Self-focusing of whistlers

V. I. Karpman and R. N. Kaufman

Institute of Terrestrial Magnetism, Ionosphere, and Radiowave Propagation, USSR Academy of Sciences
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A general Schrödinger equation is obtained, describing the waveguide propagation of whistlers in channels (ducts) oriented along the magnetic field and having a density either higher (crests) or lower (troughs) than the density of the surrounding plasma. The self-focusing of the whistlers is considered on the basis of this equation and of the system of the plasma magnetohydrodynamic equations supplemented by taking into account the ponderomotive force due to the high-frequency field. A one-parameter family of solutions of the complete system of equations is obtained and describes the stationary self-focused beams. It is shown that the beams accompanied by formation of ducts with plasma-density crests should be attenuated by the leakage of the wave due to wave tunneling. This explains why only ducts with troughs are produced by self-focusing in experiment, a result that agrees qualitatively with the solutions obtained.

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1. INTRODUCTION

The existing theories, based on the Schrödinger equation, of whistler self-focusing in a plasma lead to two main conclusions: 1) formation of asymptotically stationary wave beams as a result of self-focusing is possible only at $\omega < \omega_c/2$ (ω_c is the electron gyrofrequency); 2) these beams should propagate in waveguide ducts having a density higher than the surrounding plasma (called hereafter crests). Both conclusions contradict qualitatively the known experimental facts (see, e.g., Ref. 1), since formation of self-focusing troughs was observed both at $\omega < \omega_c/2$ and $\omega > \omega_c/2$.

Whistler propagation studies, based on WKB solutions of the complete system of Maxwell's equations in both planar[2-5] and axisymmetric[6] geometries, have shown that, generally speaking, a number of effects are missed when the Schrödinger equation is used. One of them is the leakage of the wave from the crest as a result of its tunneling into another mode.[2,3] This effect becomes particularly strong when the duct width is of the order of the longitudinal wavelength.[4-6] It turns out as a result that the linearly self-compressed crest should rapidly "leak out." Another result of the WKB solutions of Maxwell's equations is the feasibility of waveguide propagation in troughs not only at $\omega < \omega_c/2$ but also at $\omega > \omega_c/2$. Clearly, self-focusing theory must take these circumstances into account. At the same time it must be simple enough to yield analytic solutions.

The present paper is devoted to the development of the basics of such a theory. It consists of two parts. The first (Secs. 2 and 3) is devoted to the derivation of simplified electromagnetic-field equations on the basis of Maxwell's equations under the assumption that the relative change of the plasma density is small. We arrive as a result to two different Schrödinger equations. The first describes beams with wave vectors "almost parallel" to the external magnetic field. This equation was first derived in Refs. 7 and 8 and served so far as the traditional one for the investigation of the self-focusing of whistlers along a magnetic field. It leads to waveguide propagation in crests at $\omega < \omega_c/2$ and in troughs at $\omega > \omega_c/2$. The second Schrödinger equation describes beams with wave vectors that are grouped along a cone with apex angle $\theta = \arccos(2\omega/\omega_c)$, where θ is the angle between the wave vector and the external magnetic field $\mathbf{B}_0$. (The group velocity components, on the other hand, are almost parallel to $\mathbf{B}_0$ in both cases.) It is important here that in the second case the Schrödinger equation describes waveguide propagation in troughs at $\omega < \omega_c/2$. Both Schrödinger equations can be unified into one that can describe whistler propagation in troughs at both $\omega < \omega_c/2$ and $\omega > \omega_c/2$, as well as in crests at $\omega > \omega_c/2$ (if tunneling is neglected).

In the second part of the paper (Sec. 4) we add to the Schrödinger equations the plasma hydrodynamic equations supplemented by terms containing the ponderomotive forces due to the pressure of the RF field of the whistlers on the plasma. The system obtained permits a self-consistent treatment of whistler propagation and of slow plasma motion. We obtain next for this system a family of solutions that depend on a single parameter σ which describes stationary finite-amplitude wave beams propagating in the ducts produced by them. At $\sigma = 0$ this family leads to a previously obtained solution.[8,9] This particular solution, however, holds only in the frequency region $\omega < \omega_c/2$ and describes stationary beams in crests. As already indicated, such beams are transformed into a defocusing branch because of the tunneling, which becomes quite intense when the channel width is small enough. Similar results are obtained for all $\sigma < 1$. The solutions at $\sigma > 1$ describe stationary beams in ducts with troughs. These were precisely the beams observed in experiment.

Understandably, as in any self-focusing theory based on the Schrödinger equation, our equations describe the main processes only in the stage when the self-focusing is still weak (the beam width greatly exceeds the longitudinal wavelength). They can be regarded, however, as the starting point for strong self-focusing theories which should be based on the complete system of the Maxwell equations and on the equations of nonlinear hydrodynamics (and possibly also kinetics). They should also make extensive use of numerical methods. These questions, however, are way outside the scope of the present paper.

