Plasma transport drift equations

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The problem of describing a collision-dominated plasma in a strong magnetic field in terms of the transport equations is investigated. It was pointed out earlier that the standard Braginskiĭ-type equations are limited in scope, and various generalizations of these equations to make them applicable to a plasma in a magnetic field with straight force lines were proposed. In contrast, the present authors derive transport equations for an arbitrary magnetic-field geometry, using as a basis a drift kinetic equation with a collision term. The obtained system of transport equations reduces to equations for the density, longitudinal velocity, and temperature for each plasma component. These equations contain the so-called drift fluxes and forces, constituting certain combinations of higher moments. By way of example of the use of the drift transport equations, the problem of plasma rotation in a tokamak is considered. The authors propose that the new transport equations will be found useful also for an analysis of the role of viscosity and thermal conductivity in the instabilities and transport phenomena in a plasma confined by a curvilinear magnetic field.

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§1. INTRODUCTION

It is known that a collision-dominated plasma can be described with the aid of the equations for a certain number of the first moments of the distribution function of the corresponding particle species (transport equations). In the presence of a strong magnetic field the standard transport-equation systems[1] have a narrow range of validity, since they take into account neither the influence of the heat flux on the viscosity nor a number of other effects that play an important role in the stability and plasma transport across a magnetic field.[2,3] Several methods of modifying the transport-type equations were therefore proposed in the past.[1] Some are based on the use of a series expansion in reciprocal powers of the magnetic field,[4-6] and others on Grad's idea[7] of a multimoment description of the plasma.[8,9] It must be noted that the aim in both mentioned groups of papers was application to a plasma in a magnetic field with straight force lines. In view of progress in the research into magnetic traps (both toroidal and open), it seems important to obtain transport equations for an arbitrary magnetic-field geometry. This is the purpose of the present paper, in which we derive transport equations corresponding to the so-called drift approximation.[10-12] Our approach, as well as some others,[4-6] is thus based on an expansion in reciprocal powers of the magnetic field. Interest attaches also to further refinement of the multimoment approaches of the type described in Refs. 7–9, with an aim at taking into account effects that are outside the scope of the drift approximation.

The starting point in our equation is the collisional drift kinetic equation. This equation is derived in §2. We take into account here two types of collision effects connected with the averaged and oscillating parts of the distribution functions. The collision effects of the second type are well known in the usual scheme[1] and are of order ν/ω_B^2 in the continuity and heat-balance equations, and of order ν in the transverse-motion equation (ν is the collision frequency and ω_B is the cyclotron frequency). This is apparently the first time that account is taken of these effects in the drift kinetic equation scheme.

Another distinguishing feature of our drift kinetic equation is that it includes effects of a strong electric field (strong in the sense defined by Morozov and Solov'ev[13]). In this respect our approach is close to that of Hazeltine and Ware.[14] In particular, we take into account electric centrifugal drift effects.

In §3 we use the frequent-collision approximation and replace the kinetic equation by a system of equations for the density, longitudinal velocity, and temperature. We elucidate also the structures of these equations (drift transport equations) and the so-called drift fluxes and drift forces they contain. Concrete expressions for these fluxes and forces are given in §4.

An example of the use of our transport equations is given in §5, where plasma rotation in a tokamak is considered. The results are discussed in §6.

§2. COLLISIONAL DRIFT KINETIC EQUATION

To derive the collisional drift kinetic equation we used the results of the drift theory of charged particle motion, presented in Refs. 10–13.

We represent the particle velocity $\mathbf{v}$ in the form

$$\mathbf{v}=\mathbf{V}_E+v_\parallel\left(1+\frac{v_\perp^2}{2\omega_B v_\parallel}\mathbf{h}\,\mathrm{rot}\,\mathbf{h}\right)\mathbf{h}+\mathbf{v}_\perp\left(1-\frac{v_\parallel}{2\omega_B}\mathbf{h}\,\mathrm{rot}\,\mathbf{h}\right),\quad \mathbf{v}_\perp=v_\perp(\mathbf{n}\cos\vartheta+\mathbf{b}\sin\vartheta). \tag{2.1}$$

Here $v_\parallel$ and $v_\perp$ are the "smoothed" longitudinal velocity and transverse oscillatory velocity of the particle; ϑ is the phase of the particle cyclotron rotation; $\mathbf{V}_E=c\mathbf{E}\times\mathbf{h}/B$ is the particle drift velocity in crossed magnetic and electric fields $\mathbf{B}$ and $\mathbf{E}$; $\mathbf{h}=\mathbf{B}/B$ is a unit vector along the magnetic field; $\mathbf{n}$ and $\mathbf{b}$ are mutually perpendicular unit vectors transverse to the magnetic field, $\omega_B=eB/mc$ is the particle cyclotron velocity, e and m are the particle charge and mass, and c is the speed of light.

We represent the distribution function of each particle species in the form

$$f=\bar{f}+\tilde{f}, \tag{2.2}$$

where $\bar{f}$ is independent of the angle ϑ, and $\tilde{f}$ is linear in

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the first harmonics of ϑ.

