Effects of partial wave-front reversal during the reflection of waves in randomly inhomogeneous media

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The effect of partial reversal of the wave front during the reflection of a wave in a randomly inhomogeneous medium is discussed in detail. It is noted that the presence of the reversed component in the field of the reflected wave leads to the appearance of long-range correlations of the reflected field and the enhancement of the mean intensity of the reflected wave in the focal plane of a lens.

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As is well known, the backscattering of waves in a large-scale randomly inhomogeneous medium leads to the appearance of specific effects connected with the double transmission of the forward and backward waves through the same inhomogeneities. These are primarily the mean-backscattering-intensity enhancement effect,[1,2] the multichannel coherence effects,[3] and the similar effects of phase-fluctuation intensification and reflected-wave-intensity enhancement (see Ref. 4 and the references cited therein). In the present paper we discuss another class of phenomena connected with the double transmission: the effects of the partial reversal of the wave front during the reflection of a wave in a randomly inhomogeneous medium.

§1. PARTIAL PHASE REVERSAL IN THE CASE OF TWO POINT SOURCES

Let us consider the statistical properties of waves reflected from obstacles located in a large-scale randomly inhomogeneous medium whose permittivity $\tilde{\varepsilon}(\boldsymbol{\rho}, x)$ (where x and $\boldsymbol{\rho}$ are the longitudinal and transverse coordinates respectively) undergoes fluctuations. For definiteness, we shall describe the propagation of the wave incident on the reflector and the wave reflected backwards in the approximation of the parabolic-equation method (PEM).[5]

Let the complex amplitude of the wave at the beginning of the path (i.e., in the $x = 0$ plane) be equal to $u_0(\boldsymbol{\rho})$. If a reflector with a local reflection coefficient $f(\boldsymbol{\rho})$ is located at the end of the path (i.e., in the $x = L$ plane), then the complex amplitude of the reflected wave in the $x = 0$ plane is equal to[6,7]

$$u_s(\boldsymbol{\rho}) = \int u_0(\mathbf{q})\, g(\mathbf{q}, \boldsymbol{\rho})\, d\mathbf{q}. \tag{1}$$

Here

$$g(\mathbf{q}, \boldsymbol{\rho}) = \int f(\mathbf{p})\, G(\mathbf{q}, 0; \mathbf{p}, L)\, G(\boldsymbol{\rho}, 0; \mathbf{p}, L)\, d\mathbf{p}, \tag{2}$$

where $G(\boldsymbol{\rho}', x', \boldsymbol{\rho}, x)$ $(x' \leqslant x)$ is the stochastic Green function, which satisfies within the framework of the PEM the equation

$$2ik\frac{\partial G}{\partial x} + \Delta_\rho G + k^2 \tilde{\varepsilon}(\boldsymbol{\rho}, x)\, G = 0, \qquad G(\boldsymbol{\rho}', x'; \boldsymbol{\rho}, x') = \delta(\boldsymbol{\rho} - \boldsymbol{\rho}'). \tag{3}$$

The function $g_{12} = g(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)$, (2), possesses the reciprocity property:

$$g_{12} = g_{21}, \tag{4}$$

a consequence of which is, in the final analysis, the partial wave-front reversal effects.

Let us, using the general formula (1), consider the case in which the incident wave is emitted by two point sources located at the points $\boldsymbol{\rho} = \boldsymbol{\rho}_1$ and $\boldsymbol{\rho} = \boldsymbol{\rho}_2$ in the $x = 0$ plane:

$$u_0(\boldsymbol{\rho}) = u_1 \delta(\boldsymbol{\rho} - \boldsymbol{\rho}_1) + u_2 \delta(\boldsymbol{\rho} - \boldsymbol{\rho}_2).$$

For such a prescription of the primary field, the complex amplitude (1) of the reflected wave is equal to

$$u_s(\boldsymbol{\rho}) = u_1 g(\boldsymbol{\rho}_1, \boldsymbol{\rho}) + u_2 g(\boldsymbol{\rho}_2, \boldsymbol{\rho}). \tag{5}$$

