Gauss-Tolman theorem and positivity of the total mass–energy for a nonstatic sphere in the general theory of relativity[1)]

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A generalization is given of Tolman's potential form of the integral of the total mass–energy for nonstatic, centrally symmetric matter distributions in R regions of V_4. A local conservation law for the total energy of the gravitational field and matter in general relativity is discussed on the basis of the quasi-Newtonian Gauss-Tolman theorem. It is shown that the Schwarzschild mass, the equivalent of the total energy, is positive in the presence of T regions in "semiclosed" models of a sphere, and thus the gravitational mass defect cannot exceed the total self-energy of the matter.

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INTRODUCTION

Besides numerous applications in relativistic astrophysics and cosmology,[3,4] spherically symmetric nonstatic fields in general relativity[2] are of great methodological interest.[5,6] Analysis of their characteristic properties is also important for establishing different aspects of the relationship between the Einsteinian and Newtonian theories of gravitation (as is clearly shown by the simple example of the Tolman-Friedmann "dust"models[7,8]), and especially in the investigation of gravitational binding energy (the mass defect)[3,7–10] and the problem of the positive definiteness of the active Schwarzschild mass, which is the equivalent of the conserved total energy of any sphere in general relativity.[11]

In Sec. 1, we consider in detail centrally symmetric solutions of the Einstein equation in R regions of space-time V_4 (where the closest connection and similarity to Newtonian theory obtains).[3,10] On the basis of the chronometrically invariant formulation of general relativity,[12] and in a physically chosen polar R system of Schwarzschild coordinates,[2] we obtain general-relativistic analogs of the Poisson equation with 3-scalar potential $U=\sqrt{g_{00}}$ and a corresponding quasi-Newtonian Gauss theory.[2,6,13] This last leads to a generalization of Tolman's well-known formula[2–5] and gives a new integral representation for the total Schwarzschild mass-energy for a nonstatic sphere expressed in terms of the instantaneous distribution of the material sources alone within an R region of V_4.[1a]

On the basis of the Gauss-Tolman theorem, we briefly discuss a local conservation law for the active mass, the equivalent of the total energy of the matter and the gravitational field for isolated systems in general relativity.

In Sec. 2, using a convenient formulation of the Einstein equations in terms of the Lemaître invariant mass function[14] for general spherically symmetric matter distributions in a comoving system, we analyze the gravitational mass defect in the presence of essentially relativistic T regions of V_4 (see Refs. 3 and 10), and we then prove positivity of the total mass-energy of "semiclosed" models of a bounded sphere (for which Tolman's formula and the other pseudotensor expressions for the energy integral are no longer valid).

It is known[2,4] that the gravitational field of V_4 in general relativity does not possess a localized self-energy density, but nevertheless, through the specific nonlinearity of the Einstein equations, the gravitational field makes a perfectly definite contribution to the conserved total mass-energy of an isolated system with asymptotically flat Schwarzschild metric at spatial infinity.[11] For an equilibrium sphere, this delocalized field contribution is negative and represents the gravitational binding energy-the mass defect corresponding in Newtonian theory to the potential energy of the self-attraction of matter, $\Omega_0 \propto G\mathscr{M}_0^2/R_0$, which has no lower bound as $R_0 \to 0$. But in general relativity a static sphere with given self-mass $\mathscr{M}_0$ cannot have radius less then the gravitational radius $R_g = 2GM/c^2$, and its Schwarzschild mass is $M \leqslant \mathscr{M}_0$.[3,4] For general nonstatic matter distributions it is to be expected that the negative contribution of the gravitational binding energy must decrease the active mass $M = E/c^2$ (the source of the field) and progressively weaken its self-interaction, so that already in the quasi-Newtonian approximation, when allowance is made for the equivalence principle, the total mass-energy of a sphere,

$$M=\mathscr{M}_0-\alpha GM^2/R_0, \quad \mathscr{M}_0=\text{const}, \quad \alpha\sim 1,$$

always remains positive even in the limit $M \to 0$ of maximally complete binding of the matter.[8b,11]

As was shown by Novikov[10] and Zel'dovich,[3,9] it is possible in general relativity to have "semiclosed" models of a nonstatic sphere (for example, part of a closed Friedmann model) in which the gravitational binding energy of individual spherical layrs exceeds their self-energy, so that there is a self-screening effect, which takes the form of a weakening of the exterior Schwarzschild field due to the decrease in the total active mass when shells of matter are added. The general-relativistic mass defect of a sphere is manifested characteristically through the curvature of the space V_3 which is comoving with the matter[7,8] and in semiclosed models leads to an essentially non-Euclidean topology of the V_3 with a "throat," which is necessarily situated in T regions of V_4 (see Refs. 3, 9, and 10), and at which the radius of curvature of the

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Lagrangian spheres and the active mass reach maxima and then decrease with increasing distance from the "center."

It would appear that, because of the predominant contribution of the gravitational binding energy of such matter layers after the throat, one could have a negative total mass-energy of a sphere. In fact, general relativity does not preclude a negative sign of the mass M in the Schwarzschild metric, but the nature and global properties of the exterior field in vacuum then differ radically, since there is no pseudosingularity ($R = R_g > 0$), no null horizons, and no T regions of V_4 when $M < 0$. It follows from the requirement of correct matching of the semiclosed models to the exterior Schwarzschild field (which is possible only through vacuum T regions when $M > 0$) that the total mass-energy of any sphere in general relativity must be positive or vanish for the special case of spatial closure[2] (as in a closed Friedmann model).[9]

1. GAUSS THEOREM AND GENERALIZED TOLMAN FORMULA FOR NONSTATIC, CENTRALLY SYMMETRIC SYSTEMS

Among the nonstatic Einstein gravitational fields, the centrally symmetric fields in R regions of V_4,[2–4] like static fields,[2,5,6] are closest to and most similar to Newtonian fields on account of Birkhoff's theorem, which precludes the existence of gravitational waves. Therefore, the case of spherical symmetry is interesting for studying the connection between the relativistic and classical theories of gravitation, especially with regard to noninvariant concepts such as the potential and "intensity" of the field, the inverse square law, Gauss's theorem, and the distribution of the gravitational "charge"—the active mass of the material sources with energy-momentum-stress tensor $T_{ik}(x)$. These questions have previously been discussed only for static and usually spherically symmetric[6,13] (and stationary[15]) systems in an invariantly defined "rigid" frame of reference that is comoving with the matter and based on a congruence of trajectories of the "static" group G_1 with family of hypersurfaces orthogonal to them—the spatial sections V_3 at each chosen universal time $x_0 = t = \text{const}$. Then $U = \sqrt{g_{00}} = (\xi^i \xi_i)^{1/2}$ (the norm of the Killing vector) plays the role of a relativistic potential, and it satisfies the generalized Poisson equation (see Refs. 2,6):

$$\Delta U = (\tilde{\nabla}^\alpha \tilde{\nabla}_\alpha) U = {}^1/_2 \varkappa (T_0{}^0 - T_\alpha{}^\alpha); \quad \varkappa = 8\pi G/c^4; \quad \alpha, \beta = 1, 2, 3,$$

while the spatial acceleration vector of test bodies at rest,

$$-f_i = \gamma^k \nabla_k \gamma_i = \delta\gamma_i/\delta s = \partial_i \ln U, \quad \gamma_i = U\delta_i{}^0, \; \gamma_i\gamma^i = 1; \; i, \; k = 0, 1, 2, 3$$

measures the intensity of the Einstein gravitational field.

