin, with $\mathbf{r}$ an arbitrary point in space. Charged particles with specified generalized angular momentum l also moves along the same trajectories in a homogeneous magnetic field $H=-l/eR^2$. The circles passing through diametrically opposite points (relative to an arbitrary chosen force center) correspond to one and the same angular momentum. The particle-focusing property in such a magnetic field can be easily proved by a purely geometric method.

Another system for which all the calculations can be carried through to conclusion is defined by condition (5a). In the scalar case it corresponded to the so-called Luneburg lens. A plane-parallel beam of charged particles is focused here into a circle of radius R.

The inversion procedure (19) yields for this case

$$g=[2-r^2/R^2]^{1/2},$$

and the potential is

$$A(r)=\frac{l}{er}(1-1/[2-r^2/R^2]^{1/2}),\quad r<R. \tag{22}$$

The inversion problem is solved just as accurately in the case of backward reflection of charged particles with a given orbital momentum l. Here

$$\chi(b)=\begin{cases}\pi, & b<R\\ 0, & b>R,\end{cases}$$

and we obtain from (19)

$$g=(2R/r-1)^{1/2}.$$

Hence the magnetic cat's-eye lens

$$A(r)=\frac{l}{er}(1-1/[2R/r-1]^{1/2}),\quad r<R. \tag{23}$$

The magnetic potentials (21)–(23) can be obtained by the indicated magnetoelectric analogy from the corresponding scalar potentials $V(r)$ obtained in Ref. 3.

7. CONCLUSION

The new inversion formulas and new focusing fields obtained here extend greatly the earlier results in this field. They can be of interest for research in electron optics, as well as for the heretofore little-investigated connection between the focusing properties of electric or magnetic fields and their internal symmetries, the existence of additional integral of motion etc. in the problem of particle motion in these field (this question has been elucidated fully enough only for the Maxwell "fish-eye" problem, Ref. 4).

As already noted, it is of interest to ascertain how many different explicit inversion formulas exist. It should be noted, finally, that when using a fixed angular momentum we come closest to the quantum formulation of the inverse problem of reconstructuring the potential from the scattering phase shift. It appears that it is precisely here that it is easiest to track the transition from the classical to the semiclassical and further to the quantum inverse problem.

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