The function $\tilde{f}$ is connected with $\bar{f}$ by the relation

$$\tilde{f}=\tilde{f}_0+\tilde{f}^c; \tag{2.3}$$

$$\tilde{f}_0=\mathbf{v}_\perp\left\{\left[\frac{\mathbf{h}}{\omega_B}\times\nabla\bar{f}\right]+\mathbf{u}_R\left(\frac{\partial}{\partial v_\parallel^2/2}-\frac{\partial}{\partial v_\perp^2/2}\right)\bar{f}\right.$$
$$\left.-(\mathbf{u}_E^\perp+\mathbf{u}_E^\parallel)\frac{\partial\bar{f}}{\partial v_\perp^2/2}\right\}+\Lambda_\parallel\frac{\partial\bar{f}}{\partial v_\parallel}, \tag{2.4}$$

$$\tilde{f}^c=\frac{1}{\omega_B}\frac{\partial}{\partial\vartheta}C(\tilde{f}_0). \tag{2.5}$$

Here C is the collision term, and $\mathbf{u}_R$, $\mathbf{u}_E^\perp$, and $\mathbf{u}_E^\parallel$ are the velocities of the particle centrifugal drifts due respectively to the curvature of the magnetic field, to the electric drift, and to the joint influence of both:

$$\mathbf{u}_R=v_\parallel^2\boldsymbol{\sigma}^\parallel,\quad \mathbf{u}_E^\perp=\frac{1}{\omega_B}[\mathbf{h}\times(\mathbf{V}_E\nabla)\mathbf{V}_E],\quad \mathbf{u}_E^\parallel=v_\parallel\boldsymbol{\varkappa}_E;$$
$$\boldsymbol{\sigma}^\parallel=\frac{1}{\omega_B}[\mathbf{h}\times(\mathbf{h}\nabla)\mathbf{h}],\quad \boldsymbol{\varkappa}_E=\frac{1}{\omega_B}\mathbf{h}\times[(\mathbf{V}_E\nabla)\mathbf{h}+(\mathbf{h}\nabla)\mathbf{V}_E]. \tag{2.6}$$

Finally,

$$\Lambda_\parallel=\frac{1}{\omega_B}\{[\mathbf{v}_\perp\times\mathbf{h}](\mathbf{V}_E\nabla)\mathbf{h}-\mathbf{h}([\mathbf{v}_\perp\times\mathbf{h}]\nabla)\mathbf{V}_E\}. \tag{2.7}$$

The equation for $\bar{f}$ is

$$\frac{\partial\bar{f}}{\partial t}+\frac{1}{G}\left[\operatorname{div}(G\boldsymbol{\xi})+\frac{\partial}{\partial v_\parallel}(G\eta)+\frac{\partial}{\partial v_\perp^2/2}(G\zeta)\right]=C(\bar{f}). \tag{2.8}$$

Here

$$\boldsymbol{\xi}=\bar{f}\frac{d\mathbf{r}}{dt}+\frac{1}{\omega_B}[\mathbf{a}\times\mathbf{h}],\quad \zeta=\bar{f}\frac{dv_\perp^2/2}{dt}-\mathbf{a}(\mathbf{u}_R+\mathbf{u}_E),$$
$$\eta=\bar{f}\frac{dv_\parallel}{dt}+v_\parallel\boldsymbol{\sigma}^\parallel\mathbf{a}+\frac{1}{\omega_B}\{[\mathbf{a}\times\mathbf{h}](\mathbf{V}_E\nabla)\mathbf{h}-\mathbf{h}([\mathbf{a}\times\mathbf{h}]\nabla)\mathbf{V}_E\}, \tag{2.9}$$
$$\mathbf{a}=\langle\mathbf{v}_\perp C(\tilde{f}_0)\rangle,\quad \mathbf{u}_E=\mathbf{u}_E^\perp+\mathbf{u}_E^\parallel,\quad G=1+\frac{v_\parallel}{2\omega_B}\mathbf{h}\operatorname{rot}\mathbf{h},$$

$\langle\ \rangle$ denotes averaging over ϑ.

The quantities $d\mathbf{r}/dt$, $dv_\parallel/dt$, $d(v_\perp^2/2)/dt$ are defined by the relations

$$\frac{d\mathbf{r}}{dt}=v_\parallel\mathbf{h}+\mathbf{V}_E+\mathbf{u}_D+\mathbf{u}_E,$$
$$\frac{dv_\parallel}{dt}=\frac{e}{m}E_\parallel+\frac{v_\perp^2}{2}\operatorname{div}\mathbf{h}-\frac{v_\perp^2}{2v_\parallel}\mathbf{u}_R\nabla\ln B+v_\parallel\mathbf{V}_E(\mathbf{h}\nabla)\mathbf{h}+\mathbf{V}_E(\mathbf{V}_E\nabla)\mathbf{h}, \tag{2.10}$$
$$d(v_\perp^2/2)/dt=-\{v_\parallel\operatorname{div}\mathbf{h}+\operatorname{div}\mathbf{V}_E+\mathbf{V}_E(\mathbf{h}\nabla)\mathbf{h}-u_R\nabla\ln B\}v_\perp^2/2.$$

