Theory of small-scale magnetic fields

S. A. Vaĭnshtein

Institute of Terrestrial Magnetism, Ionosphere, and Radiowave Propagation, Siberian Division, USSR Academy of Sciences

(Submitted 5 March 1982)
Zh. Eksp. Teor. Fiz. **83**, 161–175 (July 1982)

An equation for the correlation tensor of the magnetic field in a high-conductivity turbulent liquid is obtained on the basis of the Lagrangian approach. This equation is analyzed for all presently known types of turbulence. It is shown that an exponential growth of the magnetic fluctuations (a turbulent dynamo) takes place in a Kolmogorov turbulence at high magnetic Reynolds numbers. The relation between the kinematic viscosity and the diffusion coefficient of the magnetic field is of no significance in this problem. An analysis of few-mode turbulence shows that a field is generated in this turbulence, too. Finally, a concrete form of the equation for the magnetic-field correlation tensor is obtained for turbulence with intermittence and it is shown that the latter also gives rise to the turbulent-dynamo effect.

PACS numbers: 47.25.Jn, 05.40. + j

Magnetic-field generation under astrophysical conditions is among the most pressing problems. There are many so-called laminar models, wherein the generation is effected by simple motion of a conducting plasma. More extensively used in applications is development of methods of field generation in a turbulent medium. The turbulence is characterized by an external scale l. The magnetic field is called large-scale if the averaged component $\langle \mathbf{H} \rangle$ does not vanish and in this case the field scale $L \gg l$. The dynamics of a large-scale field has by now been relatively well investigated. An exact theory that describes the field $\langle \mathbf{H} \rangle$ has been developed. However, even without an exact theory, the main dynamic processes in a large-scale field can be obtained from dimensionality considerations (just as the macroscopic equations are deduced heuristically in electrodynamics), and are therefore subject to no doubt.

Small-scale magnetic fields (SSMF) are defined as those whose scale is comparable with or smaller than l. Even though turbulence theory started out precisely with the consideration of SSMF,[1] the theory subsequently developed dealt almost exclusively with large-scale fields. The point is that only an exact SSMF theory could be developed, and neither approximate nor heuristic approaches could prove anything in this case.[2] Likewise, nothing was proved by the analogy between the (nonlinear) equation for curl $\mathbf{v}$ ($\mathbf{v}$ is the velocity) and the (linear) equation proposed for H in Ref. 1. The difficulties of the problem were recognized following the publication of Ref. 2. We indicate here the principal ones. The dynamics of a field of a given scale (smaller than l) is determined by two factors. First, the field is generated in this scale; second, its scale decreases and as a result the energy of a field of a given scale decreases. The rates of these two competing processes are of the same order of magnitude, and the theory must determine which of these processes prevails. These velocities, of course, do not depend on the viscosity ν and on the electric conductivity σ at the scales of interest to us, where the freezing-in of the magnetic field in the moving plasma is substantial. Since there was no exact theory until recently, the question of which of the two competing processes predominates remained open. In other words, the principal question of the SSMF theory, whether a turbulent medium generates SSMF, remained unanswered.

The development of an exact theory of large-scale fields[3,4] made it possible to study the SSMF. A description of the dynamics of SSMF was first presented for a highly conducting plasma (large magnetic Reynolds numbers $R_m \gg R$, where R is the Reynolds number).[5,6] In essence, the problem of the turbulent dynamo was by the same token solved, since, as already stated, the rates of the two competing processes are independent of the electric conductivity. Nonetheless, a rigorous consideration of the case $R_m \leqslant R$ was necessary.[7] The results of the two cases $R_m \gg R \gg 1$ and $R > R_m \gg 1$, as expected, coincide: in both cases the turbulent liquid generates a field. Nonetheless, no complete theory was developed in Refs. 5–7. In the present article we develop a consistent SSMF theory that allow us to consider a much larger class of turbulent motions than heretofore. In particular, we consider below turbulence with few modes (of the strange-attractor type) and turbulence with intermittence, as generators of magnetic fields.

§1. SSMF DYNAMICS EQUATIONS

SSMF dynamics in a highly conducting medium (in which the freezing-in condition is satisfied) is defined by the exact solution

$$H_i({}^1\mathbf{x}, t) = \frac{\partial\, {}^1x_i}{\partial\, {}^1a_j} H_j({}^1\mathbf{a}, 0) = \lim_{{}^2\mathbf{a} \to {}^1\mathbf{a}} \frac{{}^2x_i - {}^1x_i}{{}^2a_j - {}^1a_j} H_j({}^1\mathbf{a}, 0). \tag{1}$$

Here ${}^1\mathbf{x}$ and ${}^2\mathbf{x}$ are the coordinates, at the instant of time t, of the liquid particles that have emerged from the points ${}^1\mathbf{a}$ and ${}^2\mathbf{a}$ at the instant $t = 0$. We confine ourselves to a homogeneous, isotropic, reflection-invariant turbulence. In addition, the first moments are equal to zero: $\langle \mathbf{v} \rangle = 0$ and $\langle \mathbf{H} \rangle = 0$. The correlation-tensor time evolution is described by the expression

$$B_{ij} = \langle H_i({}^1\mathbf{x}, t) H_j({}^3\mathbf{x}, t) \rangle = \int \lim_{\substack{{}^2\mathbf{a} \to {}^1\mathbf{a} \\ {}^4\mathbf{a} \to {}^3\mathbf{a}}} \frac{{}^2x_i - {}^1x_i}{{}^2a_m - {}^1a_m} \frac{{}^4x_j - {}^3x_j}{{}^4a_n - {}^3a_n} \times p({}^\alpha\mathbf{x} | {}^\beta\mathbf{a}, t) B_{mn}({}^1\mathbf{a}, {}^3\mathbf{a}, 0)\, d\,{}^1\mathbf{a}\, d\,{}^3\mathbf{a}\, d\,{}^2\mathbf{x}\, d\,{}^4\mathbf{x}. \tag{2}$$

We have introduced here the distribution function $p({}^\alpha\mathbf{x} | {}^\beta\mathbf{a}, t)$ (the Greek indices run through the values 1, 2, 3, and 4), namely the density of the probability of finding the liquid particles at the point ${}^{\alpha}\mathbf{x}$ under the condition that they were located at the point ${}^{\alpha}\mathbf{a}$ at $t=0$. Equation (2) incorporates averaging over the initial coordinates of the liquid particles (integration with respect to ${}^{1}\mathbf{a}$ and ${}^{3}\mathbf{a}$), over close trajectories (integration with respect to ${}^{2}\mathbf{x}$ and ${}^{4}\mathbf{x}$) and over the initial distribution of the magnetic field. The latter is assumed to be uncorrelated at $t=0$ with the velocity field. This is why B_{mn} is separated in (2), and for the same reason expression (2) should be considered at $t\gg\tau$, where τ is the "memory" time (in the present case τ is the correlation time), when the system "forgets" the initial data and goes into a universal regime.

