On the conductivity of anisotropic inhomogeneous media

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The conductivity of highly anisotropic media with macroscopic dielectric or perfectly-conducting inclusions is considered. It is shown that, when the matrix is highly anisotropic, so that it can be characterized by a quantity $\gamma \gg 1$, the approximation linear in the inclusion concentration c is violated in a number of cases even at low $c \sim \gamma^{-1} \ll 1$, since the expansion parameter is the quantity $c\gamma$. A significant isotropization of the properties of the system is found to occur in the case $c\gamma \gg 1$, for which an order-of-magnitude analysis is performed with the aid of the diffusion analogy. The metal-dielectric phase transition region for such systems is considered. It is concluded that the properties of highly anisotropic two-component media in the vicinity of the transition point are close to the properties of isotropic systems. Certain size effects in highly anisotropic samples with low inclusion concentrations are discussed.

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1. INTRODUCTION

The permittivity, conductivity, and other characteristics of isotropic media with low spherical-inclusion concentrations c can be calculated in standard fashion[1] in the approximation linear in c. The generalization of the result obtained in Ref. 1 to the anisotropic case for a crystal with "normal" anisotropy (i.e., one of the order of unity) does not lead to any qualitatively new phenomena, amounting only to a quantitative refinement; as in the case of the isotropic medium, the expansion parameter is the inclusion concentration $c \ll 1$.

In a number of cases, however, the medium can be highly anisotropic. Strong anisotropy of natural origin is possessed, for example, by layered crystals of the graphite type of filamentary crystals of the TCNQ type. Anisotropy can also be artificially created in a system: by introducing identically oriented prolate inclusions into the isotropic matrix. Finally, any conducting medium becomes highly anisotropic when it is placed in a sufficiently strong magnetic field. Allowance for the anisotropy in these cases leads to qualitatively new results. Thus, if a polycrystalline sample located in a magnetic field H contains "special" crystallites whose conductivity does not depend on H, then the current distribution (as $H \to \infty$) will be highly inhomogeneous, and the effective transverse conductivity will differ significantly from the mean conductivity.[2,3] The expansion parameter in this case is the quantity $c\beta$, where $\beta \propto H$ ($\beta \gg 1$) and c is the concentration of the special crystallites ($c \ll 1$); therefore, the approximation linear in the concentration is inapplicable even when $c \sim \beta^{-1} \ll 1$.

In the present paper we consider the conductivity of uniaxial media possessing natural anisotropy, and containing dielectric or perfectly conducting inclusions. We find here, just as Dreizin and Dykhne found in their investigations,[2,3] that in a number of cases the expansion parameter at low inclusion concentrations c is not $c \ll 1$, but the quantity $c\gamma$ (where $\gamma \gg 1$ characterizes the anisotropy of the matrix), i.e., such anomaly is a general property of highly anisotropic media. We shall elucidate the cause of the appearance of the parameter $c\gamma$ in the particular case of the filamentary crystals ($\sigma_x = \sigma_y \ll \sigma_z$) containing dielectric inclusions. Let the mean current be directed along the z axis. Then the flux perturbation that arises during the flow around an obstacle propagates along the z axis over a distance not $\sim R$ (R is the radius of an inclusion), as in the isotropic case, but $\sim \gamma R \gg R$; $\gamma = (\sigma_z/\sigma_x)^{1/2} \gg 1$. The "stagnation" region has an anomalously large volume $\sim \gamma R^3$, and makes to the resistance a contribution $\sim c\gamma$. For $c\gamma \gtrsim 1$ the expression for the effective conductivity is negative, which indicates the inapplicability in this region of the approximation linear in c. Indeed, the inclusion closest to the one in question in the direction of the z axis is at a distance of $\sim R/c$. For $\gamma R \sim R/c$ the flux perturbation regions for these two inclusions overlap, and the approximation based on the consideration of the isolated inclusion is inapplicable. Consequently, in the case of the longitudinal conductivity σ_{zz} the approximation linear in c is valid when $c\gamma \ll 1$. For the opposite limiting case $c\gamma \gg 1$, a qualitative, order-of-magnitude analysis has been performed with the aid of the diffusion analogy.[3] The longitudinal conductivity σ_{zz} has a much lower value in this concentration region ($\gamma^{-1} \ll c \ll 1$). If, on the other hand, the inclusions are perfectly conducting, then the transverse conductivity σ_{xx} increases sharply on going from $c\gamma \ll 1$ to $c\gamma \gg 1$. In both cases there occurs a significant isotropization of the properties of the medium: the anisotropy of the system as a whole is significantly weaker than that of the matrix.

This isotropization increases as the inclusion concentration is increased further. The analysis of the region of concentrations close to the critical concentration with the aid of the ideas of percolation theory[11] allows us to conclude that the isotropization of the system in the critical region will, apparently, be practically total. The properties of randomly inhomogeneous anisotropic media in the vicinity of the metal–dielectric phase transition are in many respects similar to the properties of isotropic systems. For example, they have the same critical exponents, which are largely determined by a purely geometric factor: the randomly inhomogeneous isotropic distribution of the components (in particular, the topology of the infinite cluster).[11] Let us note that highly anisotropic layered crystals (i.e., those for which $\sigma_x = \sigma_y \gg \sigma_z$) undergo another metal–dielectric phase transition: at the "two-dimensional" critical concentration $c = \frac{1}{2}$.

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Connected with the characteristics of the flow of current in an inhomogeneous sample with $c\gamma \gg 1$ are certain size effects that occur in highly anisotropic media. In the case of filamentary crystals with dielectric inclusions the effective longitudinal conductivity σ_{ze} of the sample exhibits a significant dependence on the sample thickness d in the direction of the z axis. When $d \lesssim d_c \sim R/c$ the quantity $\sigma_{ze} \sim \sigma_z$. For greater thicknesses $d \gtrsim d_c$ the longitudinal conductivity has much lower values: $\sigma_{ze} \sim \sigma_x c^{-2} \ll \sigma_z$. If, on the other hand, the inclusions are perfectly conducting, then the size effect exists for the effective transverse conductivity σ_{xe}: this conductivity increases sharply when we go over from $d \lesssim d_c$ to $d \gtrsim d_c$. Here, as above, d is the sample thickness in the z direction and $d_c \sim R/c$.

The investigation of the conductivity of two-component media with magnetic-field-induced anisotropy, i.e., the study of their galvanomagnetic properties, is complicated in the general case by the fact that the conductivity tensor has an antisymmetric part (Hall components), which must be considered separately. If, however, the Hall components do not depend on the coordinates, i.e., if they are the same for the matrix and the inclusions, then they practically drop out from the problem, and can be discarded in the computation of the symmetric part $\hat{\sigma}_{es}$ of the effective conductivity tensor,[4] so that the results of the present paper are applicable in the case of $\hat{\sigma}_{es}$. It is precisely this case that is considered in Refs. 2 and 3 for the problem of polycrystals in a strong magnetic field **H**. Therefore, the results of the present paper concerning highly anisotropic filamentary crystals with perfectly conducting inclusions of low concentration are entirely similar to the results of Refs. 2 and 3 pertaining to polycrystals as $H \to \infty$. The replacement of the special crystallites by perfectly conducting inclusions is unimportant in the order-of-magnitude estimates.