Here $E_\parallel=\mathbf{E}\times\mathbf{h}$ is the longitudinal component of $\mathbf{E}$, $\mathbf{u}_D=\mathbf{u}_R+\mathbf{u}_B$,

$$\mathbf{u}_B=\boldsymbol{\sigma}^\perp v_\perp^2/2,\quad \boldsymbol{\sigma}^\perp=[\mathbf{h}\times\nabla\ln B]/\omega_B. \tag{2.11}$$

The appearance of the weighting factor G in (2.8) is due to the use of the smoothed variables $v_\parallel$ and $v_\perp$. The volume element in the smoothed-velocity space is defined as

$$d\mathbf{v}=Gv_\perp dv_\perp av_\parallel d\vartheta. \tag{2.12}$$

The quantity $\operatorname{div}(G\boldsymbol{\xi})$ in (2.8) contains terms that are quadratic in the spatial gradients. These terms correspond to well-known drift effects.[2,10] When account is taken of these effects we must retain, generally speaking, also terms with $\mathbf{u}_R\nabla\ln B$ in expressions (2.10) for $dv_\parallel/dt$, $d(v_\perp^2/2)/dt$. These terms can also be important in problems of high-pressure plasma in a curvilinear magnetic field. In that case we must take into account the difference between G and unity. At low plasma pressure or if the magnetic-field curvature is neglected, we can set $G=1$ and neglect the terms with $\mathbf{u}_R\nabla\ln B$ in $dv_\parallel/dt$ and $d(v_\perp^2/2)/dt$. This is the approximation most frequently employed.

The term $C(\bar{f})$ in the right-hand side of (2.8) is standard in the collisional drift kinetic equation scheme (see, e.g., Refs. 2 and 5). Terms with the vector $\mathbf{a}$ in the functions ξ, η, and ζ [see (2.9)] describe collision processes connected with the oscillating part of the distribution function. They are responsible for the effects of order ν/ω_B^2 noted in §1 (see also §4).

§3. DRIFT TRANSPORT EQUATIONS

In analogy with Ref. 1, we express the longitudinal particle velocity in the form $v_\parallel=w_\parallel+V_\parallel$, where $w_\parallel$ and $V_\parallel$ are respectively the random and directed velocities. In the variables $(t,\mathbf{r},w_\parallel,v_\perp)$ Eq. (2.8) means that

$$\frac{\partial\bar{f}}{\partial t}+\frac{1}{G}\left\{\operatorname{div}(G\boldsymbol{\xi})+\frac{\partial}{\partial w_\parallel}G\eta+\frac{\partial}{\partial(v_\perp^2/2)}G\zeta\right.$$
$$\left.-\left[\frac{\partial V_\parallel}{\partial t}\frac{\partial}{\partial w_\parallel}G\bar{f}+\nabla V_\parallel\frac{\partial}{\partial w_\parallel}G\boldsymbol{\xi}\right]\right\}=C(\bar{f}). \tag{3.1}$$

It is necessary accordingly to make in (2.4) the substitution $\nabla\to\nabla-\nabla V_\parallel\partial/\partial w_\parallel$.

We put

$$\bar{f}=F(1+\Phi)/G. \tag{3.2}$$

Here

$$F=n(m/2\pi T)^{3/2}\exp(-mw^2/2T); \tag{3.3}$$

$w^2=w_\parallel^2+v_\perp^2$; n and T are the density and temperature of the corresponding plasma component; Φ is a certain small increment satisfying the conditions

$$\int(1,w_\parallel,w^2)F\Phi v_\perp dv_\perp dw_\parallel=0. \tag{3.4}$$

Integrating (3.1) with respect to velocity with weights 1, $mw_\parallel$, $mw^2/2$, we obtain

$$\partial n/\partial t+\operatorname{div}n\mathbf{U}=0, \tag{3.5}$$
$$mn(\partial V_\parallel/\partial t+\mathbf{U}\nabla V_\parallel)+\operatorname{div}\mathbf{P}=R_\parallel, \tag{3.6}$$
$$^3/_2\partial(nT)/\partial t+\operatorname{div}\mathbf{S}+\mathbf{P}\nabla V_\parallel=Q. \tag{3.7}$$

