Solitary toroidal vortices

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We find smooth solutions in the form of toroidal vortices, which decrease exponentially rapidly at infinity, in contrast to solutions known previously. They are self-similar and are therefore possibly the structural elements for hydrodynamic and magnetohydrodynamic turbulence. We suggest a numerical method to find such solutions and give their classification.

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1. INTRODUCTION

In connection with the slowing down in the progress in the strong-turbulence theory based upon the assumption of complete stochasticity of all degrees of freedom in a fluid medium, and also in connection with the appearance of a number of experimental indications, it has become ever clearer that one should consider turbulence as a random set of ordered formations (structures). These structures realize extrema of the Hamiltonian in phase space and therefore, even if they are unstable, the system spends considerably more time in their neighborhoods than far from them. Solutions in the form of solitary waves (if such exist) can serve as the most appropriate structures in such a model of turbulence. An illustrative picture of such structural turbulence is observed in the boundary layer of Poiseuille flow,[1] in a film of a viscous fluid which is draining off,[2] and in other cases.

In three-dimensional hydrodynamic and magnetohydrodynamic turbulence (in particular, in the turbulent-dynamo problem), the structures in the form of the solitary vortices, considered in what follows, must play an important role.

Two basic kinds of vortex motions are known—filamentary (open) and toroidal (closed). A study of such motions is of general physical interest. The first studies of vortex motions were carried out back in the last century. Filamentary vortices have by now been thoroughly studied. However, the theory of toroidal vortices is far from its completion. Its development has been hindered by the complexity of the equations describing these vortices. Well known is the solution in the form of Hill's toroidal vortex,[3,4] in which the vorticity is proportional to the distance from its axis inside a sphere, outside of which the velocity field is potential. The problem is then reduced to the solution of two linear equations and the joining of the solutions on the surface of the sphere. The perturbed velocity with increasing distance from such a vortex decreases as the inverse cube of the distance.

In theoretical studies of the containment of plasma in magnetic traps, solutions have been obtained in the form of toroidal equilibrium configurations.[5,6] The magnetic field in those was found for a given coordinate-dependence of the vorticity (electrical current). These solutions also decrease insufficiently rapidly and it is impossible to consider them to be solitary.

One can also give a different statement of the problem when the vorticity is given in the form of a function of the stream function. In that case the vortex equation becomes similar to a soliton equation and the toroidal vortices themselves can be considered to be solitons.

In the present paper we propose a simple method for obtaining solutions of the equations for toroidal vortices in the form of three-dimensional solitons with axial symmetry. A study of toroidal vortices has become particularly topical in connection with the development of powerful quasistationary plasma-static and plasma-dynamic traps, where the hydrodynamic motion coexists with a vortical magnetic field.[5,7] In our discussion we use therefore as an example the magnetohydrodynamic plasma equations.

2. TYPES OF TOROIDAL VORTICES

We look for a stationary solution of the magnetohydrodynamic equations in the form of a solitary toroidal vortex with axial symmetry. By solitary we mean here that the plasma parameters tend to constant values with increasing distance from the vortex.

In the rest frame of the vortex we can write the magnetohydrodynamic equations in the form

$$[\mathbf{q}\times\operatorname{rot}\mathbf{q}]-[\mathbf{B}\times\operatorname{rot}\mathbf{B}]=\nabla(4\pi p+\mathbf{q}^2/2); \tag{2.1}$$

$$\mathbf{q}=(4\pi\rho)^{1/2}\mathbf{v},\quad \mathbf{q}\nabla\rho=0,$$
$$\operatorname{div}\mathbf{q}=0,\quad \mathbf{B}\nabla\rho=0,\quad \operatorname{rot}[\mathbf{q}\times\mathbf{B}]=0. \tag{2.2}$$

We have assumed here that the plasma density ρ is constant along the stream lines and along the magnetic-field lines, i.e., these lines lie on the constant density surfaces. There are three possibilities. The first one is when $\mathbf{q}$ and $\mathbf{B}$ are proportional to one another. We call such vortices parallel. (This case was considered for the first time in Refs. 8 and 9). In the other cases either the velocity or the magnetic field has only a toroidal component. We shall call them, respectively, a magnetic or a dynamic nonparallel vortex. We note that these configurations are particular cases of helical plasma flow in a magnetic field with helical symmetry, considered in Ref. 10.