It is natural to introduce this last by analogy with the Newtonian concept of a potential field of the gravitational force as a 3-vector $F^\alpha = Uf^\alpha = \gamma^{\alpha\beta}\partial_\beta U$ in the Riemannian space V_3 ($t = \text{const}$) of the invariant "rigid" frame of reference as follows:

$$\text{Div}\, \mathbf{F} = \tilde{\nabla}_\alpha F^\alpha = \Delta U = {}^1/_2 \varkappa \mu^*, \quad \mu^* = (T_0{}^0 - T_\alpha{}^\alpha) U, \tag{1}$$

so that the flux of its normal component through a closed surface S determines the active gravitational mass within the region, this giving the analog of Gauss's theorem in general relativity[2,6,13]:

$$-\oint_{(S)} (F_\alpha n^\alpha)\, d\Sigma = \frac{\varkappa}{2} m^*[V], \quad F_\alpha = -\partial_\alpha U;$$
$$m^*[V] = \int_{(V)} \mu^*\, dV = \int_{(V)} (T_0{}^0 - T_\alpha{}^\alpha) \sqrt{-g}\, d^3x. \tag{1a}$$

The correspondence between general-relativistic and Newtonian spherically symmetric fields is most clearly revealed in a canonical polar R system of the form[2–6]

$$-ds^2 = -U(r, t)\, dt^2 + dl^2 = -e^{\nu(r,t)} dt^2 + e^{\lambda(r,t)} dr^2 + r^2 (d\vartheta^2 + \sin^2\vartheta\, d\varphi^2), \tag{2}$$

where the chosen universal time lines with tangent velocity 4-vector $\gamma^i = U^{-1}\delta_0^i$ of the test particles coincide with the invariant static congruence of the exterior vacuum Schwarzschild field, which can be uniquely continued for the interior nonstatic R region. The radial curvature coordinate is the analog of the Euclidean radius, since the area of the sphere and the length of the circumference are, respectively, πt^2 and $2\pi r$.

Using the chronometrically invariant formalism of (3 + 1) decomposition of V_4 fields[11,12] onto the distinguished universal time t and its orthogonal spatial sections $V_3(t = \text{const})$ in the polar R system (2), we can express all the geometrical characteristics of the V_4 in terms of the 3-scalar relativistic potential $U(r,t) = e^{\nu/2}$ and the variable metric $dl^2 = \gamma_{\alpha\beta} dx^\alpha dx^\beta$ of the physical space V_3 of this chosen frame of reference:

$$R_0{}^0 = -U^{-1}\Delta U + \Theta,$$
$$\Delta U = \frac{1}{\gamma^{1/2}} \frac{\partial}{\partial x^\alpha}\left(\gamma^{1/2}\gamma^{\alpha\beta}\frac{\partial U}{\partial x^\beta}\right) = \frac{1}{r^2 e^{\lambda/2}} \frac{\partial}{\partial r}\left(r^2 e^{\lambda/2} \frac{\partial U}{\partial r}\right),$$
$$\Theta = U^{-1}\frac{\partial D}{\partial t} + D_\beta{}^\alpha D_\alpha{}^\beta = \frac{1}{2} e^{-\nu}\left(\ddot{\lambda} + \frac{\dot{\lambda}^2}{2} - \frac{\dot{\lambda}\dot{\nu}}{2}\right),$$
$$D_{\alpha\beta} = \frac{1}{2U}\frac{\partial \gamma_{\alpha\beta}}{\partial t}, \quad D = D_\alpha{}^\alpha = \frac{1}{U}\frac{\partial \ln \gamma^{1/2}}{\partial t}, \tag{3}$$
$$-R_1{}^1 = \tilde{K}_1{}^1 + U^{-1}\Delta U - \Theta - 2\Pi;$$
$$-R_2{}^2 = -R_3{}^3 = \tilde{K}_2{}^2 + \Pi = \tilde{K}_3{}^3 + \Pi, \quad \Pi = e^{-\lambda}\frac{\nu'}{r},$$
$$2G_0{}^0 = \tilde{K} + (D_\beta{}^\alpha D_\alpha{}^\beta - D^2) = \tilde{K}; \quad \tilde{K}_1{}^1 = -e^{-\lambda}\frac{\lambda'}{r},$$
$$\tilde{K}_2{}^2 = \tilde{K}_3{}^3 = e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{2r}\right) - \frac{1}{r^2}.$$

Here, $\Delta U = \tilde{\nabla}_\alpha F^\alpha = \gamma^{\alpha\beta}(\tilde{\nabla}_\alpha \tilde{\nabla}_\beta U)$ is the generalized Laplace-Beltrami operator, $\tilde{K}_{\alpha\beta}$ is the Ricci tensor of the intrinsic curvature of the V_3, and $D_{\alpha\beta}$ is the rate-of-deformation tensor, the exterior curvature of the spatial sections $t = \text{const}$; all 3-tensor operations are performed by means of the metric tensor $\gamma_{\alpha\rho}\gamma^{\beta\rho} = \delta_\alpha^\beta$, and the prime and the dot denote differentiation with respect to r and t.

In the canonical R system (2), the Einstein equations have a simple form,[2,5,6] and the time component

$$R_0{}^0=-\varkappa(T_0{}^0-{}^1/_2T)$$

of the Ricci tensor gives, when (3) is used, a generalized Poisson equation for the scalar potential $U=e^{\nu/2}$:

$$U^{-1}\Delta U=\varkappa(T_0{}^0-{}^1/_2T)+\Theta,$$

$$\Theta=U^{-1}\,\partial D/\partial t+D_\beta{}^\alpha D_\alpha{}^\beta=\frac{e^{-\nu}}{2}(\ddot\lambda+\dot\lambda^2/2-\dot\lambda\dot\nu/2). \qquad (4)$$

In the case of spherical symmetry, the Einstein equations lose their wave hyperbolic nature and admit a formal solution of quasistatic type, so that the metric potentials (2) are determined by the instantaneous, without retardation, nonstatic distribution of the material sources with the hydrodynamic energy-momentum-stress tensor, which has nonvanishing components of the form[2,6]

$$T_0{}^0=\frac{\varepsilon+\beta^2P}{1-\beta^2},\quad T_1{}^1=-\frac{P+\beta^2\varepsilon}{1-\beta^2},\quad T_2{}^2=T_3{}^3=-P,$$

$$T_0{}^1=\frac{(\varepsilon+P)}{(1-\beta^2)}e^{(\nu-\lambda)/2};\quad \beta=\left(\frac{dl}{ds}\right)=e^{(\lambda-\nu)/2}\left(\frac{\partial R}{\partial t}\right)_\chi,$$

where β is the radial velocity, and $R(t,\chi)$ is the law of motion of a fixed layer of fluid with $\chi=$ const.