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2. FUNDAMENTAL EQUATIONS

We assume that the electromagnetic field is quasi-monochromatic, i.e., can be represented in the form

$$\tfrac{1}{2}[\vec{\mathscr{E}}(\mathbf{r},t)e^{-i\omega t}+\text{c.c.}], \tag{2.1}$$

where $\vec{\mathscr{E}}$ varies slowly with the time t; the dependence on r, however, is as yet arbitrary. Maxwell's equations reduce then to the equation

$$\nabla(\nabla\vec{\mathscr{E}})-\Delta\vec{\mathscr{E}}=\frac{\omega^2}{c^2}(\hat{\varepsilon}\vec{\mathscr{E}})+\frac{i}{c^2}\frac{\partial(\omega^2\hat{\varepsilon})}{\partial\omega}\frac{\partial\vec{\mathscr{E}}}{\partial t}, \tag{2.2}$$

where $\hat{\varepsilon}$ is the dielectric tensor, and its nonvanishing components in a cold plasma take, in a coordinate frame in which the magnetic field is directed along the z axis, the form

$$\varepsilon_{xx}=\varepsilon_{yy}=\varepsilon,\quad \varepsilon_{xy}=-\varepsilon_{yx}=-ig,\quad \varepsilon_{zz}=\eta. \tag{2.3}$$

We denote by N the particle density, by N_0 the density at infinity, and by ω_p the plasma frequency corresponding to N_0:

$$\omega_p^2=4\pi e^2N_0/m_e.$$

Neglecting the ion motion in the whistler frequency range $\omega<\omega_c\ll\omega_p$, we can write the components of the tensor (2.3) in the form

$$\varepsilon=\frac{(1+\nu)\gamma^2}{1-u^2},\quad g=\frac{(1+\nu)\gamma^2}{u(1-u^2)},\quad \eta=-\frac{(1+\nu)\gamma^2}{u^2}, \tag{2.4}$$

where ν is the relative variation of the particle density

$$\nu=(N-N_0)/N_0,\quad \gamma=\omega_p/\omega_c,\quad u=\omega/\omega_c. \tag{2.5}$$

In the preceding papers[2-6] Eq. (2.2) was solved approximately by the WKB method. Assuming that

$$N=N(x),\quad \omega_c=\text{const},\quad \vec{\mathscr{E}}=\vec{\mathscr{E}}(x,z), \tag{2.6}$$

the WKB solutions of (2.2) are of the form

$$\vec{\mathscr{E}}=\vec{\mathscr{E}}_0(x)\exp\left\{i\frac{\omega}{c}\left[\pm\int_0 q(x)dx+pz\right]\right\}, \tag{2.7}$$

where $q^2=q_m^2$, $m=1,2$,

$$q_m^2(x)=(2u^2)^{-1}\{(1-2u^2)p^2-2\gamma^2[1+\nu(x)]+(-1)^m p(p-4\gamma^2[1+\nu(x)])^{1/2}\}, \tag{2.8}$$

and $\vec{\mathscr{E}}_0(x)$ is the polarization vector and varies slowly with x. Given the dimensionless longitudinal wave number p, Eq. (2.8) determines two branches of the dimensionless transverse wave number q. A detailed investigation of the properties of the WKB solutions (2.7) (particularly of the polarization vectors $\vec{\mathscr{E}}_0(x)$, of the energy flux, and of others) can be found, e.g., in Refs. 2, 5, and 6. We note here that relation (2.8) at $\nu=0$ is the consequence of the known dispersion equation for whistlers in the quasi-longitudinal approximation:

$$\omega=\frac{k^2\omega_c\cos\theta}{(\omega_p/c)^2+k^2} \tag{2.9}$$

as can be verified by putting in (2.9)

$$k^2=k_x^2+k_z^2,\quad k_x=(\omega/c)q,\quad k_z=(\omega/c)p.$$

3. THE SCHRÖDINGER EQUATIONS

We introduce as the main sought quantities the following linear combinations of the Cartesian field components:

$$\mathscr{E}_1=\mathscr{E}_x-i\mathscr{E}_y,\quad \mathscr{E}_2=\mathscr{E}_x+i\mathscr{E}_y \tag{3.1}$$

(when dealing with self-focusing of whistlers, these quantities are more useful than the previously employed ones[2-6]). The component $\mathscr{E}_z$ is easily expressed in terms of $\mathscr{E}_1$ and $\mathscr{E}_2$ with the aid of the equation $\operatorname{div}(\hat{\varepsilon}\vec{\mathscr{E}})=0$.

We assume for simplicity that assumptions (26) are valid, with N varying slowly with x and with $\nu(x)$ small. More accurately speaking, we assume that $\nu=\nu(x/a)$, where a is the characteristic spatial density scale that is large compared with the reciprocal wave number k^{-1} ($k\sim\omega_p/c$ for whistlers), with

$$\nu\sim\lambda^2,\quad c/\omega_p a\leqslant\lambda, \tag{3.2}$$

where λ is a small parameter.