Parallel vortex. In a parallel vortex we have $\mathbf{q}=\alpha\mathbf{B}$, where $\alpha=\text{const}$ is the Mach number—the ratio of the flow velocity to the Alfvén speed. The components of the magnetic field in a cylindrical coordinate system can then be expressed in terms of the stream function Ψ according to the formulae

$$rB_r=\partial\Psi/\partial z,\quad rB_z=-\partial\Psi/\partial r,\quad rB_\varphi=f(\Psi),\tag{2.3}$$

where $f(\Psi)$ is an arbitrary function of Ψ. In that case (2.1) and (2.2) are reduced to the Grad-Shafranov equation[5,6]

$$\tilde{\Delta}\Psi=-ff'-r^2F'(\Psi),\quad \tilde{\Delta}=r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}.\tag{2.4}$$

The prime indicates here differentiation with respect to Ψ, and the arbitrary function $F(\Psi)$ is connected with the plasma pressure p through the relation

$$4\pi p=(1-\alpha^2)F(\Psi)-q^2/2.\tag{2.5}$$

Hence it follows that at nonzero velocity ($\alpha\neq 0$) the plasma pressure is not a surface function.

Magnetic vortex. Here $q_r=q_z=0$, and $q_\varphi=rg(\Psi)$ where $g(\Psi)$ is an arbitrary function of Ψ, while the magnetic field has all components. We then get from (2.1), (2.2)

$$\tilde{\Delta}\Psi=-ff'-r^2F'(\Psi)-r^4gg',\tag{2.6}$$

while the pressure is given by the formula

$$4\pi p=F(\Psi)+r^2g^2/2.\tag{2.7}$$

Dynamic vortex. In that case the components of the vector $\mathbf{q}$ can be expressed in terms of the surface function Ψ by the relations

$$q_r=r^{-1}\partial\Psi/\partial z,\quad q_z=-r^{-1}\partial\Psi/\partial r,\quad q_\varphi=g(\Psi)/r.\tag{2.8}$$

The magnetic field can have only a toroidal component

$$B_r=B_z=0,\quad B_\varphi=rf(\Psi).\tag{2.9}$$

Using (2.8), (2.9) we get from (2.1), (2.2) an equation similar to (2.6):

$$\tilde{\Delta}\Psi=-gg'-r^2F'(\Psi)-r^4ff',\tag{2.10}$$

$$4\pi p=-F(\Psi)-r^2f^2(\Psi)-q^2/2.\tag{2.11}$$

The toroidal equilibrium of a plasma in a magnetic field is thus described by equations of the type (2.4) or (2.6).

3. SOLUTION OF THE EQUATIONS

We look for the simplest solitary solutions of Eqs. (2.4) and (2.6). To do this we assume that the right-hand sides of these equations are quadratic polynomials in Ψ. The following cases are then possible

$$\tilde{\Delta}\Psi=r^2\Psi-\Psi^2,\tag{3.1}$$

$$\tilde{\Delta}\Psi=r^4\Psi-r^2\Psi^2,\tag{3.2}$$

$$\tilde{\Delta}\Psi=r^4\Psi-\Psi^2,\tag{3.3}$$

$$\tilde{\Delta}\Psi=\Psi-r^2\Psi^2,\tag{3.4}$$

$$\tilde{\Delta}\Psi=r^2\Psi-r^2\Psi^2,\tag{3.5}$$

$$\tilde{\Delta}\Psi=r^2\Psi-r^4\Psi^2.\tag{3.6}$$

A characteristic feature of the equations which we have selected is self-similarity. This means that any coefficient in them can be removed by a similarity transformation. Because of this we have given these equations in dimensionless form. As an example we describe the transition from Eq. (2.4) to Eq. (3.1). In (2.3) we put $f=a\Psi^{3/2}$ and in (2.5) $F=-b\Psi^2+4\pi p_0$. Substituting these values into Eq. (2.4) and performing the transformation

$$\Psi\to(8b)^{1/2}\Psi/3a^2,\quad r\to(2b)^{-1/4}r,$$

we get Eq. (3.1).