Here $n\mathbf{U}$, $\mathbf{P}$, and $\mathbf{S}$ have the respective meanings of the drift fluxes of the density, longitudinal momentum, and thermal energy, while $R_\parallel$ and Q stand for the longitudinal force acting on the considered plasma component and for the change of the thermal energy of this component. By definition

$$(n\mathbf{U},\mathbf{P},\mathbf{S})=\int(1,mw_\parallel,mw^2/2)\boldsymbol{\xi}\,d\mathbf{v}, \tag{3.8}$$

$$(R_\parallel,Q)=m\int(\eta,w_\parallel h+\zeta)d\mathbf{v}+(R_{\parallel c},Q_c), \tag{3.9}$$

where $R_{\parallel c}$ and Q_c are the longitudinal force and the heat exchange connected with $\bar{f}$,

$$(R_{\parallel c},Q_c)=m\int(w_\parallel,w^2/2)C(\bar{f})d\mathbf{v}. \tag{3.10}$$

We can introduce also the macroscopic fluxes of the particles ($n\mathbf{V}$), of the longitudinal momentum ($\mathbf{P}^m$), and of the thermal energy ($\mathbf{S}^m$):

$$(n\mathbf{V},\mathbf{P}^m,\mathbf{S}^m)\equiv\int(1,mw_\parallel,mw^2/2)\mathbf{v}f\,d\mathbf{v}. \tag{3.11}$$

We then have (cf. Refs. 10 and 12)

$$(n\mathbf{V},\ \mathbf{P}^m,\ \mathbf{S}^m)=(n\mathbf{U},\ \mathbf{P},\ \mathbf{S})-\mathrm{rot}\{\mathbf{h}(I_V,\ I_P,\ I_S)/\omega_B\}, \qquad (3.12)$$

$$(I_V, I_P, I_S)=\int (1, mw_\parallel, mw^2/2)\,(v_\perp^2/2)\tilde{f}\,d\mathbf{v}. \qquad (3.13)$$

We can therefore replace $n\mathbf{U}$, $\mathbf{P}$, and $\mathbf{S}$ in (3.5)–(3.7) by $n\mathbf{V}$, $\mathbf{P}^m$, and $\mathbf{S}^m$.

§4. FLUXES AND FORCES

We represent the quantities $Y\equiv(\mathbf{U},\mathbf{V}_\perp,\mathbf{P},\mathbf{S},R_\parallel,Q)$ in the form

$$Y=Y^{(0)}+\hat{Y}+Y^c+Y_c. \qquad (4.1)$$

Here $Y(0)$ is calculated at $f=F$ with the collisions neglected; $\hat{Y}$ corresponds to the non-Maxwellian increment ΦF [see (3.2)]; Y^c is connected with $\tilde{f}^c$; $Y_c\equiv(R_{\parallel c},Q_c)$; $\mathbf{V}_\perp$ is the transverse part of the macroscopic velocity $\mathbf{V}$ [see (3.12)].

4.1. Collisionless Maxwellian parts of the fluxes and forces

With the aid of the equations of §§2 and 3 we obtain

$$\mathbf{U}^{(0)}=V_\parallel\mathbf{h}+\mathbf{V}_E+\mathbf{U}_D^{(0)}+\mathbf{U}_E,$$
$$\mathbf{V}_\perp^{(0)}=\mathbf{V}_E+\mathbf{U}_E+\mathbf{h}\times[\nabla p+mnV_\parallel^2(\mathbf{h}\nabla)\mathbf{h}]/mn\omega_B,$$
$$\mathbf{P}^{(0)}=p\{\mathbf{h}+\varkappa_E+2V_\parallel[\mathbf{h}\times(\mathbf{h}\nabla)\mathbf{h}]/\omega_B\}, \qquad (4.2)$$

$$\mathbf{S}^{(0)}={}^3/_2p(V_\parallel\mathbf{h}+\mathbf{V}_E+\mathbf{U}_E)+\mathbf{S}_D^{(0)},$$
$$R_\parallel^{(0)}=n\{eE_\parallel+T\,\mathrm{div}\,\mathbf{h}+mV_\parallel\mathbf{V}_E(\mathbf{h}\nabla)\mathbf{h}+m\mathbf{V}_E(\mathbf{V}_E\nabla)\mathbf{h}-TV_\parallel\sigma^\parallel\nabla\ln B\},$$
$$Q^{(0)}=-p(V_\parallel\,\mathrm{div}\,\mathbf{h}+\mathrm{div}\,\mathbf{V}_E-V_\parallel^2\sigma^\parallel\nabla\ln B).$$

Here $p=nT$,

$$\mathbf{U}_D^{(0)}=T\,\mathbf{h}\times[\nabla\ln B+(\mathbf{h}\nabla)\mathbf{h}]/m\omega_B+V_\parallel^2[\mathbf{h},(\mathbf{h}\nabla)\mathbf{h}]/\omega_B,$$
$$\mathbf{S}_D^{(0)}=\frac{p}{m\omega_B}\{{}^5/_2T\mathbf{h}\times[\nabla\ln B+(\mathbf{h}\nabla)\mathbf{h}]+{}^3/_2mV_\parallel^2[\mathbf{h}\times(\mathbf{h}\nabla)\mathbf{h}]\}, \qquad (4.3)$$
$$\mathbf{U}_E=V_\parallel\varkappa_E+[\mathbf{h}\times(\mathbf{V}_E\nabla)\mathbf{V}_E]/\omega_B.$$

Equations (4.2) and (4.3) can be obtained from Ref. 1.