From the first integral of the dynamical Einstein equations with allowance for (3),

$$G_0{}^0={}^1/_2\overset{*}{R}=-\varkappa T_0{}^0,$$

augmented by the requirement of regularity of the metric at the center $r=0$, it follows that the non-Euclidean geometry of the space $V_3(t=\text{const})$ is completely characterized by the Lemaître invariant mass function[14]

$$e^{-\lambda(r,t)}=1-\frac{\varkappa m(r,t)}{4\pi r},\qquad m(r,t)=4\pi\int_0^r T_0{}^0(r,t)r^2dr. \qquad (5)$$

This "geometrical" active mass is the total energy of the gravitating matter contained within the Lagrangian spheres with $R=R(t,\chi)$:

$$m'=4\pi r^2T_0{}^0,\quad \dot m=-4\pi r^2T_0{}^1,$$

$$\frac{dm\{R(t,\chi),t\}}{dt}=-4\pi R^2\dot RP(R,t), \qquad (6)$$

and it changes only due to the work of the pressure forces $P\neq0$, this being similar to the adiabatic behavior of the energy of a Newtonian "fluid" sphere. On the boundary with the vacuum $R_0=R_0(t,\chi)$, where $P(R_0,t)=0$, it is equal to the constant gravitational mass of the exterior vacuum Schwarzschild solution

$$-ds^2=-\left(1-\frac{\varkappa M}{4\pi r}\right)dt^2-\left(1-\frac{\varkappa M}{4\pi r}\right)^{-1}dr^2+r^2(d\vartheta^2+\sin^2\vartheta\,d\varphi^2)$$

by virtue of continuous matching of the V_4 metrics[6,15,16]:

$$M=m(R_0,t)=4\pi\int_0^{R_0(t)}T_0{}^0(r,t)r^2\,dr=\text{const},$$

which determines the total mass-energy $E=Mc^2$ of the sphere in general relativity.[2-6]

The radial component of the acceleration 4-vector of the test particles of the canonical R system (2)

$$-f^\alpha=\gamma^k\nabla_k\gamma^\alpha=U^{-2}\Gamma_{00}{}^\alpha=\gamma^{\alpha\beta}\partial_\beta\ln U$$

satisfies in accordance with the Einstein equations[2,6] a modified inverse square law:

$$-f^1=e^{-\nu}\Gamma_{00}{}^1=\frac{1}{2}e^{-\lambda}\nu'=\frac{\varkappa\tilde m(r,t)}{8\pi r^2},$$

$$\tilde m(r,t)=m(r,t)-4\pi r^3T_1{}^1, \qquad (7)$$

where the attracting mass of the interior "liquid" sphere $\tilde m(r,t)$ contains the additional contribution of the pressure, $P\neq0$, and for a homogeneous distribution in Friedmann models with $m(R,t)=4\pi\epsilon R^3/3$ the effective density of the active gravitational mass is $(\epsilon+3P)$.[3]

As in the Newtonian case, the relativistic potential of a centrally symmetric matter distribution can be written in two different forms by integrating the Einstein equations[2,6]:

$$\ln U(r,t)=\nu(r,t)=-\lambda(r,t)+\int_{R_0(t)}^r re^{\lambda(r,t)}[T_0{}^0(r,t)-T_1{}^1(r,t)]dr$$

$$=\frac{\varkappa}{4\pi}\int_\infty^r\frac{dr}{r^2}e^{\lambda(r,t)}\tilde m(r,t). \qquad (8)$$

As in the classical theory of gravitation, the acceleration of a given spherical layer is determined solely by the interior sources, although the potential $U=e^{\nu/2}<1$ also depends on the exterior parts of the distribution, reaching a minimum at the center $r=0$; further, the centrally symmetric motion of the matter does not influence the exterior Schwarzschild field in vacuum at all. However, in general relativity the distribution of the sources $T_{ik}(x)$ cannot be taken arbitrarily independent of the V_4 field they produce, and both must be found by simultaneous solution of a self-consistent problem, since the Einstein equations also contain the equations of motion of the matter in the form of the Bianchi identities $T^k_{i;k}=0$.[2,6] One of these conservation laws ($i=0$) is equivalent to (6), and the dynamical equation of the radial motion ($i=1$) makes it possible to write down in a simple manner the additional term θ in the Poisson equation (4) due to the nonstatic nature of the interior field, expressing it in terms of the energy-momentum-stress tensor of the material sources:

$$\frac{2}{\varkappa}\Theta=\frac{r}{\sqrt{-g}}\frac{\partial}{\partial t}(\sqrt{-g}T_1{}^0)=re^{-(\nu+\lambda)/2}\frac{\partial}{\partial t}(e^{(\nu+\lambda)/2}T_1{}^0)$$

$$=2(T_2{}^2-T_1{}^1)+r\left[\frac{1}{2}(T_0{}^0-T_1{}^1)\nu'-\frac{\partial T_1{}^1}{\partial r}\right]. \qquad (9)$$

For static distributions, we have $\Theta=0, T_1^0=0, T_0^0=\epsilon(r), T_1^1=T_2^2=T_3^3=-P(r)$ and Eq. (9) goes over into the condition of hydrostatic equilibrium of a gravitating fluid with $P\neq0$.[2-5]

Like the Newtonian potential, the relativistic potential $U=e^{\nu/2}$ in the polar R system (2) satisfies the generalized Poisson equation (4):

$$\Delta U=\frac{1}{r^2e^{\lambda/2}}\frac{\partial}{\partial r}\left(r^2e^{-\lambda/2}\frac{\partial U}{\partial r}\right)=\frac{\varkappa}{2}\mu^*,$$
$$\mu^*=\left(T_0^{\ 0}-T_\alpha^{\ \alpha}+\frac{2}{\varkappa}\Theta\right)U. \tag{10}$$

Therefore, it is natural to regard the distribution of its effective sources, $u^*(r,t)$, as the density of the gravitational charge—the active "potential" mass. This last depends on the potential U and, when (9) and also Eqs. (5) and (8) are taken into account, can be expressed soley in terms of two basic components of the energy-momentum-stress tensor for a centrally symmetric, nonstatic matter distribution in the form

$$\mu^*=\left\{(T_0^{\ 0}-3T_1^{\ 1})+r\left[\frac{1}{2}(T_0^{\ 0}-T_1^{\ 1})\nu'-\frac{\partial T_1^{\ 1}}{\partial r}\right]\right\}U. \tag{10a}$$