We consider asymptotic solutions of (2.2) in the form

$$\begin{aligned}\mathscr{E}_1&=E(\lambda x,\lambda^2 z)\exp[i(\omega/c)(Qx+Pz)],\\ \mathscr{E}_2&=G(\lambda x,\lambda^2 z)\exp[i(\omega/c)(Qx+Pz)],\end{aligned} \tag{3.3}$$

where Q and P are parameters that will be determined below.

We seek now E and G in the form

$$E=E^0+\lambda E'+\lambda^2E''+\ldots,\quad G=G^0+\lambda G'+\lambda^2G''+\ldots. \tag{3.4}$$

We substitute (3.3) and (3.4) in (2.2) and equate terms of like order in λ, confining ourselves to terms up to λ^2 inclusive. Leaving out, for lack of space, the calculation details (they are described in Ref. 10), we present here only the final results. In zeroth order, the condition for solvability of the inhomogeneous system of equations for E^0 and G^0 yield for Q an expression that coincides with (2.8), if we make in the latter the substitutions $q\to Q$ and $p\to P$, and set $\nu=0$. Solving next the system for E^0 and G^0, we obtain

$$G^0=RE^0, \tag{3.5}$$

where R a polarization factor for which the final expressions are given below. We next obtain for E' and G', in first order in λ, a system of equations which can be solved if P, Q, and R have the following values:

$$P^2=\gamma^2/u(1-u),\quad R=0, \tag{3.6}$$

$$Q=Q_1=0\ (u<1/2),\quad Q=Q_2=0\ (u>1/2) \tag{3.7}$$

or

$$P^2=4\gamma^2,\quad R=(1+u)(1-2u)/(1-u)(1+2u), \tag{3.8}$$

$$Q^2=Q_1^2=Q_2^2=Q_0^2\equiv\gamma^2(1-4u^2)u^{-2}\quad (u<1/2). \tag{3.9}$$

We have thus two cases. In the first the polarization in the zeroth approximation is circular ($R=0$) and the wave vector is directed along the magnetic field.

In the second case $R\neq 0$, i.e., the wave is elliptically polarized even in the zeroth approximation, and the two branches merge and can propagate only at $u<\frac{1}{2}$, as seen from (3.9). As for the wave vector $\mathbf{k}$, its direction and magnitude in the zeroth approximations are given by

$$\cos\theta=P(Q^2+P^2)^{-1/2}=2u,\quad k=\omega_p/c. \tag{3.10}$$

We note that in both cases the group velocity $\mathbf{v}_g$ corresponding to the wave vector $\mathbf{k}$ is directed along the magnetic field, as follows directly from the general equation

$$\mathbf{v}_{g\perp}=\frac{\partial\omega}{\partial\mathbf{k}_\perp}=\frac{\omega_c k_\parallel \mathbf{k}_\perp(\omega_p^2/c^2-k^2)}{k(\omega_p^2/c^2+k^2)^2}. \qquad (3.11)$$

Finally, comparing the terms with λ^2, we obtain for E'' and G'' a system of equations, the solvability conditions for which can lead in both cases to a certain differential equation for E_0, which takes the Schrödinger-equation form:

$$iv_g\frac{\partial E^0}{\partial z}+\frac{1}{2}S\frac{\partial^2E^0}{\partial x^2}-(N-N_0)\left(\frac{\partial\omega}{\partial N_0}\right)_{\mathbf{k}=\mathbf{k}_0}E^0=0, \qquad (3.12)$$

$$v_g=\left(\frac{\partial\omega}{\partial k_\parallel}\right)_{\mathbf{k}=\mathbf{k}_0},\quad S=\left(\frac{\partial^2\omega}{\partial k_\perp^2}\right)_{\mathbf{k}=\mathbf{k}_0}, \qquad (3.13)$$

where the wave vector $\mathbf{k}_0$ is determined from the condition $v_{g\perp}(\mathbf{k}_0)=0$. It is clear from the preceding that this takes place either at $\theta=0$, i.e., under conditions (3.6) and (3.7), or else if (3.10) is satisfied, i.e., under conditions (3.8) and (3.9).

Thus, the Schrödinger equation (3.12) describes the diffraction of the wave beams in all cases when they propagate with a group velocity almost parallel to the external magnetic field.[1)] For the two characteristic directions of $\mathbf{k}_0$ we have

$$v_g=2u^{1/2}(1-u)^{1/2}c/\gamma,\quad S=(1-2u)(1-u)c^2/\gamma^2\omega_c\quad(\theta=0), \qquad (3.14)$$

$$v_g=c/2\gamma,\quad S=-(1-4u^2)uc^2/\gamma^2\omega_c\quad(\cos\theta=2u). \qquad (3.15)$$

Thus, in both cases $S\sim 1-2u$ as $u\to\frac{1}{2}$.