2. THE TWO-DIMENSIONAL CASE

Let us consider an anisotropic film (matrix) lying in the (x,y) plane. Let us orient the coordinate axes along the principal axes of the conductivity tensor $\hat{\sigma}_1$ of the matrix, and let us denote the corresponding principal values of $\hat{\sigma}_1$ by σ_{x1} and σ_{y1} ($\sigma_{x1} \leq \sigma_{y1}$). Finally, let the film contain circular inclusions with isotropic conductivity σ_2. To compute the effective conductivity tensor of such a system by means of the Landau–Lifshitz method,[1] we must know the electric field inside an inclusion when a homogeneous field $\mathbf{E}_0$ is prescribed in the region far from it. In the case of an inclusion of circular shape the field inside it has the form (Ref. 1; see problem No. 6 at the end of §13):

$$E_\alpha = \sigma_{\alpha 1}[\sigma_{\alpha 1} - n_\alpha(\sigma_{\alpha 1} - \sigma_2)]^{-1} E_{0\alpha}. \tag{1}$$

Here the n_α ($\alpha = x,y$) are the depolarization coefficients for an elliptical cylinder with semiaxes $a_x = R\sigma_{x1}^{-1/2}$ and $a_y = R\sigma_{y1}^{-1/2}$, where R is the inclusion radius. The quantities n_x and n_y have a simple form:

$$n_x = a_y/(a_x + a_y), \quad n_y = a_x/(a_x + a_y),$$

from which after substituting the values of a_x and a_y we obtain

$$n_x = \sqrt{\sigma_{x1}}/(\sqrt{\sigma_{x1}} + \sqrt{\sigma_{y1}}), \quad n_y = \sqrt{\sigma_{y1}}/(\sqrt{\sigma_{x1}} + \sqrt{\sigma_{y1}}). \tag{2}$$

The computation of the effective conductivity tensor $\hat{\sigma}_e$ of the system under consideration is carried out in the same way as is done in Ref. 1. As a result, we obtain for the principal values of $\hat{\sigma}_e$ the expression ($\alpha = x,y$)

$$\sigma_{\alpha e} = \sigma_{\alpha 1}\left[1 - c\frac{\sigma_{\alpha 1} - \sigma_2}{\sigma_{\alpha 1} - n_\alpha(\sigma_{\alpha 1} - \sigma_2)}\right], \tag{3}$$

where $c \ll 1$ is the dimensionless inclusion concentration: $c = n\pi R^2$, n being the number of inclusions per unit area. The expression (3), together with (2), gives the solution to the formulated problem in the approximation linear in c. In the case of an isotropic medium $\sigma_{\alpha 1} = \sigma_1$, $\sigma_{\alpha e} = \sigma_e$, and $n_x = n_y = \frac{1}{2}$, and from (3) we obtain the usual result

$$\sigma_e = \sigma_1[1 - 2c(\sigma_1 - \sigma_2)/(\sigma_1 + \sigma_2)]. \tag{4}$$

For dielectric (d) inclusions we find from (4) that

$$\sigma_e^{(d)} = \sigma_1(1 - 2c). \tag{5}$$

The corresponding expression for the case of perfectly conducting (s) inclusions (i.e., for $\sigma_e^{(s)}$) is obtained from (5) by changing the sign in front of 2c. For a highly anisotropic film with dielectric inclusions ($\sigma_2 \ll \sigma_{x1} \ll \sigma_{y1}$) we find from (3) and (2) that

$$\text{a) } \sigma_{xe}^{(d)} \approx \sigma_{x1}(1-c); \quad \text{b) } \sigma_{ye}^{(d)} \approx \sigma_{y1}(1 - c\gamma); \quad \gamma = (\sigma_{y1}/\sigma_{x1})^{1/2} \gg 1. \tag{6}$$

Notice that, according to (6b), the quantity σ_{ye} vanishes when $c\gamma = 1$, and is negative when $c\gamma > 1$. It is clear that the expression (6b) is inapplicable in the region $c\gamma \sim 1$.

In order to elucidate the meaning of the results (6), let us consider the pattern of flow of a uniform current around a dielectric obstacle. In the case of an isotropic matrix the perturbations introduced by an inclusion of radius R into the flow propagate over distances $\sim R$. The effective area that is then excluded from participating in the conduction (the stagnation region) is, according to (5), equal to twice the area of the inclusion. In the anisotropic case, when the current is directed along the x axis ($\sigma_{x1} \ll \sigma_{y1}$), the perturbations along this axis propagate over small distances $\sim R/\gamma \ll R$; the effective area is roughly equal to the area of the inclusion [see (6a)]. On the other hand, if the current is directed along the y axis, then the flux perturbations along y propagate over long distances $\sim R\gamma \gg R$, and the effective area, which is of the order of γR^2, is anomalously large, so that the contribution to the resistance is equal to $c\gamma$ [see (6b)].

In deriving the expression (3), we made essential use of the "single-particle" character of the problem, i.e., the fact that the contribution to the resistance of any inclusion is independent of the contributions of all the remaining inclusions. This is valid if the regions of perturbed fluxes from different inclusions do not overlap. In the case of an isotropic matrix, this condition will be fulfilled if the inclusion radius R is small compared to the mean inclusion spacing $\sim Rc^{-1/2}$. Therefore, the condition for the expression (5) to be applicable is that the concentration should be low: $c \ll 1$. These arguments are valid also for an anisotropic ma-

trix with current flowing along the x axis.

If the current is directed along the y axis, then, as has already been noted, the flux perturbation is strongly extended along this axis over a distance $\sim\gamma R$. For the expression (6b) to be applicable, the quantity γR should be small compared to the mean distance $\langle L\rangle$ between inclusions lying closest to each other in the y direction. (The distance between the centers of these inclusions in the direction of the x axis should not exceed a quantity $\sim R$.) To determine $\langle L\rangle$, let us draw a straight line of length $\mathcal{L}$ along the y axis. This line will intersect a certain number of inclusions along chords, whose lengths we denote by l_i $(i=1,\ldots,N; N$ being the number of intersected inclusions). Then, according to Ref. 5, as $\mathcal{L}\to\infty$

$$c=\mathcal{L}^{-1}\sum_i l_i=\mathcal{L}^{-1}N\langle l\rangle. \tag{7}$$

Here c is the fraction of the area occupied by the inclusions, i.e., their concentration and $\langle l\rangle$ is the mean chord length. From (7) we determine the quantity $\langle L\rangle$, noting that as $\mathcal{L}\to\infty$ we have the equality $\langle L\rangle+\langle l\rangle=\mathcal{L}/N$. From this and (7) we obtain

$$\langle L\rangle=\langle l\rangle(1-c)/c. \tag{8}$$

For a circle of radius R, $\langle l\rangle=\pi R/2$ (see Ref. 5), so that for $c\ll 1$ we finally obtain from (8) the expression

$$\langle L\rangle\approx{}^1/_2\pi R/c. \tag{9}$$

As noted above, the condition of applicability of the formula (6b) is $\gamma R\ll\langle L\rangle$, or, with allowance made for (9), $c\gamma\ll 1$. Thus, in the case of a highly anisotropic matrix (i.e., for $\gamma\gg 1$) the expansion parameter for σ_{ye} is the quantity $c\gamma$, and the expression (6b) is inapplicable even at relatively low concentrations $c\sim\gamma^{-1}\ll 1$. Nevertheless, the formula (6b) indicates that we should expect a significantly reduced σ_{ye} value when $c\sim\gamma^{-1}$.

Let us use the diffusion analogy[3] to investigate the case $c\gamma\gg 1$. The gist of this analogy is that the equation for the direct current and the steady-state diffusion equation essentially coincide: the first equation goes over into the second when the conductivity is replaced by the diffusion coefficient and the electric potential is replaced by the density of the duffusing particles. At the same time, the diffusion phenomenon is more visualizable, since we can analyze it by following the behavior of a single particle. Furthermore, we can determine the effective diffusion coefficient by considering the time picture if we use the well-known formula

$$\overline{x_\alpha^2}\sim D_\alpha t. \tag{10}$$

Here $\overline{x_\alpha^2}$ is the mean squared displacement of a particle along the x_α coordinate axis during the time t and D_α is the corresponding principal value of the diffusion tensor. The transition to the conductivity problem is effected by making the substitution $D_\alpha\to\sigma_\alpha$ in the final formulas. Below we shall, for brevity, also make this substitution in the intermediate calculations.