By direct integration of the Poisson equation in the polar coordinates (2) we obtain the following expressions for the norm of the stress vector $F=(F_1F^1)^{1/2}e^{-\lambda/2}U'$ and the potential $U=e^{\nu/2}$:

$$F(r,t)=\frac{1}{2}e^{-(\lambda+\nu)/2}\nu'=\frac{\varkappa m^*(r,t)}{8\pi r^2},$$
$$U(r,t)=\frac{\varkappa}{4\pi}\int\limits_\infty^r\frac{dr}{r^2}e^{\lambda(r,t)/2}m^*(r,t). \tag{11}$$

We here have an inverse square law for the radial field "intensity" if we take as the active potential mass a quantity different from the one in (7) and (8), namely,

$$m^*(r,t)=4\pi\int\limits_0^r\mu^*(r,t)r^2e^{\lambda(r,t)/2}\,dr,\quad dV=4\pi r^2e^{\lambda/2}\,dr. \tag{12}$$

Comparing (7) and (11) and using the relation $F=Uf$, we obtain a nontrivial relation between the active "geometrical" mass (5) and the active potential mass (12) of the centrally symmetric matter distribution in general relativity:

$$\tilde{m}(r,t)=m(r,t)-4\pi r^3T_1^{\ 1}(r,t)=e^{-(\lambda+\nu)/2}m^*(r,t). \tag{13}$$

For arbitrary choice of the coordinates x^α in the spatial sections $V_3(t=\text{const})$ of a Schwarzschild R frame of reference of the type (2) with $dl^2=\gamma_{\alpha\beta}dx^\alpha dx^\beta$, the Laplace operator can be represented as a 3-covariant divergence of the potential vector $F_\alpha=\partial_\alpha U$ of the intensity:

$$\Delta U=\operatorname{Div}\mathbf{F}=\tilde{\nabla}_\alpha F^\alpha=\frac{1}{\gamma^{1/2}}\frac{\partial}{\partial x^\alpha}(\gamma^{1/2}F^\alpha)$$
$$=\frac{1}{\gamma^{1/2}}\frac{\partial}{\partial x^\alpha}\left(\gamma^{1/2}\gamma^{\alpha\beta}\frac{\partial U}{\partial x^\beta}\right)=\frac{\varkappa}{2}\mu^*(t,x^\alpha). \tag{14}$$

In accordance with the generalized Green's formula for the Riemannian V_4,[2,6]

$$\int\limits_{(V)}\Delta U\,dV=\oint\limits_{(S)}\left(\frac{\partial U}{\partial x^\alpha}n^\alpha\right)d\Sigma\ , \tag{15}$$

the volume integral over the distribution of the gravitational charges $\mu^*(x^\alpha,t)$ in an open, simply connected region of the space $V_3(t=\text{const})$ can be transformed into a surface integral of the flux of the gradient of the potential of the intensity 3-vector through the bounding closed 2-surface. Here, $dV=\gamma^{1/2}d^3x$ is the proper volume of the V_3 elements, and $d\Sigma$ is the area of the element of the boundary S defined in accordance with the condition relating the dual 3-vector of the element of surface to its exterior unit normal:

$$\gamma^{1/2}d\sigma_\alpha=n_\alpha d\Sigma,\quad d\sigma_\alpha={}^1/_2\gamma^{1/2}\varepsilon_{\alpha\beta\gamma}\delta x^\beta\delta x^\gamma,\quad n_\alpha n^\alpha=1.$$

Integrating the Poisson equation (10),(14) over any closed region of the spatial sections $V_3(t=\text{const})$ and using the Green's formula (15), we obtain the relativistic variant of the Gauss flux theorem for a centrally symmetric nonstatic field in a R frame of reference of the type (2):

$$\frac{\varkappa}{2}m^*[V]=-\oint\limits_{(S)}(F_\alpha n^\alpha)\,d\Sigma=\oint\limits_{(S)}\left(\frac{\partial U}{\partial x^\alpha}n^\alpha\right)d\Sigma,$$
$$m^*[V]=\int\limits_{(V)}\mu^*\,dV. \tag{16}$$

Thus, the flux of the normal component of the field "intensity" through an arbitrary simply connected surface S is a measure of the 3-scalar active gravitational mass $m^*[V]$ contained within it. For any surface in vacuum completely surrounding the interior region of a centrally symmetric matter distribution, the integral of the intensity flux has a constant value and is proportional to the gravitational mass parameter of the exterior Schwarzschild field with $e^\nu=e^{-\lambda}=1-(\varkappa M/4\pi r)$:

$$-\oint\limits_{(S)}(F_\alpha n^\alpha)\,d\Sigma=\oint\limits_{(S)}\left(\frac{\partial U}{\partial x^\alpha}n^\alpha\right)d\Sigma=\frac{\varkappa}{2}M=\text{const}. \tag{17}$$

The curvature coordinates (2) are not admissible in the sense of Lichnerowicz[15,16] but the weaker boundary conditions obtained by matching the interior and exterior solutions of the Einstein equations in the manner proposed by Israel or O'Brien and Synge[6] in the Schwarzschild R system ensure continuity of the potential $U=e^{\nu/2}$ and the field intensity F_α. Thus, the Gauss theorem (16) together with the requirement of continuity of the "intensity" flux (17) on the boundary $R_0=R_0(t)$ of the sphere gives the following integral representation for the total gravitational mass of a nonstatic sphere in general relativity:

$$M=m^*\{R_0(t),t\}=\int\left(T_0^{\ 0}-T_\alpha^{\ \alpha}+\frac{2}{\varkappa}\Theta\right)\sqrt{-g}\,d^3x,$$
$$\sqrt{-g}=U\sqrt{\gamma}. \tag{18}$$

Here, the integral is taken only over the interior region $V_3(t=\text{const})$ occupied by matter, $T_{ik}\neq0$, since the density of the gravitational "charge" vanishes in vacuum ($\Theta\equiv0$) due to the uniqueness and static nature of the vaccum Schwarzschild metric for the R system (2). Since the metric potentials are determined by the instantaneous distribution of the material sources in accordance with (5), (8), and (11), we can use the relations (9), (10), and (12) to express the conserved mass-energy (18) of the nonstatic sphere in terms of the characteristics of the gravitating matter alone; namely, the two essentially independent components of its energy-momentum-stress tensor at any fixed time:

$$M=m^*\{R_0(t),t\}=4\pi\int_0^{R_0(t)}\left\{(T_0{}^0-3T_1{}^1)+r\left[\frac{1}{2}(T_0{}^0-T_1{}^1)\nu'-\frac{\partial T_1{}^1}{\partial r}\right]\right\}e^{(\lambda+\nu)/2}r^2\,dr. \quad (18a)$$

This new representation of the total Schwarzschild mass-energy for a centrally symmetric distribution and motion of the fluid together with the formula

$$M=m\{R_0(t),\,t\}, \quad (6a)$$

which is obtained from the active geometrical mass (6), is well known in the static case of equilibrium configurations of a gravitating sphere[2-6]:

$$M=\int(\varepsilon+3P)\sqrt{-g}\,d^3x=4\pi\int_0^{R_0}(\varepsilon+3P)\,e^{(\lambda+\nu)/2}\,r^2\,dr=4\pi\int_0^{R_0}\varepsilon r^2\,dr. \quad (19)$$