We have assumed so far that the medium is stationary and homogeneous along the external magnetic field. We can obtain similarly also a more general equation, when the wave amplitude, the density, and the average magnetic field have a weaker dependence on the coordinates and on the time, and the derivative is of the order of λ with respect to x and of λ^2 with respect to t and z. In this case, generally speaking the plasma must be assumed to move with velocity $\mathbf{V}$. A detailed analysis, which we omit here, shows that Eq. (3.12) is replaced here by

$$i\left(\frac{\partial E}{\partial t}+v_g\frac{\partial E}{\partial z}\right)+\frac{1}{2}S\frac{\partial^2E}{\partial x^2}-\Delta\omega E=0, \qquad (3.16)$$

where we write E in lieu of E^0; the quantities v_g and S in (3.16) are given as before by expressions (3.13), and

$$\Delta\omega=\nu N_0\left(\frac{\partial\omega}{\partial N_0}\right)_{\mathbf{k}=\mathbf{k}_0}+\mathbf{b}B_0\left(\frac{\partial\omega}{\partial\mathbf{B}_0}\right)_{\mathbf{k}=\mathbf{k}_0}+\mathbf{V}\left(\frac{\partial\omega}{\partial\mathbf{V}}\right)_{\mathbf{k}=\mathbf{k}_0}. \qquad (3.17)$$

Here N_0 and $\mathbf{B}_0$ are the constant values of the density and of the magnetic field at $|x|=\infty$ (the z axis is directed along $\mathbf{B}_0$), $\mathbf{b}=(\mathbf{B}-\mathbf{B}_0)/B_0$, and it is assumed that $b\sim\nu\sim V\sim\lambda^2$.

The derivatives $\partial\omega/\partial N_0$ and $\partial\omega/\partial\mathbf{B}_0$ are determined from (2.9). To calculate $\partial\omega/\partial\mathbf{V}$ we must start from the dispersion equation in a moving medium. In the questions considered here this term turns out to be inessential (see Sec. 4).

We shall call (3.16) the general nonstationary Schrödinger equation for whistlers propagating along the magnetic field. For $\theta=0$ it was considered earlier in many studies of the self-action and diffraction of whistlers, beginning with Refs. 7 and 8.

To cast light on how informative the generalized Schrödinger equation can be, we use it to consider stationary waveguide propagation of whistlers. We assume for simplicity that $b=0$ and that the relative variation of the density $\nu(x)$ is an even function with one extremum $\nu(0)\lessgtr 0$. Assuming that E is proportional to $\exp[i(\omega/c)\Delta pz]$, we obtain from (3.12), (3.14), and (3.15)

$$\frac{\partial^2E}{\partial x^2}+\rho^2(x;\Delta p)E=0, \qquad (3.18)$$

$$\rho^2=\varkappa[\nu(x)-2\Delta p/P], \qquad (3.19)$$

$$\varkappa=\frac{2\omega_p^2u}{(1-2u)c^2}\quad(\theta=0);\quad \varkappa=-\frac{\omega_p^2}{(1-4u^2)c^2}\quad(\cos\theta=2u). \qquad (3.20)$$

A discrete spectrum of Δp, which determines the waveguide modes, exists obviously only at $\varkappa\nu(0)>0$. It follows therefore that in the waveguides with crests

$$0<2\Delta p/P<\nu(0), \qquad (3.21)$$

and in the waveguides with troughs

$$\nu(0)<2\Delta p/P<0. \qquad (3.22)$$

The first of these conditions is realized at $\theta=0$ and $u<\frac{1}{2}$, and the second at $\theta=0$, $u>\frac{1}{2}$ and $\theta=\arccos 2u$, $u<\frac{1}{2}$. Thus, the value $u=\frac{1}{2}$ is critical: it corresponds to a transition to another waveguide propagation regime.

It follows from (3.18)–(3.20) that the general Schrödinger equation is valid only under the condition

$$|\nu(0)|\ll(1-2u)^2. \qquad (3.23)$$

Indeed, at $(1-2u)^2\sim|\nu(0)|$ it follows from (3.18)–(3.20) that $\partial E/\partial x\sim|\nu|^{1/4}$, whereas in the derivation of the Schrödinger equation it was assumed that $\partial E/\partial x\sim|\nu|^{1/2}$ [see (3.3)].

We continue the analysis by comparing the WKB solutions of Eq. (3.12)

$$E(x,z)=C\rho^{-1/2}(x)\exp\left\{i\left[\pm\int_0^x\rho(x)\,dx+\frac{\omega}{c}\Delta pz\right]\right\} \qquad (3.24)$$

with the WKB solutions (2.7) of Maxwell's equations. It is easy to verify that if we put in (2.7) and (2.8) $p=P+\Delta p$, where P corresponds to $\theta=0$ [see (3.6)], we find in first order in the small parameter $\nu/(1-2u)^2$ that at $\theta=0$

$$(\omega/c)q_1\approx\rho\quad(u<{}^1/_2);\qquad(\omega/c)q_2\approx\rho\quad(u>{}^1/_2). \qquad (3.25)$$

In the case $\theta=\arccos 2u$, putting $P=2\gamma$ [see (3.8)], we obtain in the same order of magnitude

$$(\omega/c)q_{1,2}\approx(\omega/c)Q_0\mp\rho, \qquad (3.26)$$

where Q_0 is defined in (3.9).