Accordingly, the cross-coherence function for the reflected-wave field at the points where the emitters are located contains four terms:

$$\langle u_s(\boldsymbol{\rho}_1)\, u_s^*(\boldsymbol{\rho}_2)\rangle = |u_1|^2 \langle g_{11} g_{12}^*\rangle + |u_2|^2 \langle g_{21} g_{22}^*\rangle + u_1 u_2^* \langle g_{11} g_{22}^*\rangle + u_1^* u_2 \langle g_{21} g_{12}^*\rangle. \tag{6}$$

Let us locate the sources such that the distance between them is greater than the coherence length ρ_c of a spherical wave that has traversed a path of length L through the randomly inhomogeneous medium (i.e., such that $|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2| \gg \rho_c$). The quantity $\rho_c(L)$ is determined from the well-known equation[5]

$$D_\psi(\rho_c, L) = 1,$$

where

$$D_\psi(\boldsymbol{\rho}, L) = \frac{k^2}{4}\int_0^L D\left(\boldsymbol{\rho}\frac{x}{L}\right) dx, \quad D(\boldsymbol{\rho}) = A(0) - A(\boldsymbol{\rho}),$$

$$A(\boldsymbol{\rho}) = \int_{-\infty}^{\infty} \langle \tilde{\varepsilon}(\boldsymbol{\rho}', x)\, \tilde{\varepsilon}(\boldsymbol{\rho}' + \boldsymbol{\rho}, x + \tau)\rangle\, d\tau.$$

Let us also assume that the random phase advances that occur on a randomly inhomogeneous path of length L are not great, so that $\langle g\rangle = 0$. When these conditions are fulfilled, all the mean quantities on the right-hand side of (6),

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except the last term, are equal to zero on account of the uncompensated random-phase advances occurring before and after the reflection. Thus, the equality (6) assumes, when allowance is made for the reciprocity property (4), the form

$$\langle u_s(\rho_1)u_s^*(\rho_2)\rangle = u_1^* u_2 \langle I(\rho_1, \rho_2)\rangle, \qquad (7)$$

where $I(\rho_1, \rho_2)$ is the intensity at the point $\rho = \rho_2$ in the $x = 0$ plane of the reflected wave generated by the unit source at the point $\rho = \rho_1$. Similarly, the mean reflected-wave intensity at, for example, $\rho = \rho_1$ is equal to

$$\langle I_s(\rho_1)\rangle = \langle |u_s^2(\rho_1)|\rangle = |u_1^2|\langle I(\rho_1, \rho_1)\rangle + |u_2^2|\langle I(\rho_1, \rho_2)\rangle.$$

If the strengths of the sources are comparable, i.e., if $|u_1| \approx |u_2|$ and $\langle I(\rho_1, \rho_1)\rangle \approx \langle I(\rho_1, \rho_2)\rangle$, then the correlator (7) is smaller than the mean reflected-wave intensity by only a factor of two. Thus, there is long-range correlation between the waves that have traversed in opposite directions the path $\rho_1 \rightleftarrows$reflector$\rightleftarrows \rho_2$. Let us note that a similar cross-coherence of waves that have traversed the same inhomogeneities in opposite directions is pointed out in Ref. 8.

Evidently, the high degree of coherence of the reflected-wave fields at the widely separated points ρ_1 and ρ_2 decreases sharply as soon as the distances between the points of observation and the points where the emitters are located are greater than ρ_c. Figure 1 shows the qualitative shape of the reflected-wave-coherence function in the case when the points of observation are located symmetrically along the straight line joining the emitters:

$$\Gamma_s(\rho) = \langle u_s(\tfrac{1}{2}\rho_1(1+\rho/R) + \tfrac{1}{2}\rho_2(1-\rho/R)) \cdot$$
$$u_s^*(\tfrac{1}{2}\rho_1(1-\rho/R) + \tfrac{1}{2}\rho_2(1+\rho/R))\rangle, \quad R = |\rho_1 - \rho_2|.$$