As an example of the application of the diffusion analogy, let us estimate the dimension of the perturbed-current region during the flow along the y axis. In order for the current to bypass an obstacle of radius R, a particle should move along the x axis through a distance $\sim R$, which will require a time $t\sim R^2/\sigma_{x1}$. The diffusion of the particle should begin at a distance $y\sim(\sigma_{y1}t)^{1/2}$ from the inclusion, whence we obtain the dimension of the perturbed-current region along the y axis:

$$y\sim(\sigma_{y1}t)^{1/2}\sim R(\sigma_{y1}/\sigma_{x1})^{1/2}=R\gamma.$$

Let us now consider the diffusion of the particle in the case when the mean current is directed along the y axis and $c\gamma\gg 1$. In the zeroth approximation, when the displacement along the x axis can be neglected, the particle executes one-dimensional diffusion along the y axis between two nearest (in the y direction) inclusions. Consequently, for $\gamma=\infty$, every particle of the diffusing material is "locked" inside a region of length $\sim\langle L\rangle$ (along y) and width $\sim R$ (along x). The transfer of the particles through the sample then does not on the whole occur, and the effective diffusion coefficient (and, consequently, σ_{ye}) is equal to zero. (In the language of the conductivity problem, we have the following picture. For $\sigma_{x1}=0$ the current lines are straight lines parallel to the y axis. In a sample of infinite dimensions, any such current line will, with probability equal to unity, "abut" against one of the inclusions, whose distribution we assume to be random; from this it follows that $\sigma_{ye}=0$.) When γ is large but finite, there occurs, besides the fast motion along the y axis, slow diffusion in the transverse direction (along the x axis). As the particle diffuses along the x axis through a distance $\sim R$, it rapidly moves along the y axis through a distance $\sim\langle L\rangle$, and then gets locked in the next region.

Let us estimate the lifetime of the particle in such a region with the aid of (10):

$$\tau_d\sim\overline{x^2}/\sigma_{x1}\sim R^2/\sigma_{x1}. \tag{11}$$

Thus, the motion of the particle is a random walk (on the average, along the y axis) with step $\sim\langle L\rangle$ and hopping time $\sim\tau_d$. The diffusion coefficient, which coincides with σ_{ye}, for such a random walk can be estimated with the aid of a formula similar to (10):

$$\sigma_{ye}^{(d)}\sim\langle L\rangle^2/\tau_d. \tag{12}$$

The substitution into (12) of the expressions (9) and (11) leads to the following order-of-magnitude estimate for the effective conductivity σ_{ye}:

$$\sigma_{ye}^{(d)}\sim\sigma_{x1}/c^2,\quad \gamma^{-1}\ll c\ll 1. \tag{13}$$

The formula (13) is valid up to a numerical factor. Let us note that, for σ_{xe} in the concentration region $\gamma^{-1}\ll c\ll 1$, we have, as before, the formula (6a), i.e., $\sigma_{xe}\approx\sigma_{x1}$.

Comparison of the expressions (6b) and (13) shows that the value of σ_{ye} decreases by a factor of $(c\gamma)^2\gg 1$ when we go over from $c\ll\gamma^{-1}$ to $c\gg\gamma^{-1}$. There occurs therewith a sharp decrease in the anisotropy of the conductivity of the system as a whole, i.e., the increase in the concentration of the dielectric inclusions leads to a significant isotropization of the properties of the medium. Notice that the expression (13) can be rewritten in the form $\sigma_{ye}\sim\sigma_{y1}/(c\gamma)^2$. This fact and (6b)

allow us to postulate that the conductivity σ_{ye} for arbitrary $c\gamma$ (but when $c \ll 1$ and $\gamma \gg 1$) can be written in the form

$$\sigma_{ye}^{(d)} \approx \sigma_{y1}\psi_{yd}(c\gamma), \tag{14}$$

where the function $\psi_{yd}(\xi)$ has the following asymptotic forms: $\psi_{yd}(\xi) \approx 1 - \xi$ for $\xi \ll 1$ and $\psi_{yd}(\xi) \sim \xi^{-2}$ for $\xi \gg 1$.

Let us now consider an anisotropic film with perfectly conducting inclusions. For $\sigma_2 \to \infty$ and $\gamma \gg 1$, we find from (3) and (2) that

$$\text{a) } \sigma_{xe}^{(s)} \approx \sigma_{x1}(1+c\gamma); \quad \text{b) } \sigma_{ye}^{(s)} \approx \sigma_{y1}(1+c), \tag{15}$$

where γ is the same as in (6). In this case the correction to the quantity σ_{xe} is large, which is connected with the characteristics of the flow of the current in the direction of the x axis in the vicinity of a perfectly conducting inclusion in the anisotropic matrix. The perturbed-current region has a dimension $\sim\gamma R$ along y and $\sim R$ along x. Inside this region the current lines are directed largely along the "easy" (y) axis, along which the resistance is low, so that, for the quantity σ_{xe}, the whole region plays the role of a "perfectly conducting inclusion" with area $\sim\gamma R^2$. The condition of applicability of (15a) is $\gamma R \ll \langle L\rangle$, i.e., $c\gamma \ll 1$.

As the parameter $c\gamma$ increases, the perturbed-current regions stretch out along the y axis, overlapping when $c\gamma \sim 1$. As a result, the current distribution in the sample is highly inhomogeneous when $c\gamma \gg 1$. The current lines are directed along the applied electric field (along the x axis) only inside the inclusions; outside the inclusions the current flows largely parallel to the y axis. In the language of the diffusion problem we have the following picture (the corresponding arguments are similar to those adduced in Ref. 3). A particle, on reaching an inclusion, is instantaneously displaced along the x axis through a distance $\sim R$. After this, it is "advantageous" (when $c\gamma \gg 1$) for it to diffuse along the y axis through a distance $\sim\langle L\rangle$ until it encounters the next inclusion, after which a jump again occurs along the x axis, and so on. Thus, in this case the particle executes a random walk (on the average, along the x axis) with step $\sim R$ and hopping time $\tau_s \sim \langle L\rangle^2/\sigma_{y1}$. Estimating the effective diffusion coefficient from the formula $\sigma_{xe} \sim R^2/\tau_s$, we finally obtain for the quantity σ_{xe} the estimate

$$\sigma_{xe}^{(s)} \sim \sigma_{y1}c^2 = \sigma_{x1}(c\gamma)^2, \quad \gamma^{-1} \ll c \ll 1. \tag{16}$$

For the conductivity $\sigma_{ye}^{(s)}$ in the concentration region under consideration we have, as before, the formula (15b), i.e., $\sigma_{ye}^{(s)} \approx \sigma_{y1}$. Notice that, as in the case of dielectric inclusions, the transition from $c \ll \gamma^{-1}$ to $c \gg \gamma^{-1}$ is accompanied by substantial isotropization of the properties of the system as a whole (in the present case as a result of a sharp increase in the quantity σ_{xe}). We can also conjecture the form of σ_{xe} for an arbitrary value of the parameter $c\gamma$ ($c \ll 1$, $\gamma \gg 1$) in the case of perfectly conducting inclusions:

$$\sigma_{xe}^{(s)} \approx \sigma_{x1}\psi_{xs}(c\gamma), \tag{17}$$

where $\psi_{xs}(\xi) \approx 1 + \xi$ for $\xi \ll 1$ and $\psi_{xs}(\xi) \sim \xi^2$ for $\xi \gg 1$.