In the kinematic formulation of the dynamo problem, to which we confine ourselves, the motion of the conducting liquid is assumed given. In this case it is necessary to specify the function p. For real turbulence, however, the concrete form of this distribution function is unknown, it contains a large number of variables. It was found to be much simpler to specify not the distribution function itself but the equation for it. In the next section it will be shown that in the general case the equation for p takes at $t\gg\tau$ the form

$$\partial p/\partial t={}^{\alpha}\partial_i{}^{\beta}\partial_j T_{ij}({}^{\alpha}\mathbf{x},{}^{\beta}\mathbf{x})\,p \tag{3}$$

(we sum over repeated Greek and Latin indices). The tensor T_{ij} has the properties of the correlation tensor of the solenoidal stationary field.

Using (3), we can change from (2) to an equation for B_{ij}. To this end we differentiate B_{ij} with respect to t. In the right-hand side of (2), only the distribution function depends on t. We replace $\partial p/\partial t$ in accord with (3). It is next necessary to take all the differential operators ${}^{\alpha}\partial_i$ outside the limit sign and reduce all the equations to a form that coincides with the right-hand side of (2), on which the operators ${}^{\alpha}\partial_i$ act in addition. Cumbersome but straightforward calculations (see Ref. 5) lead to the following equation for B_{ij}:

$$\partial B_{ij}/\partial t=\sum_{\alpha,\beta}^{2}{}^{\alpha}\varepsilon_{iac}{}^{\beta}\varepsilon_{jbd}T_{ab}({}^{\alpha}\mathbf{x},{}^{\beta}\mathbf{x})B_{cd},$$
$${}^{\alpha}\varepsilon_{iac}=\varepsilon_{ijm}\varepsilon_{mac}{}^{\alpha}\partial_j. \tag{4}$$

Equation (4) follows directly from the exact solution (1) [with allowance for (3)], which is valid, as stated, when the freezing-in condition is satisfied. It is necessary here to refine this condition and make it more specific. In the most general case it takes the form

$$v_L L/D\gg 1, \tag{5}$$

where L is the SSMF scale, $D=c^2/4\pi\sigma$, v_L is the mean squared value of the velocity of the scale L, and $L\le l$ in accord with the definition of the SSMF. In a real turbulence v_L decreases with decreasing L (at $L<l$). For a Kolmogorov turbulence, requirement (5) leads to the condition

$$l/\min\{R^{1/4}R_m^{1/2},R_m^{3/4}\}\ll L<l,$$
$$R_m=vl/D,\quad R=vl/\nu. \tag{6}$$

Equation (6) contains two limiting scales. This is due to the presence of two regimes, $R\gg R_m\gg 1$ and $R_m\gg R\gg 1$. In the first regime, the ohmic dissipation becomes substantial even in the inertial region, while in the second the dissipation comes into play at scales much smaller than the viscous ones.

In the general case of a power-law (not necessarily Kolmogorov) turbulence spectrum, condition (6) changes but (5) remains the same. In the inertial region, where neither viscosity nor ohmic diffusion is significant, the tensor equation (4) can be transformed into the differential equation

$$\frac{1}{2}\frac{\partial B_{LL}}{\partial t}=T_\alpha r^{\alpha-2}\left[r^2\frac{\partial^2}{\partial r^2}+(\alpha+4)r\frac{\partial}{\partial r}+(\alpha^2+3\alpha)\right]B_{LL}. \tag{7}$$

Here B_{LL} is the longitudinal correlation function (see Ref. 8) of the magnetic field: it is a function of the modulus $r=|{}^{3}\mathbf{x}-{}^{1}\mathbf{x}|$ of the distance between the points and of the time t. The coefficient T_α is determined from the longitudinal correlation function $T_{LL}(r)$, which is obtained from the tensor T_{ij},

$$T_{LL}=A-T_\alpha r^\alpha, \tag{8}$$

and A is a constant. The tensor T_{ij}, just as the function T_{LL}, has the dimension of the diffusion coefficient; the exponent α can be determined from dimensionality considerations. For a kolmogorov turbulence we have $\alpha=4/3$ (the Richardson "4/3 law"[8]). For a turbulence with intermittence we have $\alpha>4/3$ (see §5). The exact values of A and T_α are unknown, and in the theory developed below it suffices to know their order of magnitude: $A=vl$, $T_\alpha=vl/l^\alpha$.

The region of applicability of (7) for a Kolmogorov turbulence is

$$l/\min\{R^{3/4},R_m^{3/4}\}<r<l. \tag{9}$$

At $R_m>R$ the region (9) coincides with the inertial region, while at $R_m<R$ the lower limit of r is determined by the ohmic damping. We shall discuss in greater detail the case $R_m>R$. The region (6) includes in this case the region (9). In the interval

$$lR^{-1/4}R_m^{-1/2}\ll L<lR^{-3/4}=l_1, \tag{10}$$

according to (6), the frozen-in condition is satisfied, and there are practically no velocity pulsations because of the viscosity. At such small values $r<l_1$ it is necessary to use in place of (8) the expansion

$$T_{ij}(r)=T\delta_{ij}-T_2\left(r^2\delta_{ij}-\frac{1}{2}r_ir_j\right),\qquad T,T_2>0. \tag{11}$$

In order of magnitude $T=vl$ and $T_2=R^{1/2}v/l$, where T_2 is the reciprocal lifetime of the smallest turbulence pulsations, whose scale is determined by the viscosity (and is equal to l_1 according to (10)). Substituting (11) in (4) we obtain the equation

$$\frac{1}{2}\frac{\partial}{\partial t}B_{ii}=T_2\left(\frac{1}{2}r^2\frac{\partial^2}{\partial r^2}+3r\frac{\partial}{\partial r}+5\right)B_{ii}, \tag{12}$$

which can be obtained from (7) by putting $\alpha=2$ and $T_\alpha=T_2/2$.

Equation (7) is thus a universal description of the behavior of the SSMF. In the region (9), for a Kolmogorov turbulence, we must put $\alpha=\frac{4}{3}$ in (7). In the scale region (10) (which exists only at $R_m>R$), Eq. (7) is valid and $\alpha=2$.

§2. DERIVATION OF EQUATION FOR THE DISTRIBUTION FUNCTION

In the general case the equation for p can be written in the form

$$\partial p/\partial t=\hat{L}p. \tag{13}$$

The problem is to determine the operator $\hat{L}$. We derive first an equation for the two-point distribution function p_2:

$$p_2({}^1\mathbf{x},{}^2\mathbf{x}|{}^1\mathbf{a},{}^2\mathbf{a},t)=\int pd^3\mathbf{x}d^4\mathbf{x}=\int pd^3\mathbf{a}d^4\mathbf{a},$$
$$\partial p_2/\partial t=\hat{L}_2p_2. \tag{14}$$