The relativistic analog of Gauss's theorem (16)–(18) leads to the Tolman form of the total mass-energy of the sphere, which is also obtained from the canonical energy-momentum pseudotensor t_i^k of the gravitational field in the form of integrals over the whole of the space $V_3(t=\text{const})$[5]:

$$E=Mc^2=\int(T_0{}^0+t_0{}^0)\sqrt{-g}\,d^3x=\int(T_0{}^0-T_\alpha{}^\alpha)\sqrt{-g}\,d^3x+\frac{1}{\varkappa}\int\sqrt{-g}\,g^{ik}\frac{\partial}{\partial x_0}\left\{\frac{\partial(\sqrt{-g}L)}{\partial(\sqrt{-g}\,g^{ik}{}_{,0})}\right\}d^3x, \quad (20)$$

where t_i^k and, in particular, the energy density t_0^0 of the gravitational field are introduced in the standard manner in terms of the Einstein Lagrangian

$$L=g^{ik}(\Gamma_{is}{}^n\Gamma_{kn}{}^s-\Gamma_{ik}{}^n\Gamma_{ns}{}^s),\quad t_k{}^i=\frac{g^{mn}{}_{,k}}{\sqrt{-g}}\frac{\partial(\sqrt{-g}L)}{\partial(\sqrt{-g}g^{mn}{}_{,i})}-\delta_i{}^kL. \quad (21)$$

Using the relations for the variation of this Lagrangian,

$$\frac{\partial(\sqrt{-g}L)}{\partial(\sqrt{-g}g^{ik}{}_{,n})}=-\Gamma_{ik}{}^n+\frac{1}{2}\delta_i{}^n\Gamma_{ks}{}^s+\frac{1}{2}\delta_k{}^n\Gamma_{is}{}^s, \quad (22)$$

and the explicit form of the Christoffel symbols in the R system (2), we can readily show that (18) and (20) are identical, since

$$g_{ik}\frac{\partial}{\partial x_0}\left\{\frac{\partial(\sqrt{-g}L)}{\partial(\sqrt{-g}g^{ik}{}_{,0})}\right\}=2\Theta=e^{-\nu}\left(\ddot\lambda+\frac{\dot\lambda^2}{2}-\frac{\dot\lambda\dot\nu}{2}\right). \quad (23)$$

Although the canonical pseudotensor (21) depends on the choice of the coordinate system, it, like the other affine complexes (for example, the Landau-Lifshitz complex[2]), yeilds an entirely satisfactory integral formulation of the conservation laws in general relativity for isolated systems with asymptotically flat Schwarzschild-Minkowski metric in an appropriate quasi-Galilean coordinate system. The integral of the total energy of the matter and the gravitational field over any volume $V_3(t=\text{const})$ is transformed into the surface flux of a combination of the first derivatives of the V_4 metric, so that the total energy is determined solely by the asymptotic behavior of the field g_{ik} at spatial infinity and for an isolated system always coincides with the conserved Schwarzschild gravitational mass $E=Mc^2=\text{const}$.[2-5]

We note that a similar and more general form of a quasi-Newtonian Gauss theorem for the general case of nonstatic V_4 fields with gravitational radiation was discussed by Pirani and Komar[17] in connection with the conservation laws in general relativity. If in V_4 there exists an invariantly chosen frame of reference, i.e., a congruence of univeral time lines $x_0=t$ or an associated family of spatial sections V_3 (distinguished by local characteristics of the geometry such as the Petrov curvature scalars or by globally asymptotic symmetries such as, for example, quasistatic behavior at infinity), then, applying the chronometrically invariant (3 + 1) formulation of general relativity,[11,12] and using a Gauss theorem of the type (16),(20), we obtain a local conservation law for the total energy of the matter and the gravitational field in the form a 3-covariant continuity equation for the gravitational charge:

$$\operatorname{Div}\mathbf{S}+\frac{1}{\gamma^{1/2}}\frac{\partial}{\partial t}(\gamma^{1/2}\mu^*)=0,\quad \mathbf{S}=-\frac{1}{\gamma^{1/2}}\frac{\partial}{\partial t}(\gamma^{1/2}\mathbf{F}),\quad F_\alpha=-U_{,\alpha}. \quad (24a)$$

Thus, in the privileged frame of reference one can give a certain physical meaning to the localization and transport of the energy of the free gravitational field if this energy is regarded as a 3-scalar density of the active mass in vacuum of the form

$$\mu_g{}^*=\frac{2}{\varkappa}\Theta=\frac{2}{\varkappa}\left(U^{-1}\frac{\partial D}{\partial t}+D_\beta{}^\alpha D_\alpha{}^\beta\right),\quad D_{\alpha\beta}=\frac{1}{2U}\frac{\partial\gamma_{\alpha\beta}}{\partial t},\quad D=D_\alpha{}^\alpha=\frac{1}{U}\frac{\partial\ln\gamma^{1/2}}{\partial t}. \quad (24b)$$

In contrast to the pseudotensor expressions for the energy, it does not depend on the choice of the spatial coordinates and is determined by the nonstatic nature of the gravitational field in terms of the contraction of the 3-tensor of the rates of deformation, i.e., the exterior curvature of the distinguished spatial sections $V_3(t=\text{const})$ with variable metric $dl^2=\gamma_{\alpha\beta}dx^\alpha dx^\beta$, though this is achieved at the price of its containing second derivatives, and it is not a positive-definite quantity. In agreement with the results of Pirani and Komar,[17] the general-relativistic analog of the Poisson equation (4) and the Gauss theorem (16),(20) for arbitrary isolated matter distributions in the physically distinguished frame of reference, which is quasi-inertial at infinity, lead to a Tolman form of the total energy integral (18), which is defined as the total gravitational mass, like the charge in electrodynamics, by the flux of the intensity 3-vector $F_\alpha=-U_{,\alpha}$ and is determined by the Schwarzschild asymptotic behavior of the 3-scalar potential $U=\sqrt{g_{00}}$, which describes the longitudinal Coulomb component of the gravitational field. Together with the well-known 3-geometrical representation of the Hamiltonian of general relativity[11] (which is expressed in terms of the Cauchy data for the Einstein equations, i.e., the metric and the extrinsic curvature of any asymptotically flat spatial section $V_3(t = \text{const})$, which are related to the dynamical transverse part of the free gravitational field), the Tolman potential form of the active mass (18) can also be used to investigate the problem of the positive definiteness of the total mass-energy of isolated systems (cf. Refs. 18 and 19). Because of the specific global nonlinearity of the Einstein equations, the gravitational field occurs as an effective self-source and makes a quite definite contribution to the conserved mass-energy integral of the system; this contribution can be decomposed formally into the positive energy of the free radiation field, i.e., the gravitons, and the negative delocalized energy of the attractive Coulomb interaction of the matter and waves. The case of spherically symmetric matter distributions is a most important and instructive example, since it follows from Birkhoff's theorem that there is no gravitational radiation and the total energy integral of a bounded sphere is uniquely determined by regular matching of the interior solution of the Einstein equations to the exterior metric of the Schwarzschild field—the only one possible in vacuum, so that here it is easy to elucidate the necessary conditions guaranteeing positivity of the Schwarzschild mass-energy $E = Mc^2$ in general relativity.