The above-indicated long-range correlation of the reflected-wave fields at the widely separated points ρ_1 and ρ_2 at which the sources are located is due to the fact that the mutually coherent components of the reflected field at these points are phase-reversed relative to the fields of the sources. Let us explain the phase-reversal mechanism that leads to the situation in which the correlator (7) is proportional to $u_1^* u_2$. Let the first source emit a wave with phase ψ_1; the second, a wave with phase ψ_2. Then the mutually coherent components of the reflected-wave field at the locations of the sources are equal to

$$u_s(\rho_1) \sim e^{i\psi_2} g_{21}, \quad u_s(\rho_2) \sim e^{i\psi_1} g_{12}.$$

Let us take the random phase fluctuations on the path $\rho_1 \to$reflector$\to \rho_2$ into account by introducing a phase factor into the function g:

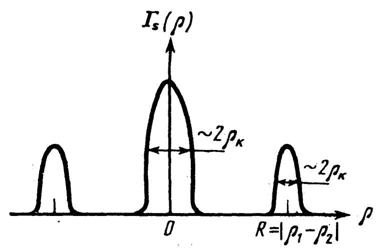

FIG. 1

$$g_{12} = |g| e^{i\psi_{12}}.$$

Then, according to (4), $\psi_{12} = \psi_{21}$, and the phases of the mutually coherent components of the reflected wave at the points ρ_1 and ρ_2 are respectively equal to $\psi_2 + \psi_{12}$ and $\psi_1 + \psi_{12}$. But

$$\psi_2 + \psi_{12} = -\psi_1 + \varphi, \quad \psi_1 + \psi_{12} = -\psi_2 + \varphi; \quad \varphi = \psi_1 + \psi_2 + \psi_{12}.$$

Consequently, up to the common term φ, the phases of the mutually coherent components of the reflected wave at the points ρ_1 and ρ_2 are respectively equal to $-\psi_1$ and $-\psi_2$, i.e., are reversed with respect to the initial phases of the emitted waves.

§2. PARTIAL REVERSAL OF THE WAVE FRONT IN THE CASE OF EXTENDED SOURCES

Let us first carry out a qualitative investigation of the problem. Let us divide the extended source of diameter $d \gg \rho_c$ into separate small beams with diameter $d_0 \sim \rho_c$. Evidently, the number of such partial beams is roughly given by the relation

$$M = d^2/d_0^2 \sim d^2/\rho_c^2 \gg 1.$$

If ρ_m and ρ_n are the centers of the mth and nth beams (the distance between them is greater than the coherence length ρ_c), then, as a result of the phase reversal during the propagation along the randomly inhomogeneous path $\rho_m \rightleftarrows$reflector$\rightleftarrows \rho_n$, the coherence function

$$\Gamma_s(\rho_m, \rho_n) = \langle u_s(\rho_m) u_s^*(\rho_n)\rangle$$

will be proportional to the product $u_0^*(\rho_m)u_0(\rho_n)$. Let us emphasize that the coherence channel $\rho_m \rightleftarrows$reflector$\rightleftarrows \rho_n$ is formed only in the case when the radiations of the partial beams in question overlap in the plane of the reflector. The radius of the spot of each partial beam in the plane of the reflector is, on account of the diffraction-induced divergence of the beam, of the order of

$$\sigma_\rho(L) \sim L/k\rho_c.$$

Thus, the partial reversal of the phases of the reflected-wave fields at the widely separated points ρ_m and ρ_n will occur only if the following condition is fulfilled:

$$\gamma = \sigma_\rho(L)/\rho_c(L) = L/k\rho_c^2 \gg 1, \qquad (8)$$

and, what is more, the greatest reflected-wave-field-correlation length due to the partial phase reversal is equal to d if $d < \sigma_\rho$, and does not exceed σ_ρ if $d > \sigma_\rho$.

Let us note further that, since only the fields of two partial beams are responsible for long-range correlations of the fields at the points ρ_m and ρ_n, whereas the intensity of the reflected-wave field is the sum of the intensities of many partial beams, $\langle I_s\rangle \gg \Gamma_s(\rho_m, \rho_n)$ (quantitative estimates for the ratio of the mean reflected-wave intensity to the magnitude of the long-range correlations will be given below). The general shape of the reflected-wave coherence function is shown in Fig. 2, in which the pedestal represents the envelope of the small correlation "spikes" of width $\sim \rho_c$, corresponding to the partial beams.