It is useful to compare the dynamics of p_2 with the correlation function of the scalar admixture $\Theta_2({}^1\mathbf{x},{}^2\mathbf{x},t)=\langle\theta({}^1\mathbf{x})\theta'({}^2\mathbf{x})\rangle$. The functions θ and θ' satisfy the equation

$$\frac{\partial}{\partial t}\theta=-\partial_jv_j\theta+\chi\Delta\theta,\quad \operatorname{div}\mathbf{v}=0, \tag{15}$$

in which χ is assumed to be very small. The function Θ_2 is uniquely determined from p_2 (see Ref. 5). On the other hand, from the equation for Θ_2 we obtain an equation p_2. Indeed, from the linearity of (15) follows a linear equation for Θ_2:

$$\frac{\partial}{\partial t}\Theta_2({}^1\mathbf{x},{}^2\mathbf{x})=\int K({}^1\mathbf{x},{}^2\mathbf{x},\mathbf{y},\mathbf{y}',t)\Theta_2(\mathbf{y},\mathbf{y}')\,dyd\mathbf{y}'. \tag{16}$$

The solution of (16) with initial admixture density

$$\theta(y)=\delta(y-{}^1a),\quad \theta'(y)=\delta(y-{}^2a)$$

is none other than $p_2({}^1\mathbf{x},{}^2\mathbf{x}|{}^1\mathbf{a},{}^2\mathbf{a},t)$ (by definition). At $t\gg\tau$ the kernel of (16) does not depend on t in stationary turbulence:

$$K(t\gg\tau)\to K({}^1\mathbf{x},{}^2\mathbf{x},\mathbf{y},\mathbf{y}',\infty).$$

Consequently, the equation for p_2 can be written in the form (16), and K is independent of the time at $t\gg\tau$.

It follows from (15) that $\theta(\mathbf{x},t+\Delta t)$ (Δt is small) depends on the first and second spatial derivatives at the point x and at the instant t, and the function $\theta(\mathbf{x},t+\Delta t)$ "knows nothing" of the points far from $\mathbf{x}$. Accordingly, the same (locality) property should apply also to Θ_2. This means that the operator in the right-hand side of (16) is a differential one, and consequently also $\hat{L}_2$ in (14) is a differential operator that depends only on ${}^1\mathbf{x}$ and ${}^2\mathbf{x}$ (since there is no dependence on ${}^1\mathbf{a}$ or ${}^2\mathbf{a}$ in (16)). Using now the exact solution of (15) as $\chi\to 0$: $\theta({}^1x,t)=\theta({}^1a,0)$ [cf. (1)], we obtain an expression for the autocorrelation function $\Theta_2'=\langle\theta({}^1\mathbf{x})\theta({}^2\mathbf{x})\rangle$—by analogy with (2), Changing now to the equation for Θ_2' (in analogy with Eq. (4) for B_{ij}), i.e., taking the derivative of Θ_2' with respect to t and using (14), we obtain for Θ_2', in place of (4), an equation that coincides exactly with (14). Thus, Eq. (14) is satisfied not only by p_2 but also by Θ_2 and Θ_2'.

Everything stated above in this section imposes such stringent limitations on the operator $\hat{L}_2$ in (14), that its form is determined without difficulty. We note first the ensuing consequences. First, the operator L_2 can be represented as an expansion in powers of the operator ${}^\alpha\partial_i$, and the expansion is confined to a sum of a finite number of terms (otherwise $\hat{L}_2$ is no longer a differential operator but is in fact integral). Second, the operator $\hat{L}_2$ is self-adjoint (see Ref. 5). The expansion contains therefore only even powers of the operator ${}^\alpha\partial_i$. Third, in the case of homogeneous turbulence the dependence of $\hat{L}_2$ on ${}^1\mathbf{x}$ and ${}^2\mathbf{x}$ should enter in the form $\mathbf{r}={}^2\mathbf{x}-{}^1\mathbf{x}$. Fourth, the operator $\hat{L}_2$ should be such that Eq. (14) preserves the properties

$$\begin{aligned}
&p_1({}^1\mathbf{x}|{}^1\mathbf{a},t)=\int p_2d^2\mathbf{x}=\int p_2d^2\mathbf{a},\\
&p_1({}^2\mathbf{x}|{}^2\mathbf{a},t)=\int p_2d^1\mathbf{x}=\int p_2d^1\mathbf{a},\\
&p_2({}^1\mathbf{x}={}^2\mathbf{x}|{}^1\mathbf{a},{}^2\mathbf{a},t)\sim\delta({}^1a-{}^2a),\\
&1=\int p_1({}^1\mathbf{x}|{}^1\mathbf{a},t)d^1\mathbf{a}=\int p_1d^1\mathbf{x}.
\end{aligned} \tag{17}$$

From the last of these equations it follows that $\hat{L}_2$ does not contain terms of zeroth power of ${}^\alpha\partial_i$. Allowance for all four corollaries leads in the operator $\hat{L}$ to a second-order term that coincides with the right-hand side of (3), where $\alpha,\beta=1,2$. (The equation for p_2 is in fact obtained from (3) by integrating with respect to ${}^3\mathbf{x}$ and ${}^4\mathbf{x}$ or with respect to ${}^3\mathbf{a}$ and ${}^4\mathbf{a}$; as a result we have an equation of the same form as (3), but with $\alpha,\beta=1,2$.)

To continue the analysis we need additional corollaries of the foregoing properties of the operator $\hat{L}_2$. We shall regard (14) as an equation for Θ_2'. It should retain the properties of the correlation function Θ_2'. This means that the form

$$F=\int C({}^1\mathbf{x})C^*({}^2\mathbf{x})\Theta_2'({}^1\mathbf{x},{}^2\mathbf{x})\,d^1\mathbf{x}d^2\mathbf{x}$$

should remain non-negative in the course of the evolution of Θ_2'. To verify this property we assume that at the instant $t=t_0\gg\tau$ we have

$$\Theta_2'=\theta'({}^1\mathbf{x})\theta'({}^2\mathbf{x}),\quad C(\mathbf{x})=C_1\delta(\mathbf{x}-\mathbf{x}_1)+C_2\delta(\mathbf{x}-\mathbf{x}_2),$$

and $F=0$. The operator $\hat{L}_2$ should be such that $\partial F/\partial t\geq 0$ at $t=t_0$, otherwise the form F becomes negative. This is the fifth corollary, and with its aid we verify that the tensor T_{ij} in (3) satisfies the properties of a correlation tensor. Finally, the sixth corollary. The solution of Eq. (15) preserves the minimum of θ, i.e., if $\theta\geq\theta(x_0)$ at $t=t_0$, then $\theta\geq\theta_0(x_0)$ at $t\geq t_0$. For Θ_2' we have similarly: if $\Theta_2'>\Theta_2'(x_0,x_0)$ at $t=t_0$, then $\Theta_2'\geq\Theta_2'(x_0,x_0)$ at $t\geq t_0$. It follows therefore that terms of fourth (and higher) order of the type $T^1\partial_i^1\partial_j^1\partial_f^1\partial_m p_2$ are absent from the operator $\hat{L}_2'$: at an arbitrary fourth derivative of the function θ' at the point $\mathbf{x}_0$ they would cause violation of the sixth corollary.