In conclusion of this section, let us consider a simple model with artificial anisotropy: an isotropic film of conductivity σ_1 with a set of impermeable scratches of length $2h$, oriented parallel to the y axis, and disposed in such a way that their centers are randomly distributed. The effective conductivity σ_{xe} in the case when the length of the scratches is small compared to the mean distance between their centers ($\sim n^{-1/2}$, n is the number of scratches per unit area of the film) can easily be found (see, for example, Ref. 6):

$$\sigma_{xe} = \sigma_1(1 - \pi n h^2), \quad nh^2 \ll 1. \tag{18}$$

In the opposite limiting case, i.e., for $nh^2 \gg 1$, we can use the diffusion analogy to estimate σ_{xe}. Let us denote the mean distance between the scratches along the x axis by $\langle L\rangle$. The diffusing particle will execute in the x direction a random walk with step of length $\sim\langle L\rangle$ and hopping time τ. Between two successive hops the particle should be displaced along the y axis through a distance $\sim h$, from which we determine τ: $\tau \sim h^2/\sigma_1$. The quantity $\langle L\rangle$ can also be determined with the aid of (8), where $\langle l\rangle$ should be regarded as denoting the mean chord length in the direction parallel to the x axis, for an ellipse with semiaxes $a \to 0$ (along x) and h (along y). It is not difficult to see that $\langle l\rangle = \pi a/2$; therefore, allowing for the equality $c = n\pi ah$, we find from (8) that $\langle L\rangle \approx (2nh)^{-1}$. As a result we obtain for the effective conductivity σ_{xe}

$$\sigma_{xe} \sim \sigma_1/(nh^2)^2, \quad nh^2 \gg 1. \tag{19}$$

When the current flows along the y axis the scratches do not offer resistance, so that $\sigma_{ye} = \sigma_1$ in the considered model.

3. RECIPROCAL RELATIONS

We can establish for a two-dimensional anisotropic medium reciprocal relations that generalize the ones found earlier by Dykhne[7] and the present author.[8] Let us consider a system for which the directions of the principal axes (x and y) of the conductivity tensor $\hat{\sigma}(\mathbf{r})$ do not depend on the coordinates (such a situation obtains, for example, for a thin isotropic film located in a magnetic field parallel to its plane), i.e., $\hat{\sigma}(\mathbf{r})$ has a diagonal form for any $\mathbf{r}$:

$$\hat{\sigma}(\mathbf{r}) = \begin{pmatrix} \sigma_x(\mathbf{r}) & 0 \\ 0 & \sigma_y(\mathbf{r}) \end{pmatrix}. \tag{20}$$

Let us, following Refs. 7 and 8, express the current density $\mathbf{j}$ and the electric field $\mathbf{E}$ relative to the primed system:

$$\mathbf{j} = \mu[\mathbf{n}\mathbf{E}'], \quad \mathbf{E} = \mu^{-1}[\mathbf{n}\mathbf{j}']. \tag{21}$$

Here $\mathbf{n}$ is the unit vector along the normal to the plane (x, y) of the system and μ is some coordinate-independent constant. The transformation (21) does not lead to a change in the system of equations for the direct current, and the conductivity tensor in the primed system has the form (20) with

$$\sigma_x'(\mathbf{r}) = \mu^2/\sigma_y(\mathbf{r}), \quad \sigma_y'(\mathbf{r}) = \mu^2/\sigma_x(\mathbf{r}). \tag{22}$$

The effective characteristics of the original (σ_{xe}, σ_{ye}) and primed ($\tilde{\sigma}_{xe}, \tilde{\sigma}_{ye}$) systems are connected by the reciprocity relations

$$\sigma_{xe}\tilde{\sigma}_{ye} = \mu^2, \quad \tilde{\sigma}_{xe}\sigma_{ye} = \mu^2. \tag{23}$$

Let us note that the relations (23) are valid also for the continuous problem, in which $\hat{\sigma}(\mathbf{r})$ is a continuous function of the coordinates.

For two-component media the quantities $\sigma_{\alpha e}$ are many-parameter functions:

$$\sigma_{\alpha e}=\sigma_{\alpha e}(p;\ \sigma_{x1},\ \sigma_{y1};\ \sigma_{x2},\ \sigma_{y2}),\quad \alpha=x,\ y, \tag{24}$$

where p is the concentration of the first component and σ_{xi} and σ_{yi} are the principal values of the conductivity tensor of the i-th component. The effective characteristics of the primed system are given by the same functions (24) with the arguments changed according to (22):

$$\delta_{\alpha e}=\sigma_{\alpha e}(p;\ \mu^2/\sigma_{y1},\ \mu^2/\sigma_{x1};\ \mu^2/\sigma_{y2},\ \mu^2/\sigma_{x2}). \tag{25}$$

If we select a dimensional factor, say σ_{x1}, in (24), the quantity $\sigma_{\alpha e}/\sigma_{x1}$ will be a function of four dimensionless parameters:

$$\sigma_{\alpha e}=\sigma_{x1}F_\alpha(p;\ \sigma_{y1}/\sigma_{x1},\ \sigma_{x2}/\sigma_{x1},\ \sigma_{y2}/\sigma_{x1}). \tag{26}$$

It follows from (23)–(26) that the parameter μ virtually drops out from the reciprocity relations. It is nevertheless convenient to fix it. If we set $\mu^2=\sigma_{x1}\sigma_{y1}$, then the reciprocity relations assume the form

$$\begin{aligned}&\sigma_{xe}(p;\ \sigma_{x1},\ \sigma_{y1};\ \sigma_{x2},\ \sigma_{y2})\sigma_{ye}(p;\ \sigma_{x1},\ \sigma_{y1};\ \sigma_{x1}\sigma_{y1}/\sigma_{y2},\ \sigma_{x1}\sigma_{y1}/\sigma_{x2})=\sigma_{x1}\sigma_{y1},\\ &\sigma_{ye}(p;\ \sigma_{x1},\ \sigma_{y1};\ \sigma_{x2},\ \sigma_{y2})\sigma_{xe}(p;\ \sigma_{x1},\ \sigma_{y1};\ \sigma_{x1}\sigma_{y1}/\sigma_{y2},\ \sigma_{x1}\sigma_{y1}/\sigma_{x2})=\sigma_{x1}\sigma_{y1}.\end{aligned} \tag{27}$$

In particular, for $\hat{\sigma}_2\to 0$, we obtain from (27) a relation between the effective characteristics of a two-dimensional system with dielectric inclusions and the properties of the same system with perfectly conducting inclusions:

$$\begin{aligned}&\sigma_{xe}^{(d)}(p;\ \sigma_{x1},\ \sigma_{y1})\sigma_{ye}^{(*)}(p;\ \sigma_{x1},\ \sigma_{y1})=\sigma_{x1}\sigma_{y1},\\ &\sigma_{ye}^{(d)}(p;\ \sigma_{x1},\ \sigma_{y1})\sigma_{xe}^{(*)}(p;\ \sigma_{x1},\ \sigma_{y1})=\sigma_{x1}\sigma_{y1}.\end{aligned} \tag{28}$$

The substitution into the second relation in (28) of the expressions (14) and (17) allow us to find a relation between the functions ψ_{xs} and δ_{yd}:

$$\psi_{xs}\psi_{yd}=1. \tag{29}$$

If the distribution of the components in the system is geometrically isotropic, then the substitution $\sigma_{xi}\rightleftarrows\sigma_{yi}$ with a simultaneous rotation of the coordinate axes through 90° does not alter the properties of the medium, i.e.,

$$\begin{aligned}&\sigma_{xe}(p;\ \sigma_{x1},\ \sigma_{y1};\ \sigma_{x2},\ \sigma_{y2})=\sigma_{ye}(p;\ \sigma_{y1},\ \sigma_{x1};\ \sigma_{y2},\ \sigma_{x2}),\\ &\sigma_{ye}(p;\ \sigma_{x1},\ \sigma_{y1};\ \sigma_{x2},\ \sigma_{y2})=\sigma_{xe}(p;\ \sigma_{y1},\ \sigma_{x1};\ \sigma_{y2},\ \sigma_{x2}).\end{aligned} \tag{30}$$

It is not difficult to see that, in the approximation linear in c, the expressions (3) and (2) satisfy the relations (27) and (30). Finally, let us not that the properties of a randomly inhomogeneous medium are not changed by the double substitution $p\to 1-p$, $\hat{\sigma}_1\rightleftarrows\hat{\sigma}_2$, whence ($\alpha=x,y$)

$$\sigma_{\alpha e}(p;\ \sigma_{x1},\ \sigma_{y1};\ \sigma_{x2},\ \sigma_{y2})=\sigma_{\alpha e}(1-p;\ \sigma_{x2},\ \sigma_{y2};\ \sigma_{x1},\ \sigma_{y1}). \tag{31}$$

The relation (31), written in the form $\sigma_{\alpha e}(p;\hat{\sigma}_1,\hat{\sigma}_2)=\sigma_{\alpha e}(1-p;\hat{\sigma}_2,\hat{\sigma}_1)$ (where $\alpha=x,y,z$), is valid in the three-dimensional case as well.