2. POSITIVITY OF THE TOTAL MASS-ENERGY OF A NONSTATIC SPHERE

A characteristic manifestation of the nonlinear self-interation effects of the gravitational field in general relativity is the partial self-screening in the relativistic Poisson equation (10), which takes the form of a dependence of the active mass density μ^* on the potential $U = e^{\nu/2}$ which generates it. In accordance with (8) and (11), the addition of a spherical layer lowers the potential and decreases the active mass (18),(19) of the interior region of the sphere, so that in general relativity distributions is violated in a distinctive manner. In the nonrelativistic limit of weak fields with

$$U=\left(1+\frac{2\Phi}{c^2}\right)^{1/2}\approx 1+\frac{\Phi}{c^2},\quad \frac{|\Phi|}{c^2}\sim\frac{R_g}{R_0}\approx\frac{GM}{c^2R_0}\ll 1$$

there follows after linearization from (4) and (10) an equation of Neumann-Seeliger type for the Newtonian potential,

$$\nabla^2\Phi-\Lambda^2\Phi=\frac{4\pi G}{c^2}\mu,\quad \Lambda=\left(\frac{4\pi G}{c^4}\mu\right)^{1/2},\quad \mu=(\varepsilon+3P)\approx\rho c^2$$

with finite gravitational screening radius of the order of the radius of curvature of the non-Euclidean space V_3 (3a),(5), this being due to the negative compensating contribution of the gravitational potential energy to the active mass density (10) of the matter distribution. Taking the simple example of an equilibrium sphere of nonrelativistic fluid in the Newtonian approximation ($|\Phi|/c^2 \ll 1, \epsilon = \rho c^2 + e, e/\rho c^2 \ll 1$), we readily see that the Tolman integral of the active Schwarzschild mass (19) includes not only the rest mass and the internal energy $\mathscr{E}$ of the fluid but also the negative gravitational energy Ω_0 (see Refs. 3 and 4):

$$E=Mc^2=\mathscr{M}_0c^2+\mathscr{E}+\Omega_0,\quad \Omega_0=-\alpha G\mathscr{M}_0^2/R_0,\quad \alpha\sim 1. \tag{25}$$

The active geometrical mass (5),(6) contains the same delocalized field contribution and differs, because of the non-Euclidean nature of the physical space $V_3(t = \text{const})$ of the static R system (2), from the total self-energy of the fluid

$$\mathscr{M}_0=\mathscr{M}(R_0)=4\pi\int_0^{R_0}\varepsilon e^{\lambda/2}r^2\,dr$$

by the negative binding energy, i.e., the gravitational mass defect[3,4]:

$$\Omega=(M-\mathscr{M}_0)c^2=4\pi\int_0^{R_0}(1-e^{\lambda/2})\varepsilon r^2\,dr$$
$$=\int_0^M\left\{1-\left[1-\frac{\varkappa m}{4\pi r(m)}\right]^{-1/2}\right\}dm. \tag{26}$$

In constrast to the Newtonian gravitational binding energy, for which there is no lower bound, the general-relativistic mass defect of a sphere made of normal matter with ϵ, $P > 0$ cannot exceed its self-energy. For example, for equilibrium relativistic states of gravitating matter of the type of neutron stars[3,4] it follows fromBondi's estimates[20a] that for $\epsilon > P > 0$ we have $e^{\lambda/2} < 2, \mathscr{M}_0 < 2M, |\Omega| \leqslant M \leqslant \mathscr{M}_0$. Thus, the total Schwarzschild mass-energy of a sphere must always be positive in accordance with the two different integral representations (6) and (19) of it, this also being true for nonstatic matter distributions in an R region of V_4 with $e^{-\lambda} > 0, \mu^* > 0, m' > 0$.

It is well known[3,20b] that if one does not require the matter distribution to be static the total energy of a sphere can be made arbitrarily smally by virtue of the gravitational mass defect. Since the negative potential enery $\Omega \sim GM^2/R_0$ of the attraction decreases the positive active mass $M > 0$ and progressively weakens its gravitational self-interaction, one would expect on qualitative arguments that the total mass-energy $E = Mc^2 > 0$ of a sphere should not change sign and in the limit of maximally strong binding $M \to 0$.[11a] However, study of this equation in the framework of the canonical R system (2) is not exhaustive, since it is unsuitable for describing the T regions within the gravitational radius of the matter distribution due to the appearance of null singularities like the Schwarzschild sphere, where $r \leqslant \kappa m(r,t)/4\pi, e^{-\lambda} < 0$, i.e., the angular coefficient of the metric (2) acquires a time-like nature and cannot be taken as a radial coordinate. In fact, the Gauss theorem (16) and the integral representations (6),(18) of the active mass are valid for only a restricted class of nonstatic matter distributions in only R regions of V_4 with $e^{-\lambda} = 1 - (\kappa m/4\pi r) > 0$.

Because of the existence of T regions of V_4 with matter,[10] it is necessary to use the formulation of the Einstein equations for general diagonal spherically symmetric metrics on the basis of the Lemaître invariant mass function[14]

$$\varkappa m(\tau,\chi)/4\pi=R(1+e^{-\sigma}\dot{R}^2-e^{-\omega}R'^2). \tag{27}$$

For the R system (2), this quantity goes over into the distribution of the active "geometrical" mass (5),(6) of the matter; it is constant in the exterior vacuum region, and due to the continuity at the boundary its value can be identified with the constant gravitational mass of the vacuum Schwarzschild field, $M = m(\tau,\chi)|_g$, which determines the integral of the total mass-energy $E = Mc^2$ of the sphere in general relativity.[14]

It is convenient to investigate the gravitational mass defect and the positive definiteness of the total mass-energy of a nonstatic sphere in the general case in comoving ($T_0^1 = 0$) frame of reference with metric[2]

$$-ds^2 = -e^{\sigma(\tau,\chi)}d\tau^2 + e^{\omega(\tau,\chi)}d\chi^2 + R^2(\tau,\chi)\,[d\vartheta^2 + \sin^2\vartheta\, d\varphi^2], \quad (28)$$

where χ is a Lagrangian radial coordinate. For hydrodynamic energy-momentum-stress tensor the conservation laws $T^k_{i;k} = 0$ and the Einstein equations take a simple form in terms of the Lemaître mass function $m(\tau,\chi)$[14]:

$$m' = 4\pi\varepsilon R^2 R', \quad \dot{m} = -4\pi P R^2 \dot{R}, \qquad (29)$$
$$e^{-\sigma}\dot{R}^2 = e^{-\omega}R'^2 - 1 + \varkappa m/4\pi R.$$