The only possible form of the fourth-order term in $\hat{L}_2$ is

$${}^1\partial_i{}^2\partial_jT_{ijfm}(\mathbf{r})\,{}^1\partial_f{}^2\partial_mp_2,$$

and does not contradict the first four corollaries. This term generates in turn second-derivative terms that act on p_2 (and were analyzed above), third-derivative terms that do not make up a positive-definite form and can cause the form F to become negative, meaning that they are not contained in the operator $\hat{L}_2$, and finally, a term in which all the four operators ${}^\alpha\partial_i$ act on p_2. But even this term is absent from $\hat{L}_2$. To verify this, it suffices to compare the form F with the function $C(\mathbf{x})$ of the type written out above. It is absolutely necessary here to use (17) (the third equation), from which it follows that $T_{ijfm}(\mathbf{r}=0)=0$. If $T_{ijfm}(\mathbf{r})$ were to have the properties of a correlation tensor, the fourth-order above term could be contained in $\hat{L}_2$. The entire point is that the vanishing of T_{ijfm} at $\mathbf{r}=0$ is not compatible with the properties of a correlation tensor. An analysis of terms of sixth and higher order is perfectly analogous to the foregoing: they must not be contained in the representation of the operator $\hat{L}_2$. The general conclusion is that Eq. (14) coincides with (3) in which $\alpha,\beta=1,2$.

We proceed now to an analysis of the general expression (13) for a four-point distribution function p. The properties of the differential operator $\hat{L}$ are determined in analogy with the properties of $\hat{L}_2$. All six foregoing corollaries (with the exception of the third) can be generalized to the four-dimensional case without any difficulty. The form F is modified in this case into

$$F=\int C({}^1\mathrm{x})\,C^*({}^2\mathrm{x})\,C({}^3\mathrm{x})\,C^*({}^4\mathrm{x})\,\Theta_4({}^1\mathrm{x},{}^2\mathrm{x},{}^3\mathrm{x},{}^4\mathrm{x})\,d^1\mathrm{x}\,d^2\mathrm{x}\,d^3\mathrm{x}\,d^4\mathrm{x},$$

$$C(\mathrm{x})=C_1\delta(\mathrm{x}-\mathrm{x}_1)+C_2\delta(\mathrm{x}-\mathrm{x}_2)+C_3\delta(\mathrm{x}-\mathrm{x}_3)+C_4\delta(\mathrm{x}-\mathrm{x}_4),$$

$$\Theta_4=\Theta'({}^1\mathrm{x})\Theta'({}^2\mathrm{x})\Theta'({}^3\mathrm{x})\Theta'({}^4\mathrm{x})\quad\text{for } t=t_0.$$

Analyzing the terms of fourth (and higher) power of the operator ${}^\alpha\partial_i$, we can show that their presence in the representation of the operator $\hat{L}$ leads inevitably to a negative form F if $F=0$ at $t=t_0$.

Let us summarize the present section. The locality property leads to a differential form of the equation for p. Numerous restrictions, mainly connected with the fact that the equation for p must preserve the properties of correlation functions, lead to the result that the operator L cannot contain derivatives of order higher than the second. Therefore the general form of the equation for p coincides with Eq. (3).

§3. GENERATION OF SSMF IN THE PRESENCE OF KOLMOGOROV TURBULENCE

All the previously considered particular cases[5-7] can be formulated and easily generalized in the general theory with the aid of the universal equation (7). For a Kolmogorov turbulence, the region of applicability of Eq. (7) is (9) and always exists. We do not know the equation for B_{LL} at arbitrary r. We can only say that in the general case it reduces to a self-adjoint form (see Ref. 5), it coincides with (7) in the region (9) and in the region of extremely small scales r, where damping manifests itself, the equation for B_{LL} is

$$\frac{\partial}{\partial t}B_{LL}=D\left(\frac{\partial^2}{\partial r^2}+4\frac{1}{r}\frac{\partial}{\partial r}\right)B_{LL}.$$

For intermediate scales, the equation for B_{LL} is unknown (with the exception of the particular case $R_m \gg R$, which will be discussed in §4). Finite dissipation ($D\neq 0$) eliminates the singularity from the equation B_{LL}, namely the vanishing of the coefficient of the senior derivative in (7). The eigenfunction problem

$$B_{LL}=B(r)e^{-Et}$$

of the general equation for B_{LL} is therefore meaningful. Since the operator is self-adjoint, all the eigenvalues E_n are real. The existence of $E_n<0$ (the analog of bound states in a sufficiently deep "well" of the Schrödinger equation) corresponds to an exponentially growing solution. In our case the analog of the potential of the Schrödinger equation is not known in all of space: it is known in the region (9) and for extremely small r. The question is whether the potential can "capture particles" into a bound state in the region (9). To obtain the answer to this question, a variational principle was used in Refs. 5–7. We describe here a simpler and more illustrative method, with which we shall operate hereafter.

Assume that there are no growing solutions. Then all $E_n>0$. We estimate the lowest eigenvalue E_0. We note for this purpose that the characteristic frequencies of the problem vary in a Kolmogorov turbulence from v/l (the reciprocal lifetime of the largest pulsations) to $\min\{R^{1/2},R_m^{1/2}\}v/l$ (the reciprocal time of rotation of the cell with the smallest scale). It is obvious that E_0 is of the order of (or less than) the lower frequency v/l; otherwise the dynamic solution, represented in the form of the eigensolutions with initial SSMF of scale l would describe too rapid a damping of the field. In fact, such a field changes within a time l/v. Replacing $\partial/\partial t$ in (7) by $-E_0$ and using the estimate of T_α (see §1) we see easily that the left-hand side of (7) is negligibly small compared with the right. Therefore the eigensolution can be sought in the form $B_0(r)\sim r^\beta$ by setting the right-hand side to zero. The result for β is

$$\beta={}^1/_2\{-(3+\alpha)\pm[(3+\alpha)^2-4\alpha(3+\alpha)]^{1/2}\}. \tag{18}$$

The value of β at $\alpha=\frac{4}{3}$ (Richardson's law) is found according to (18) to be complex. This means that $B_0(r)$ reverses sign, whereas the lower eigenfunction should be of constant sign. Thus, the assumption that there are no growing solutions has led to a contradiction. The converse, that dynamo solutions exist, does not lead to a contradiction. Indeed, in this case $E_0<0$, and the minimum E_0 has the maximum modulus. Then

$$|E_0|=\frac{v}{l}\min\{R^{1/2},R_m^{1/2}\}, \tag{19}$$

which is the maximum frequency of the problem in the region (9) and is of the order of the growth rate of the field. Replacing $\partial/\partial t$ in (7) by this value of $-E_0$, we see that in this case the left-hand side is not small compared with the right, and $B_0\sim r^\beta$ is no longer a solution of (7). In this case it is convenient to seek the solution of (7) by the WKB method: $B_0\sim\exp i\int k(r)dr$. In the first-order approximation $k^2=E_0/2T_\alpha r^\alpha$. At E_0 given by (19), this approximation is self-consistent: a constant-sign B_0 is obtained only for pure imaginary k, i.e., at $E_0<0$. Thus, a Kolmogorov turbulence generates an SSMF.

