Topological phase transition in the XY model of a spin glass

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The Villain variant of the XY model of a spin glass is considered in the three-dimensional case. It is shown that there occurs in the system of topological phase transition connected with the appearance of frustration lines of infinite length. In the phase thus produced, the correlation distance at zero temperature is infinite. At any finite temperature, however, the correlation has a finite range. The possibility of generalization to the Heisenberg model of spin glasses is discussed.

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1. INTRODUCTION

In recent years, in the theory of spin glasses, the concept of frustration has attracted much attention. Introduced by Toulouse[1] and by Villain[2] from a microscopic point of view, as a topological property of lattices with bonds (edges) of different signs, it received a macroscopic interpretation in papers of Volovik and one of the authors.[3,4] The microscopic frustration lines of Toulouse and of Villain are, from the macroscopic point of view, disclinations in a field of XY or Heisenberg spins.

The concept of frustration lines seems reasonable if the concentration of the so-called "wrong" bonds is sufficiently small. This means that we must begin with a regular ferromagnetic (antiferromagnetic) system, with positive (negative) bonds, and gradually replace the "right" positive (negative) bonds by wrong negative (positive) ones. If the concentrations of positive and of negative bonds are equal, the concept of frustration lines is of doubtful usefulness. Physically, this last case obviously includes spin glasses in dilute magnetic alloys, where everything is determined by the rapidly oscillating long-range Ruderman-Kittel-Kasuya-Yosida (RKKY) interaction.

But what happens to a system, let us say a regular ferromagnet, on gradual increase of the concentration of negative bonds? Here there are two possibilities. In the first case, at a certain concentration $c_{\rm per}$ the percolation limit will be attained, and the system will cease to be ferromagnetic. In Sec. 3 we shall discuss in more detail the properties of the phase thus produced; there is reason to suppose that it is the spin glass with finite rigidity considered by Halperin and Saslow,[5] by Andreev,[6] and by Volovik and one of the authors.[3,4] At a certain other concentration $c_{\rm top}$, the system undergoes a so-called topological transition, in which frustration lines of infinite length first appear. The spin rigidity then vanishes. It is natural to call such a phase a genuine spin glass.

In the second case, the concentration $c_{\rm top}$ at which frustration lines of infinite length appear is less than $c_{\rm per}$, and a spin glass with finite rigidity does not occur at all.

A topological transition connected with the appearance of singular lines (in our case disclinations) of infinite length has already been considered repeatedly as a model of the melting of a crystal (appearance of dislocations of infinite length), fo the superfluid transition in He^4 (by appearance of infinite vortices), etc.; a good review of the problems connected with it is given in a lecture of Halperin.[7] But there is an important difference between, for example, topological melting and the transition in spin glasses. Whereas in the first case the singular lines originate as a result of thermal excitation, in spin glasses they exist also at absolute zero, and their form (configuration) is frozen. This does not mean, however, that for spin glasses thermally excited singular lines are not also important (see, however, Sec. 4).

We shall consider a topological transition in the XY model of a spin glass with the Villain interaction.[2] A disclination in the XY model is analogous to a vortex in superfluid helium; the disclination density $\mathbf{j}(\mathbf{r})$, which is analogous to the superfluid velocity, has the usual form:

$$j_k(\mathbf{r})=2\pi N\int\delta^{(3)}(\mathbf{r}-\mathbf{R}(t))\,dR_k(t). \tag{1.1}$$

Here $\mathbf{r}=\mathbf{R}(t)$ is the equation of the disclination line, and N is its strength (charge). For frustrational disclinations, N is a half-integer: $N=\pm 1/2, \pm 3/2, \ldots$; for thermally excited vortices, an integer: $N=\pm 1, \pm 2, \ldots$. It is clear that it is sufficient to consider frustrational lines with the minimum possible strength $\pm 1/2$, treating states with higher strengths as thermal excitation. For frustrational lines, their form at $T=0$ is given by an *a priori* random distribution $W_q\{\mathbf{R}(t\}$, and the strengths are determined by the interaction energy, i.e. by a Gibbs distribution. For disclinations, both the form and the strength are given by a Gibbs distribution.