4.2. Non-Maxwellian parts of ionic fluxes and forces

We express $\hat{Y}$ in terms $\pi_\parallel$ and $q_\parallel$, where

$$(\pi_\parallel, q_\parallel)\equiv m\int(w_\parallel^2, w_\parallel w^2/2)F\Phi\,d\mathbf{v}. \qquad (4.4)$$

Here $\pi_\parallel=\pi_{hh}$ is the "longitudinally longitudinal" component of the viscosity tensor and $q_\parallel$ is the longitudinal component of the heat-flux vector.

We obtain in analogy with (4.2)

$$\hat{\mathbf{U}}=(\sigma^\parallel-\sigma^\perp/2)\pi_\parallel/mn,\qquad \hat{\mathbf{V}}_\perp=\mathbf{h}\times[3\pi_\parallel(\mathbf{h}\nabla)\mathbf{h}-\nabla\pi_\parallel]/2mn\omega_B,$$
$$\hat{\mathbf{P}}=\pi_\parallel\mathbf{h}+{}^2/_5q_\parallel(3\sigma^\parallel+\sigma^\perp),\ \ \hat{\mathbf{S}}=q_\parallel\mathbf{h},\ \ \hat{R}_\parallel=-\pi_\parallel\,\mathrm{div}\,\mathbf{h}/2, \qquad (4.5)$$
$$\hat{Q}=\frac{\pi_\parallel}{2}\left[V_\parallel\,\mathrm{div}\,\mathbf{h}+\mathrm{div}\,\mathbf{V}_E-\frac{3}{2}\mathbf{V}_E(\mathbf{h}\nabla)\mathbf{h}\right].$$

The quantities $\pi_\parallel$ and $q_\parallel$ are obtained in analogy with Ref. 9 by expanding the distribution function in Sonine-Laguerre polynomials. In an approximation with two polynomials we obtain for the ions

$$\pi_\parallel=-\frac{2nT}{3\nu}(0.96\beta-0.59\gamma),\qquad q_\parallel=-3.9\frac{nT^2}{m\nu}\mathbf{h}\varkappa_T. \qquad (4.6)$$

Here

$$\beta=-\mathrm{div}\,\mathbf{V}_E-3\mathbf{V}_E(\mathbf{h}\nabla)\mathbf{h}+V_\parallel\mathbf{h}(2\varkappa_V+\varkappa_B)+2V_\parallel\varkappa_V\varkappa_E$$
$$+\frac{T}{m}(2\sigma^\parallel-\sigma^\perp)(\varkappa_n+2\varkappa_T)+V_\parallel^2\sigma^\parallel(4\varkappa_V-\varkappa_B)+\frac{4}{5p}\mathbf{h}\nabla q_\parallel,$$
$$\gamma=-\frac{T}{m}(2\sigma^\parallel-\sigma^\perp)\varkappa_T-\frac{2}{5}\cdot 1.27\mathbf{h}\nabla q_\parallel,$$
$$(\varkappa_T,\ \varkappa_n,\ \varkappa_B,\ \varkappa_V)\equiv(\nabla\ln T,\ \nabla\ln n,\ \nabla\ln B,\ \nabla\ln V_\parallel). \qquad (4.7)$$

In (4.6), $\nu\equiv 1/\tau_i$, where τ_i is the characteristic time of the ion-ion collisions and is defined in Ref. 1.

Expression (4.6) for $q_\parallel$ coincides with that in Ref. 1. The expression for $\pi_\parallel$ contains contributions accounted for in the scheme of Ref. 1, as well as some additional terms, including terms with the gradient of the longitudinal heat flux. On the whole, therefore, expression (4.6) for $\pi_\parallel$ cannot be obtained by using the scheme of Ref. 1.

4.3. The $\tilde{f}^c$-related parts of the ion fluxes and forces

The ionic quantities are of the form

$$\mathbf{U}_i^c=\mathbf{V}_{\perp i}^c=[\mathbf{R}_{\perp i}^c\times\mathbf{h}]/m_in\omega_{Bi},$$
$$\mathbf{P}_i^c=[\mathbf{K}_{\perp i}^c\times\mathbf{h}]/\omega_{Bi},\qquad \mathbf{S}_i^c=-2\nu_inT^2\nabla_\perp\ln T/m_i\omega_{Bi}^2, \qquad (4.8)$$
$$R_{\parallel i}^c=\sigma_i^\parallel(\mathbf{K}_{\perp i}^c+V_\parallel\mathbf{R}_{\perp i}^c)+\{[\mathbf{R}_{\perp i}^c\times\mathbf{h}](\mathbf{V}_E\nabla)\mathbf{h}-\mathbf{h}([\mathbf{R}_{\perp i}^c\times\mathbf{h}]\nabla)\mathbf{V}_E\}/\omega_B,$$
$$Q_i^c=-\mathbf{R}_{\perp i}^c(V_\parallel^2\sigma_i^\parallel+\mathbf{U}_{Ei})-\mathbf{K}_{\perp i}^c\{V_\parallel\sigma_i^\parallel+\varkappa_E$$
$$-[\mathbf{h},(\mathbf{V}_E\nabla)\times\mathbf{h}]/\omega_{Bi}\}-\omega_{Bi}^{-1}\mathbf{h}([\mathbf{K}_{\perp i}^c\times\mathbf{h}]\nabla)\mathbf{V}_E.$$