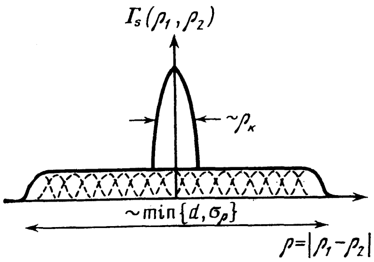

FIG. 2.

Let us note again that the condition (8) is known as the condition for saturated intensity fluctuations.[9] The region of saturated fluctuations is characterized by a multibeam wave propagation that leads to the formation of the $\boldsymbol{\rho}_m \rightleftarrows$reflector$\rightleftarrows \boldsymbol{\rho}_n$-type coherence channels that are responsible for the partial wave-front reversal and the long-range reflected-wave-field correlation.

Let us turn to the quantitative description. According to (1) and (2), the reflected-wave-coherence function can be expressed in terms of the fourth-order moment of the Green function of the incident wave[7]:

$$\Gamma_s(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \int u_0(\mathbf{q}_1) u_0^*(\mathbf{q}_2) f(\mathbf{p}_1) f^*(\mathbf{p}_2) \langle G(\mathbf{q}_1, 0; \mathbf{p}_1, L) \times G(\boldsymbol{\rho}_1, 0; \mathbf{p}_1, L) G^*(\mathbf{q}_2, 0; \mathbf{p}_2, L) G^*(\boldsymbol{\rho}_2, 0; \mathbf{p}_2, L) \rangle d\mathbf{q}_{12} d\mathbf{p}_{12}. \quad (9)$$

As is well known, the general solution to the equation for the fourth-order moment G has not yet been found.[5,9] But in the case when the saturability condition (8) for the intensity fluctuations is fulfilled, and the partial reflected-wave front reversal and long-range reflected-beam-field correlation effects manifest themselves most distinctly, the mean value $\langle G \rangle$ is practically equal to zero, and the stochastic Green functions of the incident wave are asymptotically Gaussian, so that the average of their product can be broken up with the aid of the laws of the Gaussian statistics.[9,10] As a result, the coherence function (9) breaks up into the sum of two terms:

$$\Gamma_s(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \Gamma(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) + \Gamma_{\text{rev}}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2), \quad (10)$$

where

$$\Gamma(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \int u_0(\mathbf{q}_1) u_0^*(\mathbf{q}_2) F(\mathbf{q}_1, \mathbf{q}_2; \boldsymbol{\rho}_1, \boldsymbol{\rho}_2) d\mathbf{q}_1 d\mathbf{q}_2,$$
$$\Gamma_{\text{rev}}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \int u_0(\mathbf{q}_1) u_0^*(\mathbf{q}_2) F(\mathbf{q}_1, \boldsymbol{\rho}_2; \boldsymbol{\rho}_1, \mathbf{q}_2) d\mathbf{q}_1 d\mathbf{q}_2. \quad (11)$$

Here we have introduced the notation

$$F(\mathbf{q}_1, \mathbf{q}_2; \boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \int f(\mathbf{p}_1) f^*(\mathbf{p}_2) \langle G(\mathbf{q}_1, 0; \mathbf{p}_1, L) G^*(\mathbf{q}_2, 0; \mathbf{p}_2, L) \rangle \times \langle G(\boldsymbol{\rho}_1, 0; \mathbf{p}_1, L) G^*(\boldsymbol{\rho}_2, 0; \mathbf{p}_2, L) \rangle d\mathbf{p}_1 d\mathbf{p}_2. \quad (12)$$

The first term on the right-hand side of (10) describes the reflected-wave coherence function without allowance for the correlation between the direct and reflected waves, while the second takes account of the double transmission of a wave through the same inhomogeneities of the medium, and describes the effect of partial reversal of the wave front of the reflected wave.