Disclinations interact like electric currents in magnetostatics:

$$H_{int}=\frac{1}{2}J\int\frac{\mathbf{j}(\mathbf{r}_1)\mathbf{j}(\mathbf{r}_2)}{|\mathbf{r}_1-\mathbf{r}_2|}\,d^3x_1\,d^3x_2. \tag{1.2}$$

The constant J, to within a factor, coincides with the rigidity of the standard XY model:

$$H_{XY}=\frac{1}{2}\rho_s\int(\nabla\chi)^2d^3x. \tag{1.3}$$

For given form and number of the frustration lines, the distribution of currents on them (i.e. of disclination strengths $\pm 1/2$) is determined by the partition function

$$Z_{\rm cur}=\sum_{\{\pm\}}\{-H_{int}/T\}. \tag{1.4}$$

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If it is desired to average over the form (configurations) of the frustration lines, then it is necessary to average, as always, not $Z_{\rm cur}$ but the free energy $F_{\rm cur}$. For the correctly averaged free energy F, the standard formula holds:

$$F=-T\langle \ln Z_{\rm cur}\rangle_{\rm conf}=-T\int D\mathbf{R}(t)\,W_q\{\mathbf{R}(t)\}\ln Z_{\rm cur}. \quad (1.5)$$

The best method for calculating F is the method of replicas.[8] In this method, n identical systems are introduced, but the quantity

$$\frac{1}{n}\left(\langle Z^n_{\rm cur}\rangle_{\rm conf}-1\right),$$

is calculated, and then n is made to approach zero. This technique brings the whole calculation within the framework of the standard Gibbs theory. We number the n identical systems of disclinations with the index $\alpha = 1, ..., n$. The interaction of the currents $\mathbf{j}_\alpha$ is given by the obvious formula

$$\mathscr{H}_n=\frac{1}{2}J\sum_\alpha\int\frac{\mathbf{j}_{\alpha1}\mathbf{j}_{\alpha2}}{R_{12}}\,d^3x_1\,d^3x_2. \quad (1.6)$$

One calculates the partition function of the replicas,

$$\mathfrak{Z}_n=\langle\exp\{-\mathscr{H}_n/T\}\rangle, \quad (1.7)$$

where the symbol $\langle ...\rangle$ is now understood to mean averaging both over the currents (1.4) and over the configurations (1.5) Finally, $\mathfrak{Z}_n$ is expanded in n, and the linear term is extracted.

2. THERMODYNAMICS OF FRUSTRATION LINES

The averaging over configurations of frustration lines actually coincides with the theory of polymers. A technically convenient form of it was developed by des Cloizeaux.[9] We shall use one of its variants, due to Nikomarov and one of the authors.[10] We begin with the case of high temperatures, when it is possible to neglect the interaction (1.6).

It is more convenient to begin with a discrete variant of the distribution $W_q\{\mathbf{R}(t)\}$ of (1.5) and of the energy of interaction (1.6). The distribution of frustration lines, in length and form, is then replaced by the distribution of an N-component (real) field $\varphi_\gamma(\mathbf{r}_a)$, given at the sites of a three-dimensional cubic lattice $\{\mathbf{r}_a\}$. With each edge (segment, bond), connecting adjacent sites, is associated interaction energy U. The number N mentioned above is in our case equal to

$$N=2^n, \quad (2.1)$$

where n is the number of replicas. As we shall see, this choice gives the right number of states (strenghts $\pm 1/2$) on each frustration line when $n\to 0$.

At infinite temperature, $\mathfrak{Z}_n$ is given by the continuous integral over the fields φ_γ; $\gamma = 1, ..., N$:

$$\mathfrak{Z}_n^{fr}=\int D\varphi_\gamma(\mathbf{r}_a)\prod_a\left(1+\sum_\gamma\varphi_\gamma^2(\mathbf{r}_a)\right)\times\exp\left\{-\frac{1}{2U}\sum_{\langle ab\rangle,\gamma}\varphi_\gamma(\mathbf{r}_a)\varphi_\gamma(\mathbf{r}_b)\right\}. \quad (2.2)$$

The summation in (2.2) extends over all nearest neighbors $\langle ab\rangle$. By calculating the functional integral, one can easily show[9,10] that it is the partition function of the system of closed polymers-frustration loops; each loop may exist in $N = 2^n$ different states:

$$\mathfrak{Z}_n^{fr}=\sum_{LM}C_{LM}U^LN^M. \quad (2.3)$$

Here L is the total number of segments (length) of all closed loops, M is the number of closed loops, and C_{LM} is the number of different configurations on the lattice containing M loops of L segments.