We assume the function Ψ to be smooth and to tend to zero at infinity together with its derivatives. We show that the last three equations do not have such solutions. To do that we multiply them by $r^{-3}\partial\Psi/\partial r$ and integrate over the whole volume. After integrating by parts we get on the left-hand side a negative quantity and on the right-hand side a non-negative one, whence follows that these equations cannot have solitary solutions. To solve Eqs. (3.1) to (3.3) we write Ψ as a Fourier integral

$$\Psi=\int dk\exp(ikz)\Psi_k(r).\tag{3.7}$$

Substituting this expression, for example, in (3.1) we get the following integro-differential equation

$$r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial\Psi_k}{\partial r}-(k^2+r^2)\Psi_k=\int\Psi_{k'}\Psi_{k-k'}\,dk'\equiv N_k(r).\tag{3.8}$$

It is reduced by using a Green function to an integral equation

$$\Psi_k=\int G_k(r,r')N_k(r')\,dr',\tag{3.9}$$

where G_k can be expressed in terms of confluent hypergeometric functions and can be found more simply by numerical methods. After this we solve Eq. (3.1) by a modified iteration method—the method of the stabilizing multiplier,[11] i.e., we introduce the auxiliary functional S:

$$S=(s_1/s_2)^{n/(n-1)},\quad s_1=\int|\Psi_k|^2dk\,dr,\quad s_2=\int\Psi_{-k}G_kN_k\,dk\,dr\,dr'.\tag{3.10}$$

Here n is the degree of the nonlinear right-hand side of Eq. (3.9) (in our case $n=2$).

We note that $S=1$ if Ψ is a solution of Eq. (3.9). We solve instead of (3.9) the equation

$$\Psi_k=S\int G_kN_k\,dr'\equiv\mathscr{F}[\Psi],\tag{3.11}$$

which has the same solution as (3.9), but the degree of non-linearity of its right-hand side is zero. In view of this, Eq. (3.11) solved by the iteration method gives, in contrast to (3.9), a convergent sequence

$$\Psi_k^{(m+1)}=\mathscr{F}[\Psi^{(m)}].\tag{3.12}$$

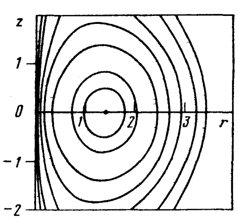

FIG. 1. Constant-level lines of the solution of Eq. (3.1). The maximum value $\Psi=8.1$ is reached at $r=1.41$. The ratios of the levels here and in Fig. 2 and Fig. 3 to the maximum value are, respectively, 0.75, 0.5, 0.2, 0.1, 0.05, 0.02, and 0.01.

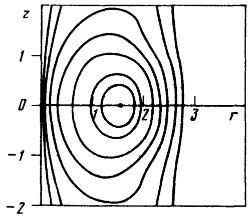

FIG. 2. Constant-level lines of the solution of Eq. (3.2). The maximum value $\Psi = 6.9$ is reached in the point $r = 1.54$.

After performing these transformations on a computer we get a solution of Eq. (3.9) and afterwards, using (3.7), a solution of Eqs. (3.1) to (3.3). We give in Figs. 1 to 3 the contour map of the solution of Eqs. (3.1) to (3.3) with quadratic nonlinearity. These equations give the distribution of the vorticity $\Omega = \tilde{\Delta}\Psi/r$ over an axial cross section of the vortex. The sign of the term which is linear in Ψ on the right-hand side is chosen such that it leads to a screening of the main part of the azimuthal current which is described by the nonlinear term. As in the case of Debye screening this leads to an exponential decrease of the field with increasing distance from the vortex. We give in Figs. 4 to 6 the distribution of the vorticity Ω which is (depending on the kind of vortex) equal to $\mathrm{curl}_\varphi \mathbf{v}$ or $\mathrm{curl}_\varphi \mathbf{B}$ after having been made dimensionless. From them it is clear that in the central part Ω is negative, while around it Ω is positive and is distributed such that there occurs a complete screening of the vortex. Such a screening did not appear in previously known solutions.

4. TOROIDAL VORTICES IN A PLASMA

1. We consider a parallel vortex in the limiting cases when the plasma velocity vanishes and when the magnetic field vanishes. Both these cases are described by Eq. (3.1). It follows from (2.4) and (2.5) that when $\mathbf{q} = 0$

$$F = -\Psi^2/2 + 4\pi p_0, \quad f = (2/3)^{1/2}\Psi^{3/2}, \tag{4.1}$$

$$p = p_0 - \Psi^2/8\pi. \tag{4.2}$$

It is clear from (4.2) that the pressure inside such a vortex is a minimum at the center of the vortex where Ψ is a maximum. Such a vortex is contained by the external plasma pressure p_0. This possibility was already indicated in Ref. 5.