It follows therefore that the phase factors of the WKB solutions (3.24) [with allowance for (3.3) and (2.7)] are equal in first order of magnitude. It can also be shown that, to the same accuracy, the polarization vectors are equal in both solutions.

It must be noted that all the solutions of Maxwell's equations are contained in the generalized Schrödinger equation. This is seen, for example, from Figs. 1 and 2, where $q(x)$ is plotted for both values of θ that follow

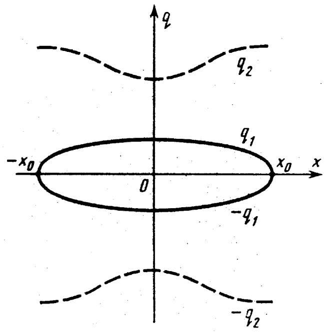

FIG. 1. Plots of $q_{1,2}(x)$ at $\omega < \omega_c/2$ for waveguides with $\nu(x) > 0$ $(\theta = 0)$. The solid lines are described by solutions of both the Schrödinger and the Maxwell equations; the dashed, only by the latter.

from (2.8). The plots that agree approximately with (3.25) and (3.26) are shown by solid lines, and the "extra" ones are dashed. Thus, at $\theta = 0$ and $u < \frac{1}{2}$ the Schrödinger equation contains no solution corresponding to the branch q_2, and at $\cos\theta = 2u$ $(u < \frac{1}{2})$ it contains no solution corresponding to the "dumbbell" of Fig. 2. At $u > \frac{1}{2}$ the Schrödinger equations yield approximately the same results as Maxwell's equations: $q_2(x) \approx \rho(x)(c/\omega)$; the $q_2(x)$ plot is an oval symmetric about the coordinate axes, and $q_1^2 < 0$.

The presence of the q_2 branch at $\nu(x) > 0$ leads to the possibility of the tunneling $q1 \to q2$ and to the ensuing damping of the q_1 branch trapped in the waveguide. The logarithmic damping decrement along the z axis is determined in the WKB approximation by the formula[2,4]

$$\mu = \frac{1}{2}\left(\int_{-x_0}^{x_0} \frac{v_{gz}(x)}{v_{gx}(x)}dx\right)^{-1} \exp\left[-2\frac{\omega}{c}\operatorname{Im}\int_0^{x_1}(q_2-q_1)dx\right], \tag{3.27}$$

where v_{gz} and v_{gx} are the group-velocity components, x_0 is the "turning" point $(q_1(x_0) = 0)$, and x_1 is the point closest to the real axis in the upper hemisphere, where $q_1(x_1) = q_2(x_1)$.

For the characteristic profile

$$\nu(x) = \nu_0 \operatorname{sech}^2(x/a)$$

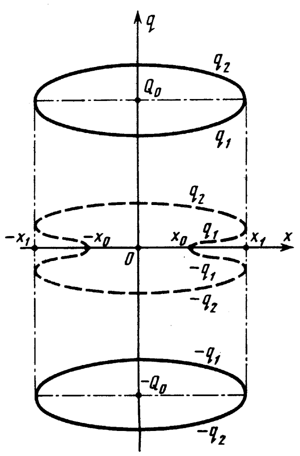

FIG. 2. Plots of $q_{1,2}(x)$ at $\omega < \omega_c/2$ for waveguides with $\nu(x) < 0$ $(\theta = \arccos(2\omega/\omega_c))$. The solid curves are realized at $(1-2u)^2 > |\nu(0)|$, and the dashed one at $(1-2u)^2 < |\nu(0)|$ and is obtained only from Maxwell's equations.

we have under condition (3.23)

$$\mu = \frac{1}{2\pi a}\left(\frac{\Delta p}{P}\right)^{1/2} T(\Delta p), \tag{3.28}$$

where $T(\Delta p)$ is the exponential in (3.27), calculated for the considered profile in Refs. 4 and 5. Equation (3.28) will be used in the next section to estimate the damping of a self-focused beam with $\nu(x) > 0$.

We note furthermore that from the general conditions of whistler waveguide propagation[5] it follows that at $\nu(x) < 0$ and $|\nu(x)| \ll 1$ a two-oval configuration is realized at frequencies

$$(1-2u)^2 > |\nu(0)|,$$

and a "dumbbell" type configuration at

$$(1-2u)^2 < |\nu(0)|.$$

Comparing this with (3.23), we see that as $u \to \frac{1}{2}$ the Schrödinger equation ceases to be valid before the transition from the ovals to the dumbbell takes place.

We note finally that the Schrödinger equation does not depend on the sign of Q; we can therefore choose as the complete solution any linear combination of expressions (3.3) with Q and $-Q$. To obtain best agreement between the results of the planar geometry considered here and the axial one of Ref. 6, we must assume that the solution is of the form

$$\begin{gathered}\mathscr{E}_1 = E(x,z,t)\exp[i(\omega/c)Pz]\{C_1\exp[i(\omega/c)Qx] \\ +C_2\exp[-i(\omega/c)Qx]\}, \quad \mathscr{E}_2 = R\mathscr{E}_1; \\ C_1 = C_2 = 1/2 \quad (Q=0); \qquad |C_1| = |C_2| = 2^{-1/2} \quad (Q = Q_0),\end{gathered} \tag{3.29}$$

where $E(x, z, t)$ is the solution of the generalized Schrödinger equation (3.16).