Using the known[5] expressions for the spherical-wave coherence functions entering into (12), we can express the reflected-wave coherence function Γ_s in terms of integrals that are easily analyzable for virtually all types of reflectors and incident waves. Here we shall limit ourselves to a discussion of the most interesting case of a collimated incident beam of radius $d \gg \rho_c$ propagating along the x axis with an initial complex amplitude $u_0(\boldsymbol{\rho})$. We shall, for simplicity, consider the reflector to be a point reflector: $f(\boldsymbol{\rho}) = f\delta(\boldsymbol{\rho})$. In this case the formulas (11) and (12) give

$$\Gamma(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = |f|^2 \left(\frac{k}{2\pi L}\right)^2 \exp\left\{\frac{ik}{2L}(\rho_1^2 - \rho_2^2)\right\} \times \exp\{-D_\psi(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2, L)\} \int |u_0(\boldsymbol{\rho}')|^2 W(\boldsymbol{\rho}'; L) d\boldsymbol{\rho}',$$
$$\Gamma_{\text{rev}}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = |f|^2 u_0^*(\boldsymbol{\rho}_1) u_0(\boldsymbol{\rho}_2) W(\boldsymbol{\rho}_1; L) W(\boldsymbol{\rho}_2, L), \quad (13)$$

where

$$W(\boldsymbol{\rho}; L) = \left(\frac{k}{2\pi L}\right)^2 \int \exp\left\{\frac{ik}{L}(\boldsymbol{\rho}\mathbf{q}) - D_\psi(\mathbf{q}, L)\right\} d\mathbf{q}. \quad (14)$$

It is convenient to interpret the function $W(\boldsymbol{\rho}; L)$ as the transverse-beam-deflection probability density: it is normalized to unity only by the condition

$$\int W(\boldsymbol{\rho}; L) d\boldsymbol{\rho} = 1$$

and has a width $\sigma_\rho \sim L/k\rho_c$, which can be interpreted here as the root-mean square deflection of the beam for the unperturbed position. The quantity σ_ρ also characterizes the diffraction-induced broadening of each partial beam in the plane $x = L$ of the reflector. Only those partial beams that fall on the reflector participate in the shaping of the mean reflected-wave intensity and in the formation of the coherence channels for the various partial beams. Thus, in the case, under consideration, of a point reflector only the partial beams occurring in the emission plane in a circle of radius $\sim\sigma_\rho$ participate in the partial reversal of the wave front. If $d < \sigma_\rho$, then the number of such beams

$$M \sim (d/\rho_c)^2.$$

If, on the other hand, $d > \sigma_\rho$, then their number is of the order of

$$(\sigma_\rho/\rho_c)^2 = \gamma^2.$$

Consequently, as follows from (13), when $d < \sigma_\rho$, the ratio $\Gamma_{\text{rev}}/\langle I_s \rangle$ is, in order of magnitude, equal to

$$1/M \sim (\rho_c/d)^2.$$

If, on the other hand, $d > \sigma_\rho$, then

$$\Gamma_{\text{rev}}/\langle I_s \rangle \sim 1/\gamma^2.$$

§3. EFFECT OF AMPLIFICATION OF THE REVERSED-COMPONENT INTENSITY UNDER CONDITIONS WHEN THE REFLECTED WAVE IS FOCUSED

In the case, considered here, of broad incident beams, i.e., beams with $d \gg \rho_c$, the double transmission of the wave through the same inhomogeneities of the medium does not lead to the effect whereby the mean intensity of the reflected wave is amplified.[1,4] But the appearance, as a result of the partial reversal of the wave front, of a highly coherent component in the field of the reflected wave creates the conditions necessary for the focusing of this wave with the aid of a lens. This gives rise to a new effect: the amplification of the mean intensity of the reflected wave in the focal plane of the lens precisely as a result of the partial reversal of the wave front of the reflected wave.

Let us illustrate this effect in the simple particular case in which the reflected wave is incident on a lens, located in the $x = 0$ plane, whose aperture coincides with that of the emitter, and is described by the function $u_0(\rho)$. The field in the focal plane of such a lens is equal to

$$v_s(\rho_F)=\frac{k}{2\pi iF}\int u_0(\rho)u_s(\rho)\exp\left\{-\frac{ik}{F}(\rho_F\rho)\right\}d\rho. \qquad (15)$$