§4. TURBULENCE WITH FEW MODES (OF THE STRANGE-ATTRACTOR TYPE)

It has become clear recently that stochasticity can occur in a system when the number of interacting modes is small. Such a system is called a strange attractor (see, e.g., Ref. 9). Let us consider the problem of generation of a magnetic field, if the velocity pulsation is described by a small number of modes (Fourier harmonics). It is difficult to indicate concrete applications of field generation in such a model. There are probably none in astrophysics. Nonetheless, consideration of the strange attractor as a field generator can cast light on the simplest class of motions capable of producing the dynamo effect. Much attention is being paid to the simplest generators, since they should be abundant in nature. Another aspect of the matter (possibly somewhat unexpected) is that consideration of a model with few modes permits a new examination of the generation of SSMF by a Kolmogorov turbulence in the particular case $R_m \gg R$.

Let the velocity field be described by a sum of harmonics:

$$\mathbf{v}(\mathbf{x},t)=\sum \mathbf{u}_k(t)\exp(-i\mathbf{k}\mathbf{x}), \quad \mathbf{k}\mathbf{u}_k=0,$$

Here $\mathbf{u}_k \neq 0$ if $k=k_0=2\pi/l$ and $\mathbf{u}_k=0$ if $k\neq k_0$. By the same token we consider all the modes whose wave vector has a modulus k_0 or, more accurately, is close to k_0, since a discrete set of wave vectors is used. For a more specific description of the amplitude $\mathbf{u}_k(t)$, we break up the time t into intervals (t_{n-1}, t_n):

$$0<t_1<t_2<\ldots<t_{n-1}<t_n.$$

Within each interval, $\mathbf{u}_k$ is independent of time. There are two linearly independent vectors perpendicular to the vector $\mathbf{k}$ (two polarizations). We shall assume that u in the given interval is parallel to one of them: $\mathbf{u}_k\cdot\mathbf{k}=0$ and $\mathbf{u}_k^*\cdot\mathbf{u}_k\times\mathbf{k}=0$ (linear polarization). In the next interval, the amplitude $\mathbf{u}_k$ has the same property, i.e., the velocity field represents a linearly polarized wave with zero frequency, i.e., the velocity is stationary, but with a new amplitude $\mathbf{u}_k$. The modulus of the amplitude is preserved, only the direction of k and the phase change. The new amplitude does not correlate with the old one. The field v is a process stationary in time (but not in space—therein lies the similarity with the strange attractor). The length of the interval $t_n - t_{n-1}$ is likewise a stationary random process, and the correlation time (or memory time) is obviously $\tau=\langle t_n - t_{n-1}\rangle$ averaged over the different n. The velocity field has four degrees of freedom: two angles, which specify the direction of the wave vector $\mathbf{k}$, the angle characterizing the polarization of the wave, and the phase of the wave. In a given time interval there is realized only one value of each of these quantities (a point in four-dimensional space). It must be kept in mind that at each given instant of time the velocity field is one-dimensional: it depends on direction of the vector $\mathbf{k}$. Sometimes the dynamo process is connected with gyrotropy in the velocity field. A gyrotropic field is defined as one for which the pseudoscalar $\langle \mathbf{v}\cdot\mathrm{curl}\,\mathbf{v}\rangle \neq 0$. The gyrotropy is in fact significant in the generation of large-scale magnetic fields. It will be shown below that an SSMF is generated in the considered velocity field for which $\mathbf{v}\cdot\mathrm{curl}\,\mathbf{v}\equiv 0$.

All the mean values for the given random process should be understood as averaged over the time or over an ensemble of realizations. For the dynamo to work it is necessary to satisfy the condition $R_m = vl/D \gg 1$. In this case v coincides with the constant (in absolute value) amplitude $\mathbf{u}_k$. The theory developed in §1 makes it possible to obtain an equation for B_{ij} without solving the induction equation for each time interval. In fact, the random process described is a particular case of the general one described by Eq. (3) if the averaging, meaning also the corresponding probability distribution, is understood in the sense defined above. Consequently, the equation for B_{ij} coincides with (4). The problem now is to determine the tensor T_{ij} in (4). This is easiest to solve in the case of an isotropic process corresponding to uniform distribution over all four degrees of freedom. Then T_{ij} is an isotropic solenoidal correlation tensor. This is still not enough to make its general form clear, but at $r \ll l$ its form is uniquely defined and written out in (11). The corresponding equation for the SSMF coincides with (12).

We consider three cases: 1) $\tau \ll l/v$, 2) $\tau \approx l/v$, 3) $\tau \gg l/v$. The first corresponds to the "white noise" of the velocity field and to a Markov process for the magnetic field. Only in this case can the coefficient T_2 in (11) and (12) be determined exactly. In the two other cases T_2 can be determined only in order of magnitude. However (and this is the advantage of the theory), this suffices to cast light on the main question: is an SSMF generated? The coefficient T_2 has the physical meaning of the average rate of the relative change of the distance r between the liquid particles. More concretely,

$$T_2=a\langle[\mathbf{r}(t_n)-\mathbf{r}(t_{n-1})]^2\rangle/\langle r^2(t_n)\rangle\tau.$$

Here a is a dimensionless constant of order unity. The numerator contains the mean squared distance between the particles after a time τ. Division by τ gives the rate of change of the squared distance. The quantity $\langle r^2\rangle$ in the denominator corresponds to the fact that the relative velocity is being determined. The expression for $\mathbf{r}(t)$ in each interval is obtained without difficulty. In a coordinate system in which the vector k is parallel to the x axis and v to the y axis we have

$$r_x(t)=r_x(t_{n-1}),\quad r_z(t)=r_z(t_{n-1}),$$
$$r_y(t)=r_y(t_{n-1})+[v_y(^2x_1)-v_y(^1x_1)](t-t_{n-1}). \tag{20}$$

The distance increases linearly with time. We turn to the first case of the Markov process. According to (20), $\langle[\mathbf{r}(t_n)-\mathbf{r}(t_{n-1})]^2\rangle \sim \tau^2$ (since y_y is independent of time). For the small distances $r \ll l$ of interest to use the quantity $v_y(^2x_1)-v_y(^1x_1)$ is replaced by $^1r_x\partial v_y/\partial x$. Consequently

$$T_2=a\tau\langle|\partial v_y/\partial x|^2\rangle. \tag{21}$$

It is appropriate to recall here that within the framework of a Markov process one can obtain for B_{ij} an equation that contains the Euler characteristics of the velocity. Since Markov processes have been well investigated, we can present directly the result of such an analysis. The tensor T_{ij} is expressed in this case in terms of the Euler velocity

$$T_{ij}(^\alpha\mathbf{x},{}^\beta\mathbf{x})={}^1/_2\int_{-\infty}^{\infty}\langle v_i(^\alpha\mathbf{x},t)v_j(^\beta\mathbf{x},t+s)\rangle ds,$$

(see Ref. 5), the coefficient T_2 is defined in the expansion (11) of this tensor in powers of $\mathbf{r}$ and is indeed of the same order as (21). The equation for B_{ij} in the region $r_D \ll r \ll l$ (r_D is determined by the ohmic dissipation) agrees with (12). The advantage of the Markov process is that it is possible to obtain for it an equation for B_{ij} at arbitrarily small r. The general equation coincides at $r \ll l$ with (12), except that the right-hand side contains an additional term $D\Delta B_{ij}$. Thus, the general theory of SSMF corresponds in the limit $\tau \ll l/v$ to a Markov description obtained for the SSMF in an entirely different manner.