4. STATIONARY SOLUTIONS OF THE SELF-FOCUSING EQUATIONS

So far we have assumed that the plasma parameters (density, magnetic field) are given. In this section we consider the self-consistent problem, i.e., it is assumed that the plasma state averaged over the RF oscillations is altered by the ponderomotive force. In a collisionless plasma with small $\beta = 8\pi NT/B^2$ the general expression for this force per unit volume is[12]

$$\begin{gathered}\mathbf{f} = \frac{1}{16\pi}\{(\varepsilon_{ij}-\delta_{ij})\nabla(\mathscr{E}_i^*\mathscr{E}_j) + M_k\nabla B_k + [\mathbf{B}\times\operatorname{rot}\mathbf{M}]\} \\ + \frac{i}{16\pi\omega}\left\{\frac{\partial}{\partial t}[(\hat{\varepsilon}-I)\vec{\mathscr{E}}\times\operatorname{rot}\vec{\mathscr{E}}^*] + \omega\left[\frac{\partial\hat{\varepsilon}}{\partial\omega}\frac{\partial\vec{\mathscr{E}}}{\partial t}\times\operatorname{rot}\vec{\mathscr{E}}^*\right] + \text{c.c.}\right\}.\end{gathered} \tag{4.1}$$

Here $\vec{\mathscr{E}}$ is connected with the total electric field strength by Eq. (2.1). $\mathbf{M}$ is the density of the plasma magnetic moment induced by the RF field[13]:

$$\mathbf{M} = \frac{1}{16\pi}\frac{\partial\varepsilon_{ij}}{\partial\mathbf{B}}\mathscr{E}_i\mathscr{E}_j^*,$$

$\mathbf{B}$ is the averaged (over the RF oscillations) magnetic-induction vector, $\hat{\varepsilon}$ is the dielectric tensor, and $\hat{I}$ is a unit matrix.

We confine ourselves to self-focusing in the Schrödinger-equation approximation. We express $\mathscr{E}_x$ and $\mathscr{E}_y$ in terms of $\mathscr{E}_1$ and $\mathscr{E}_2$, assuming that the latter are given by (3.29), and obtain $\mathscr{E}_z$ from the equation $\operatorname{div}(\hat{\varepsilon}\vec{\mathscr{E}}) = 0$.

Substituting now $\mathscr{E}$ in (4.1) and taking (2.3) and (2.4) into account, we obtain for the ponderomotive-force components the expressions

$$f_x=-A_1\partial|E|^2/\partial x,\quad f_y=0,$$
$$f_z=A_2\frac{\partial|E|^2}{\partial z}+\frac{k\cos\theta}{\omega}\left\{A_2-\frac{1}{2}\operatorname{tg}^2\theta\left[A_1+\frac{\gamma^2R}{16\pi(1-u^2)}\right]\right\}\frac{\partial|E|^2}{\partial t},\quad (4.2)$$

where

$$A_1=\frac{\gamma^2}{16\pi(1-u)^2}\left[1+R^2\left(\frac{1-u}{1+u}\right)^2\right],\quad A_2=\frac{\gamma^2}{16\pi u(1-u)}\left(1-R^2\frac{1-u}{1+u}\right),\quad (4.3)$$

and R is determined by (3.6) or (3.8).

We now write the plasma equations of motion in the magnetohydrodynamic approximation with account taken of the ponderomotive force. Since it is assumed throughout that the relative deviation of the density $\nu(x)$ is small, we neglect terms of order ν^2. The equations that describe the plasma motion and the magnetic-field evolution take then the form[13]

$$\begin{gathered}\frac{\partial V_x}{\partial t}+c_s^2\frac{\partial\nu}{\partial x}+c_A^2\left(\frac{\partial b_z}{\partial x}-\frac{\partial b_x}{\partial z}\right)=\frac{f_x}{\rho_0},\\ \frac{\partial V_z}{\partial t}+c_s^2\frac{\partial\nu}{\partial z}=\frac{f_z}{\rho_0},\quad \frac{\partial\nu}{\partial t}+\frac{\partial V_x}{\partial x}+\frac{\partial V_z}{\partial z}=0,\\ \frac{\partial b_x}{\partial t}-\frac{\partial V_x}{\partial z}=0,\quad \frac{\partial b_z}{\partial t}+\frac{\partial V_x}{\partial x}=0;\\ c_s=(T_e/m_i)^{1/2},\quad c_A^2=B_0^2/4\pi\rho_0,\quad \rho_0=(m_i+m_e)N_0.\end{gathered}\quad (4.4)$$

Here $\mathbf{V}$ is the plasma velocity, c_s is the ion-sound speed, and ρ_0 is the plasma unperturbed mass density. The system (4.4) does not contain the projections of the basic equations on the y axis; the latter describe the pure Alfven branch, which does not interact with the magnetosonic motions described by the system (4.4), since $f_y=0$.