If we introduce auxiliary quantities of the "radial" velocity,

$$v = e^{-\sigma/2}\dot{R} = e^{-\sigma/2}\left(\frac{\partial R}{\partial \tau}\right)_\chi = D_\tau R,$$

$$W = e^{-\omega/2}R' = e^{-\omega/2}\left(\frac{\partial R}{\partial \chi}\right)_\tau = D_\chi R,$$

then in the new notation the dynamical equation

$$D_\tau^2 R = D_\tau v = -\frac{\varkappa}{4\pi R^2}(m + 4\pi P R^3) - \frac{W^2}{P+\varepsilon}\left(\frac{\partial P}{\partial R}\right)_k \qquad (30)$$

is the relativistic equivalent of the inverse square law, like (7), and its first integral

$$v^2 = W^2 - 1 + \varkappa m/4\pi R \qquad (31)$$

is the analog of the Newtonian energy equation $f = W^2 - 1$ for a spherical layer with $\chi = \text{const}$. Therefore, the function

$$W(\chi,\tau) = e^{-\omega/2}R' = (1 + v^2 - \varkappa m/4\pi R)^{1/2}$$

can be interpreted as the relativistic specific energy of Lagrangian spherical shells, which, besides the self-energy of the fluid, includes the kinetic energy of its "radial" motion and the gravitational potential binding energy. We shall restrict ourselves here to considering centrally symmetric matter distributions for which the spatial sections $V_3(\tau = \text{const})$ in the comoving system (28) have a topological center $\chi = 0$, where $R(0,\tau) = 0$ and the metric (28) is locally regular and Euclidean: $e^{\omega/2} = R'$, $W(0,\tau) = 1$, $v(0,\tau) = 0$, $\dot{v}(0,\tau) = 0$, $m(0,\tau) = 0$. Then the Lemaître active gravitational mass

$$m(\tau,\chi) = 4\pi\int_0^\chi \varepsilon R^2 R'\, d\chi = 4\pi\int_0^\chi \varepsilon W e^{\omega/2} R^2\, d\chi = \int_0^\chi \varepsilon W\, dV \qquad (32)$$

gives the total energy contained within the Lagrangian sphere, and due to the non-Euclidean nature of V_3 for $W \neq 1$ it is not equal to the self-energy of the fluid in the sphere,

$$\mathcal{M}(\tau,\chi) = 4\pi\int_0^\chi \varepsilon R^2 e^{\omega/2}\, d\chi = \int_0^\chi \varepsilon\, dV, \qquad (33)$$

so that even for "dust" with $P = 0$ (see Refs. 7 and 8) it differs from the total conserved rest mass

$$\mathcal{M}_0(\chi) = 4\pi\int_0^\chi \rho R^2 e^{\omega/2}\, d\chi = \int_0^\chi \rho\, dV.$$

When an individual spherical layer is added, the ratio of their increments

$$(\partial m/\partial \mathcal{M})_\tau = R' e^{-\omega/2} = W(\chi,\tau) = \pm(1 + v^2 - \varkappa m/4\pi R)^{1/2} \qquad (34)$$

differs for $W < 1$ by the amount of the negative gravitational binding energy and for $W > 1$ by the excess kinetic energy of the matter. These general-relativistic nonlinear effects—the gravitational mass defect and the gravitation of kinetic energy—are manifested through the on-Euclidean nature of the comoving space V_3, since the active specific energy $W = m'/\mathcal{M}'$ determines also the geometry of the spatial sections $\tau = \text{const}$ in (28):

$$dl^2 = \frac{dR^2}{W^2(R)} + R^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2), \quad e^\omega = \frac{R'^2}{W^2}. \qquad (35)$$

In particular, the sign of the quasi-Newtonian energy $f = W^2 - 1$, which characterizes the infinite ($f > 0$) or finite ($f < 0$) types of motion of the Lagrangian layer ($\Lambda = 0$), is opposite to the sign of the scalar curvature of V_3 (see Ref. 7):

$$\mathcal{K} = \frac{2}{R^2}\frac{d}{dR}(fR).$$

For bounded centrally symmetric models, when there is an exterior vacuum region with unique (by Birkhoff's theorem) Schwarzschild field, it follows from the requirement of their continuous matching to the vacuum[16] that on the boundary sphere $\chi = \chi_0 = \text{const}$, where necessarily $P(\chi_0,\tau) = 0$, the constant (by virtue of (29)) value of the Lemaître mass function (32) gives the conserved total mass-energy of the sphere in the form of an integral[14b]: $M = m(\chi_0,\tau) = \text{const}$, and this is expressed in terms of the initial distribution of the matter alone.

From the condition for the frame to be comoving, $T_0^1 = 0$,[2] which can be written in the form

$$D_\tau \ln W = -\frac{v}{\varepsilon + P}\left(\frac{\partial P}{\partial R}\right)_\tau, \qquad (36)$$

it follows that the rate of change of the specific active energy of a Lagrangian spherical layer is determined by the pressure gradient. Therefore, in the simplest case of dust matter with $P = 0$, and also for homogeneous distributions with $P = P(\tau)$ in the comoving system (28) (for example, singular T models of a "sphere"[1b] with $W \neq 0$) we have the helpful integral

$$W(\chi) = [1 + f(\chi)]^{1/2},$$

which for isotropic Friedmann models ensures separation of the variables in the metric (28)[2-4,14a]:

$$-ds^2 = -d\tau^2 + a^2(\tau)\{d\chi^2 + S^2(\chi)\,[d\vartheta^2 + \sin^2\vartheta\, d\varphi^2]\},$$
$$R(\chi,\tau) = a(\tau)S(\chi), \quad m(\chi,\tau) = \mu(\tau)S^3(\chi),$$
$$S(\chi) = \begin{cases}\sin\chi \\ \chi \\ \operatorname{sh}\chi\end{cases} \quad k = \begin{cases}+1 \\ 0, \\ -1\end{cases} \qquad (37)$$

$$\frac{\varepsilon}{P+\varepsilon} = -\frac{3\dot{a}}{a}, \quad \dot{\mu} = -4\pi P a^2 \dot{a}, \quad \mu(\tau) = \frac{4\pi}{3}\varepsilon a^3 \neq \text{const}, \quad P \neq 0.$$