We proceed now to the case 2): $\tau \approx l/v$. According to (20)

$$[\mathbf{r}(t_n)-\mathbf{r}(t_{n-1})]^2 \approx r^2(t_n),$$

therefore $T = a/\tau \approx v/l$. Finally, for the third case $\tau \gg l/v$ we can again write $[\mathbf{r}(t_n)-\mathbf{r}(t_{n-1})]^2 \approx r^2(t_n)$ with the previous value $T_2 = a/\tau$, but now $T_2 \ll v/l$. All three cases can be combined into one by the interpolation formula

$$T_2 = a\frac{\tau(v/l)^2}{1+\tau^2v^2l^{-2}}. \tag{22}$$

Equation (12) was analyzed in Ref. 5 with the aid of variational principle. We use here the method described in §3 to find the growing solutions. We seek the solution in the form $B_0(r)e^{-E_0t}, B_0(r) \sim r^\beta$. Substituting this expression in (12) we have a condition on β:

$$\beta = {}^1/_2\{-5\pm[25-4(10+E_0/T_2)]^{1/2}\}, \tag{23}$$

from which it follows that only at $E_0 < -\frac{15}{4}T_2$ is β real, and the lower eigenfunction does not reverse sign. Thus, an exponentially growing solution does exist. The growth rate γ of the dynamo instability is of the same order as T_2 written out in (22). It is seen from this expression that as $\tau \to \infty$ we have $\gamma \to 0$ and there is no dynamo. This is a natural result: as $\tau \to \infty$ we actually arrive at one-dimensional motion during the entire time. In an unbounded medium, however, one-dimensional motion (the particular case of planar motion) does not generate a field, and γ tends therefore to zero. The growth rate is a maximum at $\tau \approx l/v$.

We recall that the very equation (12) used in the present section corresponds to the region (10) that appears in the Kolmogorov turbulence at $R_m \gg R$. The comparison is complete if the scale l and the velocity v of the present section are compared with l_1 given by (10) and with the corresponding velocity of the smallest pulsations of the Kolmogorov turbulence. The case $R_m \gg R$ was considered in Ref. 5. The derivation of the equation for p in Ref. 5 differs substantially from that given in §2. The point is that in Ref. 5 was obtained an equation for the dynamics of extremely close liquid particles ($r \ll \min\{l, l_1\}$), whereas in §2 the distances between the particles are arbitrary. Nonetheless, the general equation (3) for p can be justified by using the results of Ref. 5. In fact, it is shown in that reference that the equation for p is differential, and that the operator $\hat{L}$ [see (13)] does not contain derivatives of order higher than the second. Since Ref. 5 deals with extremely close particles, it can be stated that in that reference was proved the vanishing, as $r \to 0$ of the tensors $T_{ijfm}(\mathbf{r})$ for terms of fourth order in ${}^\alpha\partial_i$ and of analogous terms for higher orders. But the vanishing of $T_{ijfm}(\mathbf{r})$ as $r \to 0$ means identical vanishing of this tensor, since it should possess the properties of a correlation tensor (see §2).

A generator with few modes is the simplest model of turbulence and is therefore most useful for the understanding of the turbulent-dynamo process. The power spectrum of the velocity pulsations is proportional to $\delta(k-k_0), k_0 = 2\pi/l$, so that the turbulence can be regarded as "single-scale." A real turbulence is characterized by a large number of modes with different scales, but the main interaction of the SSMF with the velocity field is effected quasilocally. This means that the behavior of an SSMF of a given scale is determined by velocity pulsations of the same scale, and the interaction times of the two competing processes mentioned in the introduction is determined by the same velocity pulsations (equal in order of magnitude to the time of rotation of the cell). It is therefore not surprising that the SSMF generation does not depend on the ratio R_m/R.

§5. INTERMITTENCE OF THE TURBULENCE

Recent experimental and theoretical investigations shows that the small-scale structure of the turbulence is not spatially homogeneous within each cell of size l (see, e.g., Ref. 10). The small vortices into which a large vortex breaks up do not fill the space of this vortex completely. This leads to a non-kolmogorov turbulence spectrum. We shall use quantities averaged over spatial dimensions larger than l. Then all the possible probability distributions and characteristics of the SSMF will be homogeneous. In particular, the tensor $\langle H_i(\mathbf{x}+\mathbf{r})H_j(\mathbf{x})\rangle$ will be homogeneous (independent of x) for such averaging and for arbitrarily small r corresponding to scales that are small compared with l. The SSMF theory developed in §1 is preserved, and the dynamics of the SSMF is described by the universal equation (7).

The quantity α in (7) determines in accordance with (8) the measure of the diffusion of the pulsations of scale $r < l$. From dimensionality considerations we have

$$T_{LL}(r) \sim E(r)\tau_r,$$

$E(r)$ is the energy of pulsations of scale r, and τ_r is their lifetime. According to Ref. 10 we have

$$E(r) \sim r^{2/3+(3-N)/3}, \quad \tau_r \sim r^{2/3+(3-N)/3}.$$

Here N is the so-called similarity dimensionality. If the small vertices were to fill the entire three-dimensional space, N would be equal to three and $E \sim r^{2/3}, \tau_r \sim r^{2/3}$. The value of α in this case is $\frac{4}{3}$, and the power spectrum coincides with the Kolmogorov spectrum. The available experimental data point to $N = 2.5$ (the so-called 2.5-dimensional space). This conclusion, however, is not unambiguous. We assume therefore that N is arbitrary but less than three. Then $\alpha > \frac{4}{3}$.

We shall analyze Eq. (7) for the SSMF by the method described in §3. All the arguments that lead to relation (18) remain in force. All that changes is the region of applicability of (7)

$$l/\min\{R^{3/(1+N)}, R_m^{3/(1+N)}\} < r < l$$

[in place of (9)] and the region of variation of the characteristic turbulence frequencies from

$$\frac{v}{l} \text{ to } \frac{v}{l}\min\{R^{(5-N)/(1+N)}, R_m^{(5-N)/(1+N)}\}.$$

It is easily seen that the radicand in (18) is negative at $\alpha > \frac{4}{3}$ (i.e., at $N < 3$). Consequently, the lower eigenvalue E_0 cannot be positive. The negative eigenvalue is estimated in analogy with (19):

$$|E_0| = \frac{v}{l}\min\{R^{(5-N)/(1+N)}, R_m^{(5-N)(1+N)}\}.$$

This is in fact the growth rate of the field. At $N=3$ it coincides with (19), and a value $N<3$ gives a larger growth rate.