Here $\nabla_\perp\equiv\nabla-\mathbf{h}(\mathbf{h}\nabla)$, and

$$\mathbf{K}_{\perp i}^c=-\frac{6}{5}\frac{p\nu_i}{\omega_{Bi}}\{\mathbf{h}\times[\nabla V_\parallel+V_\parallel(\mathbf{h}\nabla)\mathbf{h}+(\mathbf{h}\nabla)\mathbf{V}_E]+h^\mu[\mathbf{h}\times\nabla]V_{E\mu}\}. \qquad (4.9)$$

The upper and lower indices $\mu=1,2,3$ denote the covariant and contravariant components of the corresponding vectors. The vector $\mathbf{R}_{\perp i}^c$ denotes the transverse friction force exerted by the electrons on the ions. An expression for $\mathbf{R}_{\perp i}^c$ will be given in subsection 4.4.

Expressions (4.8) for $\mathbf{U}_i^c$ and $\mathbf{S}_i^c$ can be obtained using the scheme of Ref. 1. To obtain expressions for $\mathbf{P}_i^c$, $R_{\parallel c}^c$, and Q_i^c with the aid of the formulas of Ref. 1 we must make the formal substitution $\mathbf{V}_\perp\to\mathbf{V}_E$.

4.4. Flux and force components that depend on the electron-electron and electron-ion collisions

Recognizing the smallness of the electron mass compared with that of the ion, we neglect the electronic quantity $\pi_\parallel$. The electronic $q_\parallel$ is assumed to be[1]

$$q_{\parallel e}=-\varkappa_\parallel\mathbf{h}\nabla T_e+0.71nT_e(V_{\parallel e}-V_{\parallel i}), \qquad (4.10)$$

where $\varkappa_\parallel^e=3.16nT_e/m_e\nu_e$; $\nu_e\equiv 1/\tau_e$, τ_e is the characteristic electron-electron collision time defined in Ref. 1. We take this $q_{\parallel e}$ into account only in the expression for $\mathbf{S}_e$. The formulas of type (4.5) in the case of electrons mean therefore

$$(\hat{\mathbf{U}}_e,\ \hat{\mathbf{V}}_{\perp e},\ \hat{\mathbf{P}}_e,\ \hat{R}_{\parallel e},\ \hat{Q}_e)=0,\qquad \hat{\mathbf{S}}_e=\mathbf{h}q_{\parallel e}. \qquad (4.11)$$

We take the electronic analog of Eqs. (4.8) and (4.9) to be (cf. Ref. 1)

$$\mathbf{U}_e^c=\mathbf{V}_{\perp e}^c=-\mathbf{U}_i^c,\qquad \mathbf{S}_e^c=-4{,}66\nu_enT_e^2\nabla_\perp\ln T_e/m_e\omega_{Be},$$
$$(\mathbf{P}_e^c, R_{\parallel e}^c, Q_e^c, \mathbf{K}_{\perp e}^c)=0. \qquad (4.12)$$

By the same token we neglect the transverse electronic viscosity, as well as the "transverse" electron heat release.

The expression for the electronic transverse friction force $\mathbf{R}_{\perp e}{}^{c} = -\mathbf{R}_{\perp i}{}^{c}$ is chosen to be[1]

$$\mathbf{R}_{\perp e}{}^{c}=-\nu_e[\mathbf{h}\times\nabla(p_e+p_i)]/\omega_{Be}+{}^3/_2 n\nu_e[\mathbf{h}\times\nabla T_e]/\omega_{Be}+m_e n\nu_e(\mathbf{U}_{Ei}+V_{\|i}{}^2\boldsymbol{\sigma}_i{}^{\|}). \tag{4.13}$$

Finally,[1]

$$R_{\|ce}=-R_{\|ci}=-0{,}51 m_e n\nu_e(V_{\|e}-V_{\|i})-0{,}71 n\nabla_\| T_e,$$
$$Q_{ce}=-Q_{ci}=-3\nu_e n(T_e-T_i)m_e/m_i. \tag{4.14}$$

Thus, all the quantities that depend on the electron collisions are obtained in accord with the scheme of Ref. 1. This is the consequence of neglecting the viscosity and the transverse thermal conductivity of the electrons.