In accordance with (29), the variation in time of the instantaneous active mass $m(\chi,\tau)$, the equivalent of the total energy of the interior "fluid sphere," is due to the work of the pressure forces on its boundary sphere, so that for dust with $P = 0$ in the Tolman-Bondi models[2,7] a gravitational mass distribution $m(\chi)$ of the type (33) is an integral of the motion together with the conserved rest mass $\mathcal{M}_0(\chi)$ and the specific relativistic energy $W(\chi) = m'/\mathscr{M}'_0 = dm/d\mathscr{M}_0$.[7,8] For $P=0$, such exact Tolman-Bondi solutions[2,7,8] admit regular matching to the vacuum in the coordinate system (28), which is comoving with the "dust," and the necessary requirement for such matching is continuity of the active mass on the boundary with the exterior Schwarzschild field. Therefore, the value of the arbitrary function $m(\chi)$ (which specifies the interior distribution of the active gravitational mass) on the boundary with the vacuum $\chi = \chi_0 = \text{const}$ determines the total mass-energy $E = Mc^2$ of the gravitating dust sphere.[7–10].

$$M=m(\chi_0)=4\pi\int_0^{\chi_0}\rho R^2R'\,d\chi=4\pi\int_0^{\chi_0}\rho WR^2e^{\omega/2}=\int_0^{\mathscr{M}_0(\chi_0)}W\,d\mathscr{M}_0. \quad (38)$$

As is shown in Refs. 9. 10. and 14a, the presence of T regions in V_4 makes it possible in principle, contrary to Bondi's opinion,[7] for there to be semiclosed centrally symmetric models with non-Euclidean topology of the spatial sections $V_3(\tau = \text{const})$ in which the radius of curvature $R(\chi,\tau)$ of the Lagrangian spheres and the active mass $m(\chi,\tau)$ vary nonmonotonically: With increasing distance from the center $\chi = 0$, they first increase, reaching a maximum at the throat, where

$$R'(\chi^*,\tau)=m'(\chi^*,\tau)=W(\chi^*,\tau)=0, \quad e^{\omega/2}=R'/W\neq 0,$$

and they then begin to decrease. This "throat" $\chi^* = \chi^*(\tau)$ (for $P=0$, it is fixed in the Tolman-Bondi models: $\chi^* = \text{const}$) is characterized by the fact that at it the relativistic specific energy changes sign and, together with the regular branch of the distribution $R'<0$, $m'<0$, $W=(\partial m/\partial\mathscr{M})<0$ it must necessarily be in a T region.[10] Therefore, in such a maximally strong field the gravitational binding energy of the matter may be greater than its self-mass and with the buildup of layers after the throat with $W<0$ the active mass of the sphere decreases, $m' = W\mathscr{M}'<0$, so that when matter is added the exterior Schwarzschild-Kruskal field becomes weaker; this is the self-screening effect.[9,10] The question naturally arises of whether one could construct such semiclosed models from normal matter ($\epsilon,P>0$) in such a way that, through the predominant contribution of the negative gravitational binding energy of the layers with $W<0$ after the throat in the interior T region, an inversion in the sign of the active mass of the sphere is obtained. However, the correct matching of the bounded semiclosed models to the vacuum must necessarily be made through the exterior T region of the maximally extended Schwarzschild-Kruskal field,[9] which exists only when $M>0$, and it is therefore impossible to obtain a regular, causally complete solution for a sphere with negative gravitational mass $M<0$.

Indeed, the boundary conditions of smooth matching of the V_4 geometry of the interior region of the sphere and the vacuum Schwarzschild field presuppose continuity of the metric (28) and its first partial derivatives, this requirement being necessary for the invariant angular coefficient $R(\chi,\tau)$ and the invariant mass function $m(\chi,\tau)$.[16] For semiclosed models, when the boundary sphere $\chi = \chi_0 = \text{const}$ lies behind the throat with $R' = m' = W = 0$, the inequalilty $R'(\chi,\tau)<0$ must also hold in the vacuum, and $R(\chi,\tau)$ first decreases with increasing χ. But for $M>0$ it is possible to have a regular change of the sign $R'<0$ at the vacuum "throat" $R'(\chi^*,\tau)=0$ in the T region of the Schwarzschild-Kruskal field with $R(\chi,\tau)\leqslant \varkappa M/4\pi$ and then a transition along the branch with $R'\geqslant 0$ to the asymptotically flat R region of V_4. The complete solution for such a sphere with positive mass-energy $M>0$ necessarily has a time singularity on the spacelike hypersurface $R(\tau,\chi)=0$ within the Schwarzschild sphere, and this singularity can be extended continuously to the interior T region as collapse of the matter: $\epsilon,P\to 0$. If $M<0$, then in the globally static Schwarzschild field there are no T regions at all, so that the derivative R' cannot change sign but decrease monotonically, and in the vacuum there is unavoidably a spatial singularity, $R=0$, of the type of a localized source with negative mass.

Thus, the physically admissible centrally symmetric distributions of normal matter ($\epsilon,P>0$) in the form of a bounded sphere that satisfy the criterion of minimal regularity of the complete "fitted" solution of the Einstein equations at least on one initial hypersurface $V_3(\tau = \text{const})$ always have positive total mass-energy $E = Mc^2>0$. Therefore, the gravitational mass defect of any nonstatic sphere in general relativity and the maximally possible energy release on its formation do not exceed the self-energy of the matter. For semiclosed models, the active Schwarzschild mass $M = m(\chi_0,\tau) = \text{const}$ can vanish together with $R(\chi_0,\tau)\to 0$ when the boundary sphere $\chi=\chi_0$ degenerates into a point. This corresponds to closing of the V_3 space with 3-spherical topology in the limit of complete gravitational binding of the entire self-energy of a finte amount of matter (as, for example, for the closed Friedmann model (37) as $\chi_0\to\pi$)[2–5,9]: $R = a\sin\chi_0$, $M=\mu\sin^3\chi_0\to 0$.

The positivity of the total conserved energy of the gravitational field and its material sources for isolated systems (when correctly determined using the asymptotically Schwarzschild behavior of the V_4 metric at spatial infinity[2–6]) has fundamental significance as the criterion for the existence of a ground state (and, in the first place, a stable vacuum) of the gravitational field and its sources and is necessary if general relativity is to be physically reasonable. This problem has been studied on a number of occasions,[11,18] primarily for free vacuum Einstein fields of the type of localized wave packets, and positivity has been established for the integral of the Schwarzschild mass-energy for both weak[18a] and strong gravitational fields in the case of Euclidean topology of asymptotically flat V_3 spaces,[19] and also in some wave solutions $R_i^k = 0$ with axial symmetry.[18b] Therefore, of greater importance is the question of the sign of the total mass-energy of isolated systems in the presence of normal matter ($\epsilon,P>0$) in view of the negativity of the contribution of its gravitational energy of the attractive Coulomb interaction due to the longitudinal nonwave part of the tensor massless field. In this respect, the most significant case is the critical case of centrally symmetric semiclosed models with non-Euclidean V_3 topology, when the total mass-energy of the sphere is a well-defined quantity, there is no positive contribution of the radiation field, and the negative gravitational binding energy of the matter is maximal.[11] Since for all physically acceptable models of a sphere the gravitational mass defect cannot exceed the matter self-energy, the existence of this lower bound for it guarantees positive definiteness of the conserved mass-energy as well for general systems with normal matter satifying the dominant energy condition, which evidently solves this problem in general relativity.[19]

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