4. THE THREE-DIMENSIONAL CASE

For three-dimensional anisotropic media with low concentrations of spherical inclusions with isotropic conductivity σ_2, the principal values of the tensor $\hat{\sigma}_e$ are also given by the formula (3), in which the n_α ($\alpha=x,y,z$) are the depolarization coefficients for an ellipsoid with semiaxes $a_\alpha=R\sigma_\alpha^{-1/2}$, where R is the inclusion radius. For an isotropic medium $\sigma_{\alpha 1}=\sigma_1$, $\sigma_{\alpha e}=\sigma_e$, and $n_\alpha=\frac{1}{3}$, and from (3) we obtain the well-known result[1]

$$\sigma_e=\sigma_1[1-3c(\sigma_1-\sigma_2)/(2\sigma_1+\sigma_2)]. \tag{32}$$

For systems with dielectric and perfectly-conducting inclusions, we respectively have from (3) the expressions

$$\text{a)}\ \sigma_{\alpha e}^{(d)}=\sigma_{\alpha 1}[1-c/(1-n_\alpha)];\quad \text{b)}\ \sigma_{\alpha e}^{(*)}=\sigma_{\alpha 1}(1+c/n_\alpha). \tag{33}$$

Below we shall consider a uniaxial matrix ($\sigma_{x1}=\sigma_{y1}\neq\sigma_{z1}$), and distinguish two cases: 1) a filamentary structure of the TCNQ type ($\sigma_{x1}=\sigma_{y1}<\sigma_{z1}$); 2) a layered structure of the graphite type ($\sigma_{x1}=\sigma_{y1}>\sigma_{z1}$).

1. Let us first consider filamentary crystals ($\sigma_{x1}=\sigma_{y1}<\sigma_{z1}$). For the depolarization coefficients this case corresponds to an oblate ellipsoid of revolution ($a_x=a_y>a_z$), for which[1]

$$\begin{aligned}&n_z=(1+\varepsilon^2)(\varepsilon-\operatorname{arctg}\varepsilon)/\varepsilon^3,\quad n_x={}^1/_2(1-n_z);\\ &\varepsilon=[(a_x/a_z)^2-1]^{1/2}=[(\sigma_{z1}/\sigma_{x1})-1]^{1/2}.\end{aligned} \tag{34}$$

In the case of strong anisotropy, i.e., for $(\sigma_{z1}/\sigma_{x1})^{1/2}\gg 1$, we find from (34) that

$$n_z\approx 1-{}^1/_2\pi(\sigma_{x1}/\sigma_{z1})^{1/2},\quad n_x\approx{}^1/_4\pi(\sigma_{x1}/\sigma_{z1})^{1/2}. \tag{35}$$

so that for dielectric inclusions we obtain from (33a) the estimates

$$\text{a)}\ \sigma_{xe}^{(d)}\approx\sigma_{x1}(1-c);\quad \text{b)}\ \sigma_{xe}^{(d)}\approx\sigma_{x1}(1-2c\gamma/\pi);\quad \gamma=(\sigma_{z1}/\sigma_{x1})^{1/2}\gg 1. \tag{36}$$

Here and below we retain the designation γ for the anisotropy characteristic $(\sigma_{z1}/\sigma_{x1})^{1/2}$ of the matrix.

The expressions (36) are similar to the formulas (6), which is connected with the similarity of the patterns of flow around the obstacles in the two cases. Thus, in the case of flow along the z axis the perturbed-current region (the stagnation region) has the shape of a prolate ellipsoid of revolution with semiaxes $\sim\gamma R\gg R$ along the z axis and $\sim R$ in the transverse direction. The distance $\langle L\rangle$ to the nearest (in the direction of the z axis) obstacle is determined with the aid of (8), where now c is the three-dimensional dimensionless concentration of the inclusions and $\langle l\rangle$ is the mean secant length. For a sphere $\langle l\rangle=\frac{4}{3}R$ (Ref. 5), so that at low concentrations $\langle L\rangle\approx\frac{4}{3}R/c$. Hence we obtain the condition of applicability of (36b): $c\gamma\ll 1$. The opposite case $c\gamma\gg 1$ is considered in exactly the same way as in Sec. 2. As a result, we arrive at a formula similar to (13):

$$\sigma_{xe}^{(d)}\sim\sigma_{x1}/c^2=\sigma_{z1}/(c\gamma)^2,\quad \gamma^{-1}\ll c\ll 1. \tag{37}$$

For arbitrary $c\gamma$ ($c\ll 1$, $\gamma\gg 1$) we can likewise put forward a hypothesis of the type (14) concerning the form of σ_{xe}:

$$\sigma_{xe}^{(d)}\approx\sigma_{x1}\Psi_{xd}(c\gamma), \tag{38}$$

where $\Psi_{xd}(\xi)\approx 1-2\xi/\pi$ for $\xi\ll 1$ and $\Psi_{xd}(\xi)\sim\xi^{-2}$ for $\xi\gg 1$.

For perfectly conducting inclusions we have from (33b) and (35) the estimates

a) $\sigma_{xe}^{(s)} \approx \sigma_{x1}(1+4c\gamma/\pi)$; b) $\sigma_{xe}^{(s)} \approx \sigma_{x1}(1+c)$; $c \ll \gamma^{-1} \ll 1$. (39)

The expression (39a) coincides with Korzh's result.[4] For $c\gamma \gg 1$ the diffusion analogy leads to an expression of the type (16) (Refs. 2 and 3):

$$\sigma_{xe}^{(s)} \sim \sigma_{z1}c^2 = \sigma_{x1}(c\gamma)^2; \quad \gamma^{-1} \ll c \ll 1. \tag{40}$$

For arbitrary $c\gamma$ we can make an assumption similar to (17):

$$\sigma_{xe}^{(s)} \approx \sigma_{x1}\Psi_{xs}(c\gamma), \tag{41}$$

where $\Psi_{xs}(\xi) \approx 1 + 4\xi/\pi$ for $\xi \ll 1$ and $\Psi_{xs}(\xi) \sim \xi^2$ for $\xi \gg 1$.

Thus, the properties of filamentary crystals turn out to be similar to those of highly anisotropic thin films. In particular, the properties of filamentary crystals also become increasingly isotropic as the inclusion concentration increases.

2. In the case of layered crystals ($\sigma_{x1} = \sigma_{y1} > \sigma_{z1}$) the depolarization coefficients correspond to a prolate ellipsoid of revolution[1]:

$$n_z = {}^1/_2(1-\varepsilon^2)\left(\ln\frac{1+\varepsilon}{1-\varepsilon} - 2\varepsilon\right)\Big/\varepsilon^3, \quad n_x = {}^1/_2(1-n_z);$$
$$\varepsilon = [1-(a_z/a_x)^2]^{1/2} = [1-(\sigma_{z1}/\sigma_{x1})]^{1/2}. \tag{42}$$

In the case of strong anisotropy we have from (42) the estimates

$$n_z \approx \gamma^2[\ln(2/\gamma) - 1], \quad n_x \approx {}^1/_2; \quad \gamma = (\sigma_{z1}/\sigma_{x1})^{1/2} \ll 1. \tag{43}$$

For dielectric inclusions we find from (33a) and (43) that

a) $\sigma_{xe}^{(d)} \approx \sigma_{x1}(1-2c)$; b) $\sigma_{ze}^{(d)} \approx \sigma_{z1}(1-c)$. (44)

In this case we do not have an anomalously large stagnation region associated with the flow around an obstacle in any direction, and the expressions (44) are valid for all $c \ll 1$.