The bond energy U is the modulus of the frozen distribution of frustration loops. The distribution of loops without currents is given by formula (2.3) with $n = 0$ ($2^n = 1$). It is given by "quenching" and in itself is of no interest to us. Therefore in all further formulas for $\mathfrak{Z}_n$, we shall understand (though, when it is not necessary, we shall not indicate this explicitly) that $\mathfrak{Z}_n$ is divided by $\mathfrak{Z}_0$. In the limit $n\to 0$, we have from (2.3)

$$\mathfrak{Z}_n^{fr}/\mathfrak{Z}_0^{fr}=1+n\sum_{LM}C_{LM}U^LM\ln 2/\mathfrak{Z}_0^{fr},$$

$$\mathfrak{Z}_0^{fr}=\sum_{LM}C_{LM}U^L,$$

whence, in accordance with the definition of replicas, we get for the entropy

$$S(T\to\infty)=\overline{M}\ln 2, \quad (2.4)$$

where $\overline{M}$ is the mean number of loops at the given U. Formula (24) shows that the definitions (2.2) and (2.1) correctly describe the double degeneracies of the frustration loops [two directions of current (1.1)]. The presence of the product

$$\prod_a\left(1+\sum_\gamma\varphi_\gamma^2(\mathbf{r}_a)\right) \quad (2.5)$$

in the functional integral (2.2) guarantees that through each site of the lattice there passes only one frustration line. In the language of the theory of polymers, we say that (2.2) is the partition function of free closed polymers with excluded volume.[9,10] If we allow one or more self-intersections, then we must replace (2.5) by

$$\prod_a(1+\varphi_a^2+\varphi_a^4+\ldots).$$

But as we shall see directly, in our case the problem with excluded volume is sufficiently general.

We pass to the continuous limit in (2.2) by making the length d of the edges tend to zero. We have

$$\mathfrak{Z}_n^{fr}=\int D\varphi_\gamma\exp\left\{-\frac{1}{2Ud^3}\int d^3x\sum_\gamma(\varphi_\gamma^2+d^2(\nabla\varphi_\gamma)^2)+\frac{1}{d^3}\int d^3x\ln\left(1+\sum_\gamma\varphi_\gamma^2\right)\right\}. \quad (2.6)$$

We expand the logarithm in (2.6) through terms of the fourth order and rewrite the result, introducing some new symbols:

$$\mathfrak{Z}_n^{fr}=\int D\varphi_\gamma\exp\left\{-\int d^3x\left[\frac{1}{2}\sum_\gamma(\tau\varphi_\gamma^2+(\nabla\varphi_\gamma)^2)+\frac{1}{4}u\left(\sum_\gamma\varphi_\gamma^2\right)^2\right]\right\}. \quad (2.7)$$

Here

$$\tau=(1-2U)/d^2, \quad u=2U^2/d, \tag{2.8}$$

and the coefficient of $(\nabla\varphi)^2$ has been transformed to unity by the choice of scale $\varphi\to(Ud)^{1/2}\varphi$.

The functional integral (2.7) for $n=0$ (number of components of the field φ_γ equal to $N=1$), as is well known,[9,10] determines the properties of closed polymer chains with allowance for excluded volume. The thermodynamics of open polymer chains corresponds to the case when the number of components of the field φ_γ is zero (de Gennes[11]).

The simplified variant (2.7) of the functional integral, in which only quaternary interaction is retained, is reasonable only at small τ, which corresponds to large lengths of the chains.[9-11] If the charge u of (2.8) is not too large, allowance for an arbitrary finite number of intersections of contours at a single point does not change the universality class of the integral (2.7).

For small positive

$$\tau\sim(c_{top}-c)/c_{top},$$

the mean length of the contours is large. Furthermore, for $\tau=0$ the topological transition mentioned above occurs; and for $\tau<0$, there appear frustration lines of infinite length, whose number is determined by the mean value of the field $\langle\varphi_\gamma\rangle$ (for details, see Appendix 1).

In the scaling region, the correlation radius $r_c\sim|\tau|^{-\nu}$, where ν is the index of the theory of an N-component field $(N=2^n)$. In particular, in the limit when the number of replicas $n\to0$, the properties of the distribution (2.7) are described by the indices of the Ising model. Thus the distribution of free frustration lines of great length is quite simple and universal and is independent of the charge u.

We shall now take into account the interaction (1.6). For this purpose we note that it is transmitted by a magnetic field with a vector potential **A**, interacting with the field φ_γ in a gauge-invariant manner. This essentially determines the interaction energy uniquely. We shall carry out the appropriate reasoning in the continuous limit. The considerably more cumbersome formulas for the case of a lattice and the proof that the distributions (1.7) and (2.14) coincide are given in Appendix 2.