§5. PLASMA ROTATION VELOCITY IN A TOKAMAK

The problem of rotation of a collision-dominated plasma in a tokamak was discussed earlier by Pogutse,[15] Kovrizhnykh,[16] Hazeltine,[17] and Rozhanskiĭ and Tsendin.[18] Of greatest interest is Hazeltine's paper. By solving the kinetic equation exactly (without the first few terms of an expansion in Sonine-Laguerre polynomials), Hazeltine[17] obtained the following expression for the plasma poloidal rotation velocity U_θ:

$$U_\theta=kU_T. \tag{5.1}$$

Here $U_T = cT_i'/eB_s$ is the so-called ion drift velocity due to the gradient temperature, the prime denotes the derivative with respect to the minor radius of the torus, B_s is the average toroidal magnetic field, θ is the poloidal angle (minor azimuth of the torus), and $k = -2.1$ is a numerical coefficient obtained in Ref. 17.

In the derivation of (5.1) it was assumed in Ref. 17 that the ratio of the toroidal plasma velocity $U_\varphi \approx V_\|$ (φ is the toroidal angle) to the ion thermal velocity is small: $V_\|/v_{Ti} \to 0$. We shall examine below a derivation of (5.1) from the drift transport equations cited above, and generalize (5.1) to include the case of finite $V_\|/v_{Ti}$. This generalization is of interest for a tokamak with longitudinal injection of fast neutral atoms.

Assuming the plasma to be electrically quasineutral, we represent the known continuity equation for the electric charge density in the form

$$\operatorname{div}\mathbf{j}=0, \tag{5.2}$$

where $\mathbf{j}$ is the electric current density. Averaging (5.2) over the magnetic surface, we obtain

$$(\sqrt{g}\,j^a)_0=0. \tag{5.3}$$

Here g is the determinant of the metric tensor, j^a is the ath derivative of the component of the vector j, and the zero subscript denotes averaging over θ. We use the coordinate system (a, θ, φ) described in Ref. 19, so that the coordinate a denotes the distance along the minor radius of the torus.

We calculate j^a using (3.12), (4.1), and (4.2). Equation (5.3) reduces then to

$$[\sqrt{g}\,n(\mathbf{U}_{Di}^{(0)}-\mathbf{U}_{De}^{(0)}+\hat{\mathbf{U}}_i)^a]_0=0. \tag{5.4}$$

The contributions of $\mathbf{U}_E$ and $\mathbf{U}^c$ to (5.4) are negligible.

Taking into account (4.3), (4.5), and the expressions given in Ref. 19 for the metric coefficients, we obtain from (5.4)

$$[(2\tilde{p}+m_iV_\|^2\tilde{n}+{}^1/_2\tilde{\pi}_\|)\sin\theta]_0=0. \tag{5.5}$$

Here $\tilde{p} = \tilde{p}_e + \tilde{p}_i$; the tilde designates that part of the quantity which oscillates relative to θ.

Summing the electron and ion equations of motion (3.6) we obtain

$$\tilde{p}+\tilde{\pi}_\|=-\varepsilon m_iV_\|^2\cos\theta, \tag{5.6}$$

where $\varepsilon = a/R$, and R is the magnetic-axis curvature radius. We neglect the oscillations of the electron temperature, $\tilde{T}_e \to 0$. Therefore

$$\tilde{p}=\tilde{n}(T_i+T_e)+n\tilde{T}_i. \tag{5.7}$$

According to (4.6)

$$\tilde{T}_i=\frac{2}{3}\frac{b}{n\nu_iqR}\frac{\partial\tilde{q}_\|}{\partial\theta};\quad b=\frac{3}{2}\cdot\frac{1}{3.9}\frac{m_i\nu_i^2q^2R^2}{T_i}. \tag{5.8}$$

Here $q = aB_s/RB_\theta$ is the tokamak margin coefficient and B_θ is the poloidal magnetic field. In deriving (5.8) we used the fact that $\partial^2\tilde{T}_i/\partial\theta^2 = -\tilde{T}_i$, since the oscillating quantities are linear functions of $\cos\theta$ or $\sin\theta$.

From the ion-heat balance equation (3.7) we obtain

$$\tilde{q}_\|=\frac{qR}{a}T_i\left[\left(U_\theta-\frac{5}{2}U_T\right)\tilde{n}+5\varepsilon U_T\cos\theta\right]. \tag{5.9}$$

In deriving (5.9) we took it into account that

$$U_\theta=\frac{V_\|}{qR}-\frac{cE_a}{B_s}+\frac{cp_i'}{enB_s}, \tag{5.10}$$

where E_a is the radial electric field.

Finally, the quantities β and γ defined in (4.7) take in the present case the form

$$\beta=\frac{3}{qR}\left[\frac{2}{5p_i}\left(\frac{\partial\tilde{q}_\|}{\partial\theta}+2T_i\frac{qR}{a}U_\theta\frac{\partial\tilde{n}}{\partial\theta}\right)-(U_\theta-U_T)\sin\theta\right],$$
$$\gamma=-\frac{1}{qR}\left(1.27\cdot\frac{4}{5p_i}\frac{\partial\tilde{q}_\|}{\partial\theta}+U_T\right). \tag{5.11}$$

Using (4.6) and (5.6)–(5.11) we obtain from (5.4) the sought equation for U_θ:

$$0.96({}^3/_2+\alpha)[U_\theta(1+0.19\alpha)+U_T(1.83+0.57\alpha)]+{}^1/_3\alpha b[\alpha(U_\theta-{}^5/_2U_T)-5U_T]=0. \tag{5.12}$$

Here $\alpha = V_\|^2/c_s^2$ and $c_s^2 = (T_e + T_i)/m_i$ is the speed of sound squared.