We consider now the class of motions of the stationary-wave type, when $|E|^2$ and all the sought quantities of the system (4.4) are functions of one independent variable η:

$$\eta=\varkappa_x x+\varkappa_z z-Ut,\quad \varkappa_x^2+\varkappa_z^2=1,\quad (4.5)$$

where $\varkappa_z$ and U are arbitrary parameters. The solution of the system (4.4) takes then the form

$$\frac{dV_x}{d\eta}=\frac{D_1}{D},\quad \frac{dV_z}{d\eta}=\frac{D_2}{D},\quad b_x=-\frac{\varkappa_z}{U}V_x,\quad b_z=\frac{\varkappa_x}{U}V_x;\quad (4.6)$$
$$\nu=(\varkappa_x V_x+\varkappa_z V_z)/U,\quad (4.7)$$

where D is the determinant of the corresponding algebraic system:

$$D=U^4-U^2(c_A^2+c_s^2)+\varkappa_z^2c_A^2c_s^2;\quad (4.8)$$
$$D_1=\frac{U}{\rho_0}[f_x(c_s^2\varkappa_z^2-U^2)-f_zc_s^2\varkappa_x\varkappa_z],$$
$$D_2=-\frac{U}{\rho_0}[f_x\varkappa_x\varkappa_zc_s^2+f_z(U^2-c_A^2-c_s^2\varkappa_x^2)].\quad (4.9)$$

The equation $D=0$ is equivalent to the dispersion equation of the free magnetosonic oscillations. In the limiting case $\beta=c_s^2/c_A^2\ll1$ we obtain from it the following relations for the velocities U_1 and U_2 of the fast and slow sound:

$$U_1\approx c_A,\quad U_2\approx c_s\varkappa_z.\quad (4.10)$$

We consider now those solutions of the system (4.4) which do not depend on t and z (they correspond to ducts oriented along the external magnetic field). To this end it is necessary to put formally $U\to0$ and $\varkappa_z\to0$ in (4.6)–(4.9). We encounter here, however, an ambiguity that depends on the ratio $U/\varkappa_z$. At different values of this ratio we obtain, generally speaking, different limiting solutions. It is important that this circumstance is preserved also in the case when the left-hand sides of the MHD equations are not linearized (as can be easily verified). This phenomenon is caused by singularities in the behavior of the characteristics of the MHD equations and will be discussed in detail elsewhere.

For the present, confining ourselves to the study of solutions that depend only on η, we assume that $\varkappa_z\to0$ and

$$U=\sigma c_s\varkappa_z,\quad (4.11)$$

where σ is an arbitrary non-negative parameter, with $\sigma\neq1$ [to avoid resonance with the slow magnetosonic waves—see (4.10)].

Substituting (4.11) and (4.2) in (4.6)–(4.9) and then letting $\varkappa_z\to0$, we obtain

$$\nu=\frac{\sigma^2c_s^2A_1+c_A^2A_2}{c_A^2c_s^2(1-\sigma^2)\rho_0}|E|^2,\quad (4.12)$$
$$b_z=-\frac{A_1(1-\sigma^2)+A_2}{c_A^2(1-\sigma^2)\rho_0}|E|^2,\quad b_x=0,\quad (4.13)$$
$$V_z=\frac{\sigma(c_s^2A_1+c_A^2A_2)}{c_A^2c_s(1-\sigma^2)\rho_0}|E|^2,\quad V_x=0.\quad (4.14)$$

From these expressions we get

$$\nu=\frac{A_2|E|^2}{(1-\sigma^2)c_s^2\rho_0},\quad b_z\sim\frac{c_s^2}{c_A^2}\nu\ll\nu\quad\left(\sigma\ll\frac{c_A}{c_s}\right),\quad (4.15)$$
$$\nu=b_z=-\frac{A_1|E|^2}{\rho_0c_A^2}\quad\left(\sigma\gg\frac{c_A}{c_s}\right),\quad (4.16)$$

and it is seen from (4.3) that $A_1>0$ and $A_2>0$. Equations (4.15) and (4.16) are quite general, since $c_A/c_s\gg1$. It is useful to note in this connection that (4.16) can be obtained by assuming that $\sigma\to\infty$ but $\sigma\varkappa_z\to0$, such that $U\to0$.

The expressions obtained allow us to express $\Delta\omega$ of (3.17) in terms of $|E|^2$. An analysis of the dispersion equation for a moving plasma[10] shows that

$$(\partial\omega/\partial\mathbf{V})\mathbf{V}\sim k_zV_z.$$

Using (4.14), we can easily verify that this term is negligibly small compared with the remaining terms in (3.17). As a result, using (2.9) and recognizing that $b_x=0$, we can write

$$\Delta\omega\approx-\omega(\omega_c\cos\theta-\omega)\nu/\omega_c\cos\theta+\omega b_z.\quad (4.17)$$

Substituting (4.17) together with (4.15) and (4.16) in (3.16), we obtain a closed equation for the field E. The stationary solutions of this equation, which are proportional to $\exp[i(\omega/c)\Delta pz]$, describe the homogeneous whistlers that result from the self-focusing.