We note that when there is only one replica, the two-component real field φ_γ may be regarded as a real spinor φ^λ; $\lambda=1,2$, constituting a real irreducible representation of the group SO(2). To avoid misunderstanding, we recall that although the group $SO(2)=U(1)$ is Abelian, nevertheless all its one-dimensional representations are complex: $e^{\pm in\theta}$. The minimal real representation is necessarily two-dimensional: $\cos n\theta \sin n\theta$. The vector potential A_k is connected with the matrix field $\hat{A}_k$ of the Yang-Mills group, proportional to the matrix $\hat{\varepsilon}$ that is the generator of the group SO(2):

$$\hat{\varepsilon}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}, \quad \hat{A}_k=\hat{\varepsilon}A_k. \tag{2.9}$$

In a rotation through angle χ, the spinor φ^λ transforms thus:

$$\varphi\to\exp(\tfrac{1}{2}\chi\hat{\varepsilon})\varphi. \tag{2.10}$$

If we assume that $\hat{A}_k$ transforms as the field of the Yang-Mills Abelian group SO(2),

$$\hat{A}_k\to\hat{A}_k+\hat{\varepsilon}\frac{\partial\chi}{\partial x_k}, \tag{2.11}$$

then

$$\hat{D}_k=\frac{\partial}{\partial x_k}-\frac{1}{2}\hat{A}_k$$

is the covariant derivative. For a single replica, the gradient-invariant generalization of the Hamiltonian in (2.7) is

$$\mathcal{H}_1=\frac{1}{2}\int d^3x\Big\{\tau\bar{\varphi}\varphi+\bar{\varphi}\Big(\nabla-\frac{1}{2}\mathbf{A}\hat{\varepsilon}\Big)^2\varphi +\frac{1}{2}u(\bar{\varphi}\varphi)^2+\frac{T}{J}(\mathrm{rot}\,\mathbf{A})^2\Big\}, \tag{2.12}$$

where T is the temperature. The symbol $\bar{\varphi}$ denotes the row conjugate to the column φ, and

$$\bar{\varphi}\varphi=\sum_\lambda(\varphi^\lambda)^2.$$

The functional integral for $\mathfrak{Z}_1$ now becomes, instead of (2.7), an integral of $\exp\{-\mathcal{H}_1\}$ with respect to $D\varphi^\lambda$ and DA_k in some fixed gauge, say in the transverse, $\mathrm{div}\,\mathbf{A}=0$ (see, for example, Ref. 12):

$$\mathfrak{Z}_1=\int D\varphi^\lambda DA_k\delta(\mathrm{div}\,\mathbf{A})\exp\{-\mathcal{H}_1\}.$$

The generalization to the case of n replicas is obvious. We introduce the n-component real spinor

$$\varphi^{\varepsilon_1\ldots\varepsilon_n}$$

and a Yang-Mills field defined for each replica:

$$\hat{A}_{k\alpha}=\hat{\varepsilon}_\alpha A_{k\alpha}; \quad \alpha=1,\ldots,n;$$
$$\hat{\varepsilon}_\alpha=1\times\ldots\times1\times\hat{\varepsilon}\times1\times\ldots\times1,$$

where, in the definition of $\hat{\varepsilon}_\alpha$, the matrix $\hat{\varepsilon}$ of (2.9) stands exactly in the α-th place. In the transformations

$$\varphi\to\exp\Big(\frac{1}{2}\sum_\alpha\hat{\varepsilon}_\alpha\chi_\alpha\Big)\varphi, \quad \mathbf{A}_\alpha\to\mathbf{A}_\alpha+\nabla\chi_\alpha \tag{2.13}$$

the covariant derivative has the form

$$\hat{D}_k=\frac{\partial}{\partial x_k}-\frac{1}{2}\sum_\alpha\hat{A}_{k\alpha}.$$

The functional (2.7) is accordingly replaced by

$$\mathfrak{Z}_n^{fr}=\int D\varphi DA\exp\Big\{-\frac{1}{2}\int d^3x\Big[\tau\bar{\varphi}\varphi+\frac{1}{2}u(\bar{\varphi}\varphi)^2 +\bar{\varphi}\Big(\nabla-\frac{1}{2}\sum_\alpha\mathbf{A}_\alpha\hat{\varepsilon}_\alpha\Big)^2\varphi+\frac{T}{J}\sum_\alpha(\mathrm{rot}\,\mathbf{A}_\alpha)^2\Big]\Big\} \tag{2.14}$$

with J from (1.2). Hereafter we shall measure temperature in units J and write T instead of T/J. By introducing, instead of the reduced temperature T, the "electric charge" $e^2=1/T$ and making the scale transformation $A\to eA$, we can rewrite (2.14) and (2.13) in the more familiar form