For perfectly conducting inclusions we have from (33b) and (43) the estimates

a) $\sigma_{xe}^{(e)} \approx \sigma_{x1}(1+2c)$; b) $\sigma_{ze}^{(e)} \approx \sigma_{z1}[1+c\gamma^{-2}/\ln(2/e\gamma)]$, (45)

where e is the base of the natural logarithms. The cause of the anomalously large correction to σ_{ze} is the same as the one indicated above. The form of this correction is somewhat different, which is connected with the difference in the patterns of flow around a perfectly conducting inclusion in the cases of filamentary and layered crystals. The condition of applicability of (45b) is $c \ll \gamma^2\ln(1/\gamma)$.

Let us consider the case $c \gg \gamma^2\ln(1/\gamma)$ with the aid of the diffusion analogy. If the mean current is directed along the z axis, then a particle, on reaching an inclusion, is instantaneously displaced along the z axis through a distance $\sim R$. After this, it diffuses in the plane perpendicular to the z axis until it reaches the next inclusion. Then the particle again hops a distance $\sim R$ in the z direction and so on. Let us, for the purpose of estimating the time between two successive jumps, note that perfectly conducting inclusions correspond to absorbing traps in the diffusion problem. Such a problem is considered in Refs. 9 and 10, where the probability for "survival" of a particle is found. The mean lifetime of a particle gives an estimate for the time τ_0 between successive jumps. In the case of filamentary crystals the corresponding diffusion is one-dimensional (along the z axis), so that an estimate with the aid of Ref. 10 yeilds $\tau_0 \sim \langle L\rangle^2/\sigma_{z1}$, a relation which was used in the derivation of the formula (40).

In the case of layered crystals the particle executes two-dimensional diffusion in the plane perpendicular to the z axis. For the estimation of τ_0 in the two-dimensional case, it is sufficient to limit ourselves to the gas approximation,[10] so that we have the order-of-magnitude estimate.[9,10]

$$\tau_0 \sim (n_2\sigma_{z1})^{-1}\ln(1/n_2R^2), \tag{46}$$

where n_2 is the number of traps per unit area. To determine the quantity n_2, let us draw a plane of area S perpendicular to the z axis. It will intersect a certain number N_2 of inclusions; let us denote the corresponding intersection areas by s_i. According to Ref. 5, as $S \to \infty$ we have

$$c = S^{-1}\sum_i s_i = n_2\langle s\rangle, \quad n_2 = N_2/S. \tag{47}$$

Here c is the three-dimensional concentration of the inclusions (the fraction of volume occupied by them). The mean intersection area for a sphere is easy to find: $\langle s\rangle = \frac{2}{3}\pi R^2$ (see Ref. 5), and from (47) we obtain

$$n_2 \sim c/R^2,$$

so that (46) gives

$$\tau_0 \sim \frac{R^2}{\sigma_{x1}c}\ln\frac{1}{c}. \tag{48}$$

Estimating σ_{ze} in the usual manner ($\sigma_{ze} \sim R^2/\tau_0$), we finally obtain

$$\sigma_{zc}^{(e)} \sim \sigma_{x1}c/\ln(1/c). \tag{49}$$

The formulas (45b) and (49) can apparently be regarded as the two limiting cases of the general expression

$$\sigma_{xe}^{(s)} \approx \sigma_{x1}\Psi(\xi); \quad \xi = c\gamma^{-2}/\ln(1/\Gamma), \quad \Gamma = \Gamma(c,\gamma), \tag{50}$$

where $\Psi(\xi) \approx 1+\xi$, $\Gamma \approx \gamma e/2$ for $\xi \ll 1$ and $\Psi(\xi) \sim \xi$, $\Gamma \sim c$ for $\xi \gg 1$. Notice that in this case also the properties of the medium become increasingly isotropic as the concentration of the perfectly conducting inclusions increases.

5. THE CRITICAL REGION

Also of interest are the properties of an anisotropic inhomogeneous system with fairly high c, especially in the vicinity of the critical concentration, i.e., in the metal–dielectric phase transition region. Here, as in the standard percolation theory,[11] we shall consider a medium with a random, geometrically-isotropic component distribution. In this case, for systems with dielectric inclusions, the flow through the sample cases in every direction at the critical concentration.[11] This statement is a purely geometrical fact, and does not depend on the properties of the matrix. From this it follows that, in the anisotropic case, the critical concentration is the metal–dielectric phase transition point, i.e., the point at which all the components of the effective conductivity tensor $\hat{\sigma}_e$ vanish simultane-

ously.

To determine the form of $\hat{\sigma}_e$ in the critical region, let us first consider a two-dimensional system with dielectric inclusions. It is convenient to perform the subsequent analysis in the particular case of the lattice model (the bond problem on an anisotropic quadratic lattice). If the concentration of the nonconducting bonds tends to the critical value (i.e., if $\tau \ll 1$, where τ is the proximity to the transition point in terms of concentration), then we have for the effective conductivity of the system in the case of an isotropic matrix the estimate[11,12]

$$\sigma_e \sim \sigma_1 \tau^t, \tag{51}$$

where σ_1 is the conductivity of a single bond (the conductivity of the matrix in the continuous case) and t is the critical exponent. The dependence of the conductivity on the concentration in (51) is determined largely by a purely geometric factor: the topology of an infinite cluster,[11] which, by assumption, is isotropic. It therefore follows that the dependence of σ_{xe} and σ_{ye} on the concentration in the case of an anisotropic matrix also has the form (51) with the same critical exponent t. The dimensional factor in σ_{xe} and σ_{ye} is determined, when $\sigma_{y1} \gg \sigma_{x1}$, by the bond with the lowest conductivity, i.e., it is equal to σ_{x1}. Finally, if the bonds with σ_{x1} and σ_{y1} are encountered equally frequently along the majority of the flow paths (which appears to be natural when $\tau \ll 1$), then the numerical coefficients in σ_{xe} and σ_{ye} will also be identical, so that ($\sigma_{x1} \ll \sigma_{y1}$, $\tau \ll 1$):

$$\sigma_{xe} \approx \sigma_{ye} \sim \sigma_{x1} \tau^t \quad (\tau > 0). \tag{52}$$

On the basis of the general ideas of percolation theory,[11] the result (52) will obtain in the continuous problem as well. Thus, virtually complete isotropization of the properties of the systems under consideration is to be expected at $\tau \to 0$. Let us emphasize that the isotropization begins (here and below) just when $\tau \ll 1$, and not upon the fulfullment of a stronger condition, say, of the type $\tau \ll \gamma^{-1} \ll 1$. This is connected with the geometry of the flow paths in the critical region: in the present case, with the topology of an infinite cluster.

Now let the second component have a low, but finite conductivity: $\hat{\sigma}_2 \ll \hat{\sigma}_1$. In highly anisotropic media this implies the satisfaction of the following chain of inequalities: $\sigma_{x2} \ll \sigma_{y2} \ll \sigma_{x1} \ll \sigma_{y1}$. The reciprocal relations (23) allow us in this case, as in the case of an isotropic matrix,[7,8] to relate the properties of the "metallic" ($\tau > 0$) and "dielectric" ($\tau < 0$) phases. Let us, using (31), write the first of the relations (23) in the form

$$\sigma_{xe}(p;\ \sigma_{x1},\ \sigma_{y1};\ \sigma_{x2},\ \sigma_{y2})\,\sigma_{ye}(1-p;\ \mu^2/\sigma_{y2},\ \mu^2/\sigma_{x2};\ \mu^2/\sigma_{y1},\ \mu^2/\sigma_{x1}) = \mu^2.$$