§6. PHYSICAL INTERPRETATION OF SMALL SCALE TURBULENT DYNAMO

1. Thus, the SSMF theory predicts an exponential growth of the magnetic field for all the known types of turbulence. The universality of the solutions of the problem for all types of spectra is due to the quasilocality, indicated in §4, of the interaction of the SSMF with the pulsations of the velocity field. The rates of the two competing processes (see the Introduction) are determined for an SSMF of scale r by the velocity-field pulsations of the same scale. The SSMF dynamics was interpreted in Ref. 2 by using the approximation of the direct Kraichnan interaction, while in Ref. 11 the diffusion approximation was used. However, as emphasized in Ref. 2, no approximate scheme or model (in particular, the Markov model) has the force of proof. The results of the theory developed above agree with the results of the approximation schemes in Refs. 2 and 11, and this circumstance is an argument favoring the latter.

2. The result of an exact analysis of the SSMF shows that the growth rate of the SSMF energy prevails in a turbulent medium over the rate of the cascaded transfer of the SSMF energy into smaller scales. We present an illustrative interpretation of this phenomenon. In the presence of a frozen-in field, the flux through the liquid contour is conserved:

$$\Phi = \int_S \mathbf{H} dS = \oint_C \mathbf{A} d\mathbf{x} = \text{const}.$$

Here $\mathbf{A}$ is the vector potential, C is the contour around the surface S. We locate the contour in such a way that the force lines of the field H cross it normally. The contour C isolates a field force tube. The streamlines are stretched in the turbulent stream, as is also the force tube. From the conservation of the mass of material in the tube it follows that lengthening of the tube is accompanied by a decrease of its cross section. The cross section is in fact the scale L of the field. The conservation of the flux Φ means that

$$HS = HL^2 = \text{const}, \quad AL = \text{const}$$

or, for the squares of the averaged quantities

$$\langle A^2\rangle \sim L^{-2}, \quad \langle H^2\rangle \sim L^{-4}. \tag{24}$$

In a number of early papers on turbulence it was assumed that the field $\mathbf{A}$ behaves like a scalar admixture and H like a gradient of the scalar admixture. This meant that $\langle A^2\rangle = \text{const}$ and $\langle H^2\rangle \sim L^{-2}$ (this is precisely the situation in the two-dimensional turbulence discussed below). The growth of the field energy was in this case due to the decrease of the scale, and a turbulent dynamo was therefore impossible. Relations (24) give a faster growth of the field. It can be shown that the field A behaves like the gradient of a scalar admixture $\nabla\theta$, and therefore the growth rate of the magnetic field exceeds the rate of decrease of its scale. An additional argument favoring this premise is the similarity of the exact solution for A

$$A_i(\mathbf{x}, t) = \frac{\partial a_j}{\partial x_i} A_j(\mathbf{a}, 0)$$

[the curl operation transforms this expression into (1)] to the solution for $\nabla\theta$. Of course, the foregoing arguments do not have the force of proof and are valuable (at least for illustration purposes) only after an exact theory is developed.

That relations (24) conform to the theory can be seen for one-scale turbulence (§4). We use for this purpose the Fourier transform of Eq. (12), i.e., the equation for the spectral function $B(k,t)$:

$$\frac{\partial B}{\partial t} = T_2\left(k^2\frac{\partial^2}{\partial k^2} + 2k\frac{\partial}{\partial k} + 4\right)B. \tag{25}$$

Since

$$\langle H^2\rangle = \int B(k)\,d\mathbf{k} = 4\pi\int_0^\infty B(k)k^2dk, \quad \langle A^2\rangle = 4\pi\int_0^\infty B(k)\,dk,$$

we have in accord with (25)

$$\frac{d}{dt}\langle A^2\rangle = 4T_2\langle A^2\rangle, \quad \frac{d}{dt}\int_0^\infty Bkdk = 6T_2\int_0^\infty Bkdk.$$

It is natural to define the scale of the field as

$$L^{-1} = \int_0^\infty Bkdk \Big/ \int_0^\infty Bdk,$$

and then

$$\frac{d}{dt}L = -2T_2L, \quad \frac{d}{dt}L^2 = -4T_2L^2.$$

From this follows, first, $d/dt\langle A^2\rangle L^2 = 0, \langle A^2\rangle \sim L^{-2}$, i.e., the first relation of (24) (while the second is a consequence of the first), and second, that the growth rate of $\langle A^2\rangle$ (the argument of the exponential is $4T_2$) is larger than the rate of decrease of the scale (the exponent is $-2T_2$).

3. From the point of view of the method, it is useful to compare the three-dimensional and two-dimensional turbulences. There is an antidynamo theorem[12] that excludes generation of a magnetic field if $\mathbf{v} = \{v_x, v_y, 0\}$. The theory developed in §1 is valid also in this case, and in particular Eqs. (3) and (4) remain in force, but the tensor T_{ij} is anisotropic: $T_{i3} = T_{3j} = 0$. The equation for the H_z component coincides with (15), and generation of this component is of course impossible. The equation for the spectral function of the field $\{H_x, H_y, 0\}$ is of the form

$$\frac{\partial B}{\partial t} = T_2\left(k^2\frac{\partial^2}{\partial k^2} - k\frac{\partial}{\partial k}\right)B. \tag{26}$$

We seek the eigenfunction of this value in the form

$B \sim r^{\beta} e^{-E_0 t}$, from which we get $\beta = 1 \pm (1 - E_0/T_2)^{1/2}$. Here, in contrast to (23), E_0 can be positive. In other words, this solution tells us nothing about the sign of E_0. For the new function $z(k) = Bk^2$ we obtain a self-adjoint (in two-dimensional k-space) equation with a positive definite operator in the right-hand side. It follows from this that all $E_n > 0$, and there is no dynamo.

We compare now the illustrative interpretations of the three-dimensional and two-dimensional cases. To investigate the H_z component we locate the liquid contour introduced in Subsec. 2 in the $z = \text{const}$ plane. Two-dimensional turbulence will not take the contour out of this plane. The force tube (the force lines are parallel to the z axis) are not stretched: there are no motions in the z direction. Therefore the area inside the contour is conserved. Consequently, $H_z S = \text{const}$, $\langle H_z^2 \rangle = \text{const}$, and generation is impossible. We turn now to the field $\mathbf{H} = \{H_x, H_y, 0\}$. We draw our contour in the following manner. Two parts of the contour lie on the surfaces $z = z_0$ and $z = z_1$. Between these surfaces the contour is connected by two straight lines parallel to the z axis. In the course of the two-dimensional motion, the contour configuration described above remains unchanged. The conservation of the flux Φ is written in the form $HS = HL(z_1 - z_0) = \text{const}$. Since the distance $z_1 - z_0$ remains unchanged, we have $H \sim L^{-1}$, $\langle H^2 \rangle \sim L^{-2}$. We arrive at the same conclusion by using the relation for the circulation of A. Since A has only one A_z component, conservation of the circulation means $A(z_1 - z_0) = \text{const}$ and $\langle A^2 \rangle = \text{const}$, and consequently $\langle H^2 \rangle \sim L^{-2}$. Thus, the field H behaves like $\nabla\theta$, and generation is impossible. This is the illustrative interpretation of the antidynamo theorem.[12]