In the case when the magnetic field vanishes we have a hydrodynamic vortex in which F and f have the same form (4.1) while the pressure in it equals

$$p = \Psi^2/8\pi - |\nabla\Psi|^2/r^2 - 2\Psi^3/3r^2 + p_0. \tag{4.3}$$

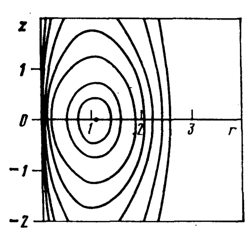

FIG. 3. Constant-level lines of the solution of Eq. (3.3). The maximum value $\Psi = 9.45$ is reached in the point $r = 1.09$.

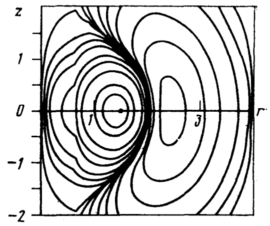

FIG. 4. Constant level lines for the vorticity $\tilde{\Delta}\Psi/r$ in the case of Eq. (3.1). The maximum value is 17.5 for $r = 2.4$, the minimum value is -61.1 and is reached at $r = 1.4$. The ratios of the levels here and in the next figures to the minimum value are, respectively, 0.75, 0.5, 0.2, 0.1, 0.05, 0.02, 0.01, 0, -0.001, -0.005, -0.01, -0.05, and -0.75.

The constant α has been eliminated here by means of a similarity transformation.

This vortex can therefore likewise not exist without a toroidal velocity component and an external pressure p_0. Usually one observes vortices without a toroidal component (vortex rings) which are described by Eq. (3.5). Calculations show that this equation has no solitary solution. Hence it follows that vortex rings apparently are supported by the external velocity field and by viscosity effects.[4]

At a Mach number equal to unity degeneracy occurs, i.e., stationary solutions of a rather arbitrary form are possible. Chandrasekhar[8] has shown that small perturbations of such a flow have only real eigenfrequencies. From this it follows apparently that the unity Mach number separates the region of stable and of unstable vortices, i.e., super-Alfvénic vortices ($\alpha > 1$) are stable and sub-Alfvénic vortices ($\alpha < 1$) are unstable.[9]

2. We consider a magnetic vortex in which the magnetic field has only a poloidal component. After a similarity transformation the equation of such a vortex reduces to (3.2). Then

$$F = \Psi^3/3, \quad g = (g_0^2 - \Psi^2)^{1/2}, \tag{4.4}$$

$$4\pi p = \Psi^3/3 + r^2(g_0^2 - \Psi^2)/8, \tag{4.5}$$

where g_0 is an integration constant. Its physical meaning consists in that far from the soliton where $\Psi \to 0$ the toroidal component of $\mathbf{q}$, equal to rg, takes the form rg_0, i.e., outside the soliton the motion has the form of solid rotation. Of course, such a motion must be bounded in space. Such a soliton recalls the magnetic field of a rotating planet with a liquid conducting core, except

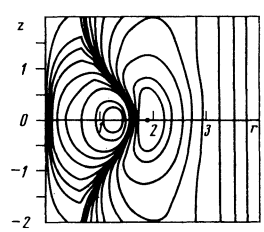

FIG. 5. Constant-level lines for the vorticity in the case of Eq. (3.2). The minimum value is -15.5 at $r = 1.27$, the maximum value is 18.6 at $r = 1.9$.

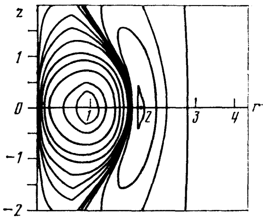

FIG. 6. Constant level lines for the vorticity in the case of Eq. (3.3). The minimum value is −82.4 at $r=0.89$, the maximum value is 1.7 at $r=1.9$.

that in our case the rotation axis is the same as the magnetic axis.