Setting here $\mu^2 = \sigma_{x1}\sigma_{y2}$, and proceeding to the limit $\sigma_{y1} \to \infty$, $\sigma_{x2} \to 0$, we obtain

$$\sigma_{xe}(p)\,\sigma_{ye}(1-p) = \sigma_{x1}\sigma_{y2}, \quad \sigma_{ye}(p)\,\sigma_{xe}(1-p) = \sigma_{x1}\sigma_{y2}, \tag{53}$$

where, for brevity, we have set $\sigma_{\alpha e}(p) = \sigma_{\alpha e}(p;\ \sigma_{x1};\ \infty;\ 0,\ \sigma_{y2})$. Also given in (53) is a second relation, which differs from the first by the substitution $p \to 1-p$. The reciprocal relations in the form (53) are, according to the results of Sec. 2, valid provided the concentration of either component is not too low:

$$1-p \gg (\sigma_{x1}/\sigma_{y1})^{1/2} \ll 1 \text{ and } p \gg (\sigma_{x2}/\sigma_{y2})^{1/2} \ll 1.$$

When the conductivity of the second component is finite (but $\hat{\sigma}_2 \ll \hat{\sigma}_1$), the expressions (52) describe the properties of the system in the metallic phase ($\tau > 0$, $p > \frac{1}{2}$) in the region lying within the $\tau \ll 1$, but outside the smearing,[12] region. The substitution of (52) into (53) allows us to determine the properties of the system in the dielectric phase ($\tau < 0$) also outside the smearing region, but where $|\tau| \ll 1$:

$$\sigma_{xe} \approx \sigma_{ye} \sim \sigma_{y2} |\tau|^{-q}; \quad q = t \quad (\tau < 0). \tag{54}$$

The result (54) can be interpreted as follows. In the $\tau < 0$ region outside the smearing region the inclusions of the first component can be considered to be perfectly conducting. For $|\tau| \ll 1$ the current flows largely through the perfectly conducting regions (finite clusters), the dimension of which increases without restriction as $|\tau| \to 0$ (Ref. 11). It is "advantageous" for the current to get over the short cross-connecting jumpers between these regions in the direction of the y axis. To elucidate the nature of the isotropy of the properties of such a medium, let us note that, using (30), we can write the equality $\sigma_{xe} = \sigma_{ye}$ for $\{\sigma_{x1}, \sigma_{y1}\} \to -\infty$, $\sigma_{x2} \to 0$ in the ($p < \frac{1}{2}$)

$$\sigma_{\alpha e}(p;\ \infty,\ \infty;\ 0,\ \sigma_{y2}) = \sigma_{\alpha e}(p;\ \infty,\ \infty;\ \sigma_{y2},\ 0), \quad \alpha = x,\ y. \tag{55}$$

On the left-hand side of the equality (55) stands the conductivity of a system in which the finite clusters are connected by jumpers formed by only the vertical bonds, while the conductivity on the right is for a system in which the clusters are joined by jumpers formed by only horizontal bonds; in both cases the conductivity of the bonds is equal to σ_{y2}. Consequently, if such "switching" of the jumpers does not change the properties of the system, the system will be isotropic.

According to (52) and (54), the considered system in the vicinity of the transition point ($|\tau| \ll 1$) outside the smearing region is isotropic. It is natural to suppose that it is also isotropic inside the smearing region. Then it follows from (53) that at the critical point ($p = \frac{1}{2}$)

$$\sigma_{xe} \approx \sigma_{ye} \approx (\sigma_{x1}\sigma_{y2})^{1/2} \quad (\tau = 0). \tag{56}$$

The dimension Δ of the smearing region is determined in the usual fashion by "matching" the expression (56) with (52) or (54), which yields $\Delta^t \sim (\sigma_{y2}/\sigma_{x1})^{1/2}$. Thus, the properties of the considered anisotropic system in the critical region qualitatively coincide with the properties of an isotropic medium with conductivities of components σ_{x1} and σ_{y2} ($\sigma_{x1} \gg \sigma_{y2}$). At the same time we should expect the occurrence in, for example, the formula (52) (in comparison with the isotropic case) of a numerical coefficient ≈ 2, since nearly half of the bonds (with conductivity σ_{y1}) do not, when $\sigma_{x1} \ll \sigma_{y1}$, make any contribution to the resistance. Accordingly, there then arises in (54) a factor $\approx \frac{1}{2}$.

Arguing in exactly the same way as in the two-dimensional case (assuming, in particular, that the switching of the jumpers between the finite clusters does not change the properties of the system), we find in the case of filamentary crystals with $\sigma_{z1} \gg \sigma_{x1} \gg \sigma_{z2} \gg \sigma_{x2}$ that in the critical region $|\tau| \ll 1$

$$\begin{aligned}
&\text{a)}\ \sigma_{ze}\approx\sigma_{xe}\sim\sigma_{x1}\tau^{t};\quad \tau>0,\quad \Delta\ll\tau\ll 1;\\
&\text{b)}\ \sigma_{ze}\approx\sigma_{xe}\sim\sigma_{x1}(\sigma_{z2}/\sigma_{x1})^{s};\quad |\tau|\ll\Delta;\\
&\text{c)}\ \sigma_{ze}\approx\sigma_{xe}\sim\sigma_{z2}|\tau|^{-q};\quad \tau<0,\quad \Delta\ll|\tau|\ll 1.
\end{aligned}\tag{57}$$

Here the dimension of the smearing region is determined from the condition $\Delta^t \sim (\sigma_{z2}/\sigma_{x1})^s$ and the exponents t, s, and q are connected by exactly the same relation that obtains in the isotropic case[12]: $q = t(1-s)/s$. As follows from the arguments leading to (57a) and (57c), t and q are the same exponents that occur in the isotropic case. Since t, s, and q are connected by the same relation that obtains in the isotropic percolation theory, the critical exponent s coincides with the corresponding "isotropic" exponent. The case of layered crystals ($\sigma_{x1} \gg \sigma_{z1} \gg \sigma_{x2} \gg \sigma_{z2}$) can be considered in exactly the same fashion, and differs from (57) by the substitution $x \rightleftarrows z$.

The foregoing analysis [see (57)] pertained to the three-dimensional critical concentration region $p_c^{(3)} \approx 0.2$ (Ref. 11). It turns out that another metal–dielectric transition is possible in layered crystals: at the two-dimensional critical concentration $p_c^{(2)} = \frac{1}{2}$. Let the mean electric field be directed along the x axis. Then in the case of a sufficiently strong anisotropy, i.e., for $\sigma_{z1} \ll \sigma_{x1}$, the current distribution in the region of dielectric ($\hat{\sigma}_2 = 0$) component concentrations $c < \frac{1}{2}$ will be largely two-dimensional: in the planes perpendicular to the z axis. [Notice that this explains the similarity of the expressions (44a) and (5); according to (47), the fraction of the area occupied by the dielectric component in any plane is equal to the three-dimensional concentration c.] Therefore, as $p_c^{(2)}$ is approached, the quantity σ_{xe} will decrease according to the law $\sigma_{xe} \sim \sigma_{x1}(p - p_c^{(2)})^{t_2}$, where $t_2 \approx 1.3$ is the corresponding two-dimensional critical exponent.[11,12] For $p < p_c^{(2)}$ (but $p > p_c^{(3)}$), a purely two-dimensional percolation is impossible,[11] and the flow of the current has a three-dimensional character. In this case the resistance of the sample is determined by the lowest σ_{z1} plays a role similar to the role played by the conductivity of the "dielectric" component in the ordinary metal–dielectric transition.[12] It is natural to expect, therefore, that the conductivity σ_{xe} will have in the critical region $|\tau_2| \ll 1$ the form

$$\begin{aligned}
&\text{a)}\ \sigma_{xe}\sim\sigma_{x1}\tau_2^{t'};\quad \tau_2>0,\quad \Delta_2\ll\tau_2\ll 1;\\
&\text{b)}\ \sigma_{xe}\sim\sigma_{x1}(\sigma_{z1}/\sigma_{x1})^{s'};\quad |\tau_2|\ll\Delta_2;\\
&\text{c)}\ \sigma_{xe}\sim\sigma_{x1}|\tau_2|^{-q'};\quad \tau_2<0,\quad \Delta_2\ll|\tau_2|\ll 1.
\end{aligned}\tag{58}$$

Here $\tau_2 = (p - p_c^{(2)})/p_c^{(2)}$; the dimension Δ_2 of the smearing region is determined from the condition $\Delta_2^{t'} \sim (\sigma_{y1}/\sigma_{x1})^{s'}$. The critical exponents t', s', and q' are connected by the usual relation $q' = t'(1-s')/s'$, one of them being known: $t' = t_2 \approx 1.3$. Thus, there occurs at $p = p_c^{(2)}$ a metal–dielectric phase transition that is similar in form to the normal transition.[12] To the conductivity σ_{ze}, nothing apparently distinguishes the point $p = p_c^{(2)}$, and in this region of concentrations $\sigma_{ze} \lesssim \sigma_{z1} \ll \sigma_{xe}$.