4. Turbulence causes rapid generation of the field. A classification of fast and slow dynamos was given in Ref. 13. A fast dynamo is characterized by a growth rate $\gamma = v/l$, and a slow one by $\gamma = v/lR_m^n, n > 0$. It was shown above that turbulence causes rapid generation, therefore, according to (19), the growth rate is determined by the highest frequency of the velocity pulsations in which a field is still frozen-in. A large-scale field is generated much more slowly. Turbulent generation is at present the only example of rapid field generation. Moreover, even if it becomes possible to construct a model of generation by a stationary (not turbulent) velocity field with $\gamma = v/l$, the very amplification process will still last much longer than l/v. The point is that the scale of the priming magnetic field greatly exceeds as a rule the scale of the eigenfunction. The process of "attuning" the initial field to the eigenfunction, i.e., the process of decreasing the scale, depends on the time linearly and is therefore quite prolonged. In particular, if the growth rate is $\gamma = v/l$, then the scale of the eigenfunction is $l_1 = l R_m^{-1/2}$ and the time when the exponential growth is reached is $R_m^{1/2} l/v$. In a turbulent medium, the time of transfer into the region of small scales is l/v (just as for a scalar admixture, see Subsec. 3), and this is in fact the time in which an exponentially growing solution is reached.

The results of §4 shows that rapid generation of the field can be effected by an extremely simple motion (even one-dimensional at a given instant of time). It was shown in Ref. 14 that a differentially rotating cylinder is capable of generating a field. Such a motion has a high degree of symmetry: axial and planar symmetry are present. The motion is one-dimensional (the velocity depends only on the distance to the axis) and a field is nevertheless generated. A conclusion suggests itself that the significance previously attached to the antidynamo theorems is unjustified. They cannot be realized in nature quite simply. At any rate, attempts to separate (e.g., on the basis of topological properties a class of motions that cause field generation were unsuccessful. In particular, it was suggested in Ref. 15 that rapid generation is possible only if a $\mathbf{a} \cdot \mathbf{v} \neq 0$ and curl $\mathbf{a} = \mathbf{v}$. Actually, however, the quantity $\mathbf{a} \cdot \mathbf{v}$ is not a criterion for the dynamo. If $\mathbf{a} \cdot \mathbf{v} = 0$, generation may not occur (planar motion), the dynamo can be slow and, as shown in §4 (where $\mathbf{a} \cdot \mathbf{v} \equiv 0$) it can be fast.

5. An exponentially growing solution is characterized by a rather small scale determined by the dissipation. In a collision-dominated plasma dissipation is due to ohmic losses, and in a collisionless (interplanetary medium, solar wind) it is due to Čerenkov damping over scales comparable with the Larmor radius of the ion. After a time comparable with $\gamma^{-1} |E_0|^{-1}$ in accord with (19), the energy density of the magnetic fluctuations reaches the density of the kinetic pulsations in the same small scales. Further growth of these supersmall-scale magnetic fields stops: they already act on the motion. In larger scales (but still small compared with l), the growth rate of the SSMF still exceeds the rate of subdivision of the scales, naturally, and therefore the field fluctuations in these scales continue to grow until they become comparable in energy with the kinetic fluctuations. This process continues until all the SSMF scales (up to l) are enhanced. In particular, enhancement of a field of scale l takes place after a time l/v, which is long compared with γ^{-1}. The time l/v characterizes the entire linear stage of turbulence of the dynamo. After this time there is established an approximate equipartition of the magnetic and kinetic energies in the scale l. In smaller scales, the fluctuations are transformed into MHD waves against the background of a quasihomogeneous magnetic field (of scale l). The wave spectrum was obtained by Kraichnan.[16] The Kraichnan $k^{-3/2}$ spectrum differs from the Kolmogorov $k^{-5/3}$ spectrum because the nonlinear interactions in a magnetic field are not local. In a collision-dominated plasma the spectrum is cut off at lengths determined by equality of the dissipation time to the time of the nonlinear interaction. In a collisionless plasma, the cutoff occurs at lengths close to the ion Larmor radius.

The entire SSMF dynamics described above is practically independent of the presence of large-scale magnetic fields. The situation is entirely different with large-scale fields: a consistent theory of such fields can be constructed only if the dynamics of the SSMF is known. If the turbulence were not to cause generation of SSMF, the situation would be the exact opposite of the described one. Without a large-scale field, the SSMF would become dissipated within a time l/v. The large-scale field would serve as a source of weak fluctuations of the SSMF, with an increasing spectrum (of the type $k^{1/3}$, as a gradient of the scalar admixture). Finally, weak SSMF fluctuations would little influence the large-scale field. The presently available observational and experimental data (the solar photosphere, the interstellar gas of the galaxy, the solar wind) offer convincing evidence that the energy of the magnetic pulsations is larger (sometimes by several orders of magnitude) than the energy of the large-scale field, thus confirming the prediction of the theory concerning the SSMF dynamo.

[1]G. K. Batchelor, Proc. Roy. Soc. **A201**, 405 (1959).

[2]R. H. Kraichnan and S. Nagarajan, Phys. Fluids **10**, 859 (1967).

[3]R. H. Kraichnan, Phys. Rev. Lett. **41**, 1677 (1979).

[4]S. I. Vaĭnshteĭn, Izv. Vyssh. Ucheb. Zav. Radiofizika IIIV, 172 (1981).

[5]S. I. Vaĭnshteĭn, Zh. Eksp. Teor. Fiz. **79**, 2175 (1980) [Sov. Phys. JETP **52**, 1099 (1980)].

[6]S. I. Vainshtein, Usp. Fiz. Nauk **134**, 744 (1981) [Sov. Phys. Usp. **24**, 724 (1981)].

[7]S. I. Vainshtein, Magnitnaya Gidrodinamika, No. 2, 3 (1982).

[8]A. S. Monin and M. A. M. Yaglom. Statisticheskaya gidromekhanika (Statistical Hydromechanics), Part 2, Nauka, 1967.

[9]M. I. Rabinovich, Usp. Fiz. Nauk **125**, 123 (1978) [Sov. Phys. Usp. **21**, 443 (1978)].

[10]U. Firsch, P.-L. Sulem, and M. Melkin, J. Fluid Mech. **87**, 719 (1978).

[11]J. Lorat, A. Pouguet, and U. Frisch, Fully Developed MHD Turbulence . . . , Preprint, Observaorie de Nice, France, 1980.

[12]Ya. B. Zel'dovich, Zh. Eksp. Teor. Fiz. **31**, 154 (1956) [Sov. Phys. JETP **4**, 460 (1957)].

[13]S. I. Vaĭnshteĭn, and Ya. B. Zel'dovich, Usp. Fiz. Nauk **106**, 431 (1972) [Sov. Phys. Usp. **15**, 159 (1973)].

[14]S. I. Vaĭnshteĭn, Magnitnaya gidrodinamika No. 3, 3 (1981).

[15]A. AS. Ruzmaikin and D. D. Sokoloff, Geoph. Astrophys. Fluid Dyn. **16**, 73 (1980).

[16]R. H. Kraichnan, Phys. Fludis **18**, 1385 (1965).

Translated by J. G. Adashko