These solutions together with the corresponding expressions for the variation of the density are of the form

$$E=E_0\operatorname{sech}(x/a)\exp[i(\omega/c)\Delta pz],\quad \nu=\nu_0\operatorname{sech}^2(x/a),\quad (4.18)$$
$$a^2=S\omega_c\cos\theta/\nu_0\omega(\omega_c\cos\theta-\omega),\quad \Delta p=S/2v_ga^2,\quad (4.19)$$

wherein ν is connected with E by (4.15) or (4.16). We have thus obtained an entire family of stationary solution of a system that consists of the generalized Schrödinger equation and the hydrodynamics equations; this solution depends on the parameter σ. The requirement $a^2>0$ and Eqs. (4.15) and (4.16) lead to the conditions

$$\operatorname{sign} S=\operatorname{sign} \nu_0=\operatorname{sign}(1-\sigma^2). \tag{4.20}$$

It is seen from (3.15) that

$$(1-2u)S>0 \quad (\theta=0); \quad (1-2u)S<0 \quad (\cos\theta=2u). \tag{4.21}$$

We conclude from (4.20) and (4.21) that at $\omega<\omega_c/2$ the self-focusing can lead to formation of channels with both crests ($\nu>0$) and troughs ($\nu<0$). In the former case $\theta=0$ and $\sigma<1$, and in the latter $\cos\theta=2\omega/\omega_c$, $\sigma>1$. At $\omega>\omega_c/2$ the second relation of (4.21) is not realized, therefore self-focusing at $\omega>\omega_c/2$ can lead only to troughs; in this case $\theta=0$ and $\sigma>1$.

The solution obtained earlier in Refs. 8 and 9 is obtained at $\sigma=0$; it describes the beam in the crest at $u<\frac{1}{2}$. We see that it covers by far not all possibilities.

It must next be kept in mind that at $\nu>0$ (i.e., $\sigma<1$) the electromagnetic field should gradually "leak out" of the waveguide because of the tunnel-transformation effect. As a result, the amplitude should attenuate along the z axis with a logarithmic decrement μ, which is determined in the WKB approximation by Eq. (3.28) under the condition (3.23). For the soliton solutions (4.18) and (4.19) the condition (3.23) takes the form

$$1-2u\gg c^2/\omega_p{}^2a^2u, \tag{4.22}$$

and the condition for the applicability of the WKB approximation is not satisfied. Equation (3.28) can be used in this case, however, for an order of magnitude estimate, by substituting in it Δp from (4.19) and T from Refs. 4–6 at $p=P+\Delta p$. As shown in the cited references,

$$T\geqslant\exp\left\{-\frac{\pi\omega_p a}{c}\left[\frac{1-2u}{u(1-u)}\right]^{1/2}(1+\nu_0)\right\}. \tag{4.23}$$

From (3.28), (4.19), and (4.23), with allowance for (4.22) we obtain the following estimate for μ:

$$\mu\gg\frac{\pi}{4a}\left(\frac{c}{\pi a\omega_p}\right)^2\exp\left\{-\frac{\pi\omega_p a}{c}\left[\frac{1-2u}{u(1-u)}\right]^{1/2}\right\}. \tag{4.24}$$

If the condition (4.23) is not satisfied, the $\gg$ symbol must be replaced by $\gtrsim$.

These results remain qualitatively valid also for axisymmetric solutions. However, an additional investigation, which we omit here, shows that the axisymmetric analogs of the solutions (4.18) and (4.19) are unstable to self-compression (at least for channels with $\nu>0$). The compression stops when the duct width becomes of the order of the longitudinal wavelength, i.e., $a\sim c/\omega_p$. It is seen from (4.24) that in this case the decrement μ becomes large ($\mu\sim a^{-1}$), i.e., the waveguide "leaks out," so that at $\nu>0$ the self-focusing ceases before a stationary (or quasi-stationary) duct is produced, in qualitative agreement with experiment.[15]

On the basis of the foregoing, it seems to us that the solutions of principal interest are those with $\sigma>1$, which describe wave beams in troughs. They exist both at $\omega<\omega_c/2(\cos\theta=2\omega/\omega_c)$ and at $\omega>\omega_c/2(\theta=0)$, in agreement with experiment. It is possible that an investigation of the stability of these solutions will lead to further restrictions on the values of the parameter σ. This question is under study at present.

[1)] The fact that the group velocity is parallel to the magnetic field not only at $\theta=0$ but also at $\theta=\arccos 2u$ is clearly seen from the geometric optics of whistlers (see, e.g., Ref. 11). It is natural therefore that we arrive at this face when deriving the Schrödinger equation from first principles.

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