6. THE SIZE EFFECTS

Connected with the characteristics, considered in Secs. 2 and 4, of the flow of current in highly anisotropic inhomogeneous media are distinctive size effects that manifest themselves in critical dependences of the the effective characteristics of a sample on its dimension in the direction of highest conductivity in the matrix.

Let us first consider the two-dimensional case. Let the film have a finite width d in the direction of the y axis ($\sigma_{y1} \gg \sigma_{x1}$). Let us assume that the inclusions are dielectric, and that the condition $c\gamma \gg 1$ is fulfilled. Then for $d \ll \langle L\rangle \sim R/c$ we have $\sigma_{ye} \approx \sigma_{y1}$ even when $c\gamma \gg 1$. On the other hand, if $d \gg R/c$, then the effective conductivity σ_{ye} will be the same as in an unbounded sample: $\sigma_{ye} \sim \sigma_{x1}/c^2 \ll \sigma_{y1}$. Thus, there exists a critical film width $d_c \sim \langle L\rangle \sim R/c$, such that the transition from $d \lesssim d_c$ to $d \gtrsim d_c$ is accompanied by a sharp decrease in the quantity σ_{ye}.

To determine the critical width d_c more accurately, we must solve a problem that it is natural to call the problem of flow along straight lines. Specifically, setting $\gamma = \infty$, we arrive at the problem of current flow along straight lines (along the y axis) through a sample of finite width. It is clear that the conductivity will be nonzero when $d \ll \langle L\rangle \sim R/c$ and zero when $d = \infty$. The conductivity will be sharply reduced when $d \sim \langle L\rangle$. It will be interesting to determine that value of the sample dimension (in the $\gamma = \infty$ case) at which σ_{ye} vanishes, specifically, to find out whether σ_{ye} vanishes at a finite $d = d_c \sim \langle L\rangle$, or as $d \to \infty$, i.e., whether the phase transition is sharp in terms of the film width or smeared.

If the inclusions are perfectly conducting, then the size effect exists (when $c\gamma \gg 1$) for the quantity σ_{xe}, and manifests itself in the dependence of σ_{xe} on the film width d along the y axis. If $d \ll \langle L\rangle$, then the mechanism, described in Sec. 2, of flow of current along the x axis cannot be realized, and $\sigma_{xe} \approx \sigma_{x1}$. If $d \gg \langle L\rangle$, then, as in an unbounded sample, $\sigma_{xe} \sim \sigma_{y1}c^2 \gg \sigma_{x1}$. The corresponding critical width is, as in the above-discussed case, of the order of $d_c \sim \langle L\rangle \sim R/c$. Thus, the effective conductivity σ_{xe} increases sharply when we go over from $d \lesssim d_c$ to $d \gtrsim d_c$.

Entirely similar size effects exist for filamentary crystals (when $c\gamma \gg 1$) in the dependence on the sample width in the direction of the z axis. The case of perfectly conducting inclusions is entirely similar to the problem, considered by Dreĭzin and Dykhne,[2,3] of the transverse conductivity of polycrystals in a strong magnetic field.

For layered media, the size effect is manifested at $c \ll 1$ only by the quantity σ_{ze} in the case of perfectly conducting inclusions. If the sample is a circular cylinder of radius r with the generatrix along the z axis, then the conductivity σ_{ze} increases sharply from $\sigma_{ze} \approx \sigma_{z1}$ to the value given by the expression (49) when we go over from $r \lesssim r_c$ ($r_c \sim \langle L\rangle \sim R/c$) to $r \gtrsim r_c$.

7. CONCLUSION

The range of problems considered in the present paper pertains to the simplest problems of the anisotropic percolation theory. The necessity of the investigation of such problems arises both in the case of nat-

ural highly anisotropic crystals and in the study of the galvanomagnetic properties of two-component media located in strong magnetic fields. For three-dimensional systems the latter problem is, in the general case, complicated by the presence of Hall components. In the two-dimensional case (thin films) this difficulty does not arise if the magnetic field is parallel to the plane of the system. Such a system with induced and controlled anisotropy can serve as a convenient object for the experimental investigation of anisotropic inhomogeneous media. The investigation of such a system is also of interest for the reason that its properties qualitatively coincide with the properties of three-dimensional filamentary crystals.

In considering the critical region (thin films), we made essential use of the assumption that bonds with σ_x and σ_y are, on the average, encountered equally frequently along the majority of the flow paths. If the results obtained in Sec. 5 are not confirmed in experiment, then this will indicate that the dominant contribution to the effective conductivity (even in the isotropic case) is made by the flow paths with the preferred orientation, and, what is more, their number should be fairly high. Thus, the study of highly anisotropy inhomogeneous media in the region $|\tau| \ll 1$ will enable us to verify (and, if necessary, refine) the current ideas about the flow pattern at $\tau \to 0$, and can provide additional information about the structure of the infinite cluster.[11]

Also of interest is the investigation of the size effects, in particular, the problem of flow along straight lines for a sample of finite thickness (width). Here we should first and foremost elucidate the question whether there exists a sharp phase transition in terms of thickness, and determine the corresponding critical exponents. Similar questions arise in the case of perfectly conducting inclusions.

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[1]L. D. Landau and E. M. Lifshitz, Élektrodinamika sploshnykh sred (Electrodynamics of Continuous Media), Fizmatgiz, Moscow, 1959 (Eng. Transl., Pergamon Press, Oxford, 1960), §§9 and 13.
[2]Yu. A. Dreĭzin and A. M. Dykhne, Pis'ma Zh. Eksp. Teor. Fiz. **14**, 101 (1971) [JETP Lett. **14**, 66 (1971)].
[3]Yu. A. Dreĭzin and A. M. Dykhne, Zh. Eksp. Teor. Fiz. **63**, 242 (1972) [Sov. Phys. JETP **36**, 127 (1973)].
[4]S. A. Korzh, Zh. Eksp. Teor. Fiz. **59**, 510 (1970) [Sov. Phys. JETP **32**, 280 (1971)].
[5]M. G. Kendall and P. A. P. Moran, Geometrical Probability, Griffin, London, 1963 (Russ. Transl., Nauka, Moscow, 1972).
[6]B. Ya. Balagurov, Zh. Tekh. Fiz. **51**, 1146 (1981) [Sov. Phys. Tech. Phys. **26**, 651 (1981)].
[7]A. M. Dykhne, Zh. Eksp. Teor. Fiz. **59**, 110 (1970) [Sov. Phys. JETP **32**, 63 (1971)].
[8]B. Ya. Balagurov, Zh. Eksp. Teor. Fiz. **81**, 665 (1981) [Sov. Phys. JETP **54**, 353 (1981)].
[9]G. V. Ryazanov, Teor. Mat. Fiz. **10**, 271 (1972).
[10]B. Ya. Balagurov and V. G. Vaks, Zh. Eksp. Teor. Fiz. **65**, 1939 (1973) [Sov. Phys. JETP **38**, 968 (1974)].
[11]B. I. Shklovskiĭ and A. L. Éfros, Usp. Fiz. Nauk **117**, 401 (1975) [Sov. Phys. Usp. **18**, 845 (1975)].
[12]A. L. Efros and B. I. Shklovskii, Phys. Status Solidi B **76**, 475 (1976).

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