3. In the dynamic vortex the velocity has all components. We restrict ourselves to the consideration of the case where there is only a poloidal velocity. The magnetic field has only a toroidal component. Such a vortex is described by the same Eq. (3.2) as the magnetic vortex. For it we have

$$f=(f_0^2-\Psi^2)^{1/2}, \tag{4.6}$$

$$p=\Psi^3/3-r^2(f_0^2-\Psi^2)-|\nabla\Psi|^2/8\pi+p_0. \tag{4.7}$$

Here rf_0 is the magnetic field far from the soliton, where $\Psi\to 0$, i.e., it goes over into the magnetic field of a current of constant density. In contrast to the magnetic vortex the pressure in the dynamic vortex decreases away from the axis and tends to zero at the boundaries of the plasma.

5. TOROIDAL VORTICES IN AN EXTERNAL FIELD

We gave above examples of magnetic and dynamic vortices in an external magnetic field or in a velocity field, but in them the stream function Ψ decreased exponentially with increasing distance from the soliton. Vortices in which the transition to the external field does not take place so smoothly are also possible. In the general case the region of the vortex field can be surrounded by a vortexless field in which the equation for the stream function has the form

$$\Delta\Psi=0. \tag{5.1}$$

In the vortex region, on the other hand, we have one of Eqs. (3.1) to (3.5). On some surface $\Psi=$const the vortex solution must be joined to the vortexless solution. Hill's vortex is an example of such a vortex. The general axisymmetric solution of Eq. (5.1) has the form

$$\Psi=r\int_0^\infty[\Psi_{1k}I_1(kr)+\Psi_{2k}K_1(kr)]\cos kz\,dk+br^2+cr^2(r^2+z^2)^{-3/2}. \tag{5.2}$$

Here $I_1(z)$ is a modified Bessel function, $K_1(z)$ a Macdonald function, Ψ_{1k} and Ψ_{2k} are arbitrary functions of k, and b and c are arbitrary constants. This field is produced by external sources. Such a solution can be joined both to arbitrary solutions of Eqs. (3.1) to (3.6) and to solitary solutions of Eqs. (3.1) to (3.3). We see that there is here a great variety of configurations.

6. CONCLUSION

The analysis given here reveals the great variety of vortex equilibrium configurations. In most of them there is no screening and the velocity decreases according to a power law when one goes away from the center of the vortex. We have found in the present paper a class of stationary double layer vortex solutions in which the vorticity in the inner layer is screened by the vorticity of opposite sign in the outer layer. As a result the vortex decreases exponentially rapidly at infinity. These solutions are distinguished also by the fact that they are self-similar, i.e., when there is damping or buildup they are changed according to a similarity law. Therefore such solutions are realized more often than others and may serve as the structural elements of turbulent motion.

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[1]S. J. Kline, W. L. Reynolds *et al.*, J. Fluid Mech. **30**, 741 (1967).
[2]V. I. Petviashvili and O. Yu. Tsvelodub, Dokl. Akad. Nauk SSSR **238**, 1321 (1978) [Sov. Phys. Dokl. **23**, 117 (1978)].
[3]H. Lamb, Hydrodynamics, Cambridge University Press, 1932.
[4]M. A. Lavrent'ev and B. V. Shabat, Problemy gidrodinamiki i ikh matematicheskie modeli (Hydrodynamic problems and their mathematical models) Nauka, Moscow, 1977.
[5]V. D. Shafranov, Voprosy Teorii Plazmy **2**, 92 (1963) [Rev. Plasma Phys. **2**, 103 (1966)].
[6]H. Grad and H. Rubin, Proc. Second Internat. Conf. on the Peaceful Use of Atomic Energy, Atomizdat, Moscow, 1959, p. 131.
[7]A. I. Morozov and L. S. Solov'ev, Voprosy Teorii Plazmy **8**, 3 (1974) [Rev. Plasma Phys. **8**, 1 (1980)].
[8]S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability, Oxford University Press, 1961.
[9]V. D. Shafranov, Zh. Eksp. Teor. Fiz. **33**, 831 (1957) [Sov. Phys. JETP **6**, 642 (1958)].
[10]L. S. Solov'ev, Voprosy Teorii Plazmy **3**, 245 (1963) [Rev. Plasma Phys. **3**, 277 (1967)].
[11]V. I. Petviashvili, Fiz. Plazmy **2**, 469 (1976) [Sov. J. Plasma Phys. **2**, 257 (1976)].

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