It follows from (5.12) that an equation such as (5.1) holds not only in the case $V_\|/c_s \to 0$ considered in Ref. 17, but also at arbitrary $V_\|/c_s$. Only the coefficient k depends on $V_\|/c_s$. It remains therefore to consider only the $k(\alpha)$ dependence.

At $\alpha \ll 1/b$ it follows from (5.12) that

$$k=k_0=-1.83. \tag{5.13}$$

The small quantitative difference between our numerical coefficient (5.13) and $k = -2.10$ obtained in Ref. 17 is due to the fact, noted above, that Hazeltine[17] solved the kinetic equation exactly, while we used an approximation with two Sonine-Laguerre polynomials.

We note also that according to Pogutse[15] $U_\theta = 0$ both at $U_T = 0$ and $U_T \neq 0$. Pogutse's error at $U_T \neq 0$ is due to his use of Braginskii's expression[1] for the viscosity tensor, which does not hold at $\nabla T \neq 0$. Rozhanskii and Tsendin,[18] who followed Pogutse's approach,[15] are wrong for the same reason. Kovrizhnykh[16] obtain for U_θ in place of (5.1) some erroneous expression that depends not only on the temperature gradient but also on the density gradient and on the temperature. His error is due to an incorrect transformation of the expression for the viscosity tensor. Were his calculations correct, Kovryzhnykh would obtain Eq. (5) with $k = -1$, since his hydrodynamics was of the type used in Refs. 7 and 8; see also Ref. 20.

At $\alpha \approx 1/b$ Eq. (5.13) is replaced by

$$k=-1.83+(10/9\cdot 0.96)\alpha b. \quad (5.14)$$

It is seen that the plasma poloidal rotation reverses direction at $\alpha \simeq 1/b$. We note also that $b \gg 1$, since the hydrodynamic description does not hold at $b \lesssim 1$.

At $1/b \lesssim \alpha \ll 1$ we have in place of (5.14)

$$k=5\alpha b/[\alpha^2 b+9\cdot 0.96/2]. \quad (5.15)$$

According to (5.15) the rotation velocity reaches a maximum at

$$\alpha=\alpha_*={}^1/_3(2/0.96b)^{1/2}. \quad (5.16)$$

In this case

$$k_{max}={}^5/_6(2b/0.96)^{1/2}. \quad (5.17)$$

At $\alpha b \gg 1$ we have in lieu of (5.15)

$$k={}^5/_2(1+2/\alpha), \quad (5.18)$$

so that $k = 5/2$ as $\alpha \to \infty$.

§6. DISCUSSION OF RESULTS

We have derived the collisional drift kinetic equation (2.8) that takes into account the many diverse physical effects noted in §§1 and 2. The equation can be used at various degrees of collision dominance over the plasma, i.e., in the Pfirsch-Schlüter (frequent-collision), plateau, and "banana" regimes (see Ref. 21 for the regime classification). With the aid of this equation we obtained in the frequent-collision regime Eqs. (3.5)–(3.7) for the density, longitudinal velocity, and temperature; these equations depend on the so-called drift fluxes and forces defined by Eqs. (3.8)–(3.10). These forces and fluxes comprise sums of four physically different parts [see (4.1)]. One part corresponds to the approximation of a collisionless drift kinetic equation with a Maxwellian distribution function [Eqs. (4.2) and (4.3)]. The second is connected with a non-Maxwellian collision-dominated increment to the middle part of the distribution function [Eqs. (4.4)–(4.7), (4.10), (4.11), and (4.13)]. The third is determined by collision effects connected with the oscillating part of the distribution function [Eqs. (4.8) and (4.9)]. Finally, the fourth part constitutes the longitudinal friction force and the heat exchange between the ions and electrons [Eq. (4.14)].

The greatest difference between our transport equations and the Braginskii standard equations[1] is that we take into account the dependence of the viscosity on the heat flux. This was pointed out also in Refs. 4 and 5, in which, however, no prescription was given for the calculation of the viscosity in the case of a curvilinear magnetic field.

The effectiveness of our drift transport equation is demonstrated for plasma rotation in a tokamak in §5, where we have confirmed in kinetic result[17] and generalized it to the case of finite $V_\parallel/v_{Ti}$.

We assume that our drift transport equations will be found useful also for the analysis of the role of viscosity and heat conduction in the treatment of instabilities of drift-hydromagnetic type in complex magnetic configurations, as well for transport phenomena in these configurations in the Pfirsch-Schlüter regime.

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