Under the conditions of a plane incident—on the reflector—wave (i.e., for $d\gg\sigma_\rho\gg\rho_c$), when the usual amplification of the mean reflected-wave intensity clearly does not occur, we find from (15), (10), and (13) that the mean reflected-wave intensity in the focal plane of the lens is equal to

$$J_s(\rho_F)=\langle|v_s^2(\rho_F)|\rangle=J(\rho_F)+J_{rev}(\rho_F), \qquad (16)$$

where

$$J(\rho_F)=|f|^2\left(\frac{k}{2\pi F}\right)^2|u_0(0)|^2\left|u_0^2\left(\frac{L}{F}\rho_F\right)\right|^2, \qquad (17)$$

$$J_{rev}(\rho_F)=|f|^2\left(\frac{k}{2\pi F}\right)^2|u_0(0)|^4\times\exp\left\{-2D_\psi\left(\frac{L}{F}\rho_F,L\right)\right\}.$$

It can be seen from this that the mean reflected-wave intensity at the center of the focal plane of the lens is equal to twice the intensity in a homogeneous medium:

$$J_s(0)=2|f|^2\left(\frac{k}{2\pi F}\right)^2|u_0(0)|^4=2J(0).$$

This means that an absolute effect of intensity amplification with coefficient $N_F = 2$ should be observed at the center of the focal spot. It can also be seen from (16) and (17) that the amplification effect manifests itself only at a small spot of radius of the order of $\rho_c F/L$, where the reversed component of the reflected wave is focused. Figure 3 shows the intensity distribution in the focal plane of the lens. The broad pedestal, whose dimensions ($\sim dF/L$) are of the order of the dimensions of the intensity-distribution spot for a spherical wave reflected from a point reflector in a homogeneous medium, and "underfocused" in the focal plane of the lens, corresponds to the "usual" component $J(\rho_F)$. The intensity $J_{rev}(\rho_F)$, on the other hand, forms a narrow peak corresponding to the quasiplane (as a result of the reversal) reflected-wave component with coherence length of the order of σ_ρ. The appearance of the narrow peak $J_{rev}(\rho_F)$ can further be interpreted as the result of multichannel coherence effects: the interference of waves propagating along cross channels.[4]

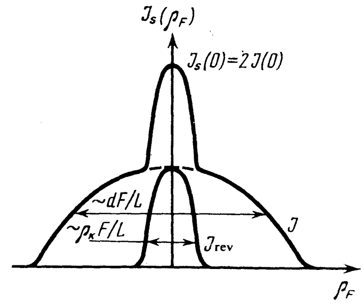

FIG. 3.

In conclusion, we express our profound gratitude to B. Ya. Zel'dovich, who drew our attention to the possibility of the presence of a reversed field component in the reflected wave, and thereby stimulated the investigation summarized in the present paper.

[1]A. G. Vinogradov, Yu. A. Kravtsov, and V. I. Tatarskiĭ, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **16**, 1064 (1973).
[2]A. G. Vinogradov and Yu. A. Kravtsov, Izv. Vyssh. Uchebn. Zaved. **16**, 1055 (1973).
[3]Yu. A. Kravtsov and S. A. Namazov, Radiotekh. Elektron. **25**, 459 (1980).
[4]Yu. A. Kravtsov and A. I. Saichev, Usp. Fiz. Nauk **137**, 501 (1982) [Sov. Phys. Usp.
[5]S. M. Rytov, Yu. A. Kravtsov, and V. I. Tatarskiĭ, Vvedenie v statisticheskuyu radiofiziku (Introduction to Statistical Radiophysics), Part 2, Nauka, Moscow, 1978.
[6]V. I. Gel'fgat, Akust. Zh. **22**, 123 (1976) [Sov. Phys. Acoust. **22**, 65 (1976)].
[7]A. I. Saichev, Izv. Vyssh. Uchebn. Zaved. **21**, 1290 (1978).
[8]K. M. Watson, J. Math. Phys. **10**, 688 (1969).
[9]V. U. Zavorotnyĭ, V. I. Klyatskin, and V. I. Tatarskiĭ, Zh. Eksp. Teor. Fiz. **73**, 481 (1977) [Sov. Phys. JETP **46**, 252 (1977)].
[10]R. Dashen, J. Math. Phys. **20**, 894 (1979).

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