$$\mathfrak{Z}_n^{fr}=\int R\varphi DA\exp\Big\{-\frac{1}{2}\int d^3x\Big[\alpha\bar{\varphi}\varphi+\frac{1}{2}u(\bar{\varphi}\varphi)^2 +\bar{\varphi}\Big(\nabla-\frac{1}{2}e\sum_\alpha\mathbf{A}_\alpha\hat{\varepsilon}_\alpha\Big)^2\varphi+\sum_\alpha(\mathrm{rot}\,\mathbf{A}_\alpha)^2\Big]\Big\}; \tag{2.15}$$

$$\varphi \to \exp\left(\frac{1}{2} e \sum_\alpha \chi_\alpha \hat{\varepsilon}_\alpha\right)\varphi, \quad \mathbf{A}_\alpha \to \mathbf{A}_\alpha + \nabla\chi_\alpha. \tag{2.16}$$

Similar problems (for $n = 1$) were treated in their time[13,14] in connection with the question of the effect of fluctuations of the magnetic field on the superconducting transition. We shall return to the problem of fluctuations a little later; but we begin with the molecular-field approximation. Here the situation is analogous to the situation with Goldstone modes, which occur when a Higgs mechanism is included in gauge theories of the field (see, for example, Ref. 15). When $\tau > 0$, i.e. when $c \ll c_{\rm top}$, the system contains 2^n massive bosons of the field φ and n massless photons $\mathbf{A}_\alpha$. When $\tau < 0$ ($c > c_{\rm top}$), infinite frustration lines appear, corresponding to a finite mean value of the field $\varphi_0^2 = -\tau/u$. All n photons $\mathbf{A}_\alpha$ become massive, with $m_{\rm ph}^2 \sim e^2\varphi_0^2$; one of the 2^n bosons φ also remains massive. Of the remaining bosons,

$$N_G = 2^n - n - 1$$

become massless Goldstone particles, and n are absorbed by the vacuum of the gauge field.[15] Thus the infinite frustration lines, equivalent to open infinite electric currents, shield the magnetic field at a finite distance $\sim 1/m_{\rm ph}$. On the other hand, currents of infinite length produce a new type of correlation of finite radius [more simply stated, the radius is equal to the length of the current (frustration) lines $L \to \infty$].

From a formal point of view, the correlations of pairs of currents $\mathbf{j}$ in different replicas, $\mathbf{j}_\alpha$, $\mathbf{j}_\beta$, are long-range (hereafter, for brevity, we shall not indicate the vector indices on the currents $\mathbf{j}$):

$$\begin{aligned}&\langle j_\alpha(x) j_\beta(x') j_\alpha(y) j_\beta(y')\rangle \\ &\quad \sim \langle \bar{\varphi}(x) \nabla_x \hat{\varepsilon}_\alpha \varphi(x) \bar{\varphi}(x') \nabla_{x'} \hat{\varepsilon}_\beta \varphi(x') \\ &\qquad \times \bar{\varphi}(y) \nabla_y \hat{\varepsilon}_\alpha \varphi(y) \bar{\varphi}(y') \nabla_{y'} \hat{\varepsilon}_\beta \varphi(y')\rangle.\end{aligned} \tag{2.17}$$

Suppose that the mean value of only one of the components of the field φ is nonzero:

$$\varphi_0 = \langle \varphi^{1\cdots 1}\rangle$$

(in our situation, this is the general case). It is easy to see that in an arbitrary order of perturbation theory, the long-range part of the correlators of the currents of a single replica, for example $\langle j_\alpha j_\alpha\rangle$, vanish in the limit $n \to 0$. In the correlator of currents from different replicas, one can separate out, in (2.17), a term containing means of the type

$$\langle \varphi_x^{1\ldots121\ldots1} \varphi_{x'}^{1\ldots121\ldots1}\rangle, \quad \langle \varphi_x^{1\ldots121\ldots121\ldots1} \varphi_y^{1\ldots121\ldots121\ldots1}\rangle,$$

and, expressly,

$$\begin{aligned}\langle j_\alpha j_\beta j_\alpha j_\beta\rangle &\sim \langle \varphi_x^{1\ldots1}\rangle \langle \nabla_x \varphi_x^{1\ldots121\ldots1} \nabla_{x'} \varphi_{x'}^{1\ldots121\ldots1}\rangle \\ \times &\langle \varphi_{x'}^{1\ldots121\ldots121\ldots1} \varphi_y^{1\ldots121\ldots121\ldots1}\rangle \langle \nabla_y \varphi_y^{1\ldots121\ldots1} \nabla_{y'} \varphi_{y'}^{1\ldots121\ldots1}\rangle \langle \varphi_{y'}^{1\ldots1}\rangle.\end{aligned}$$

In the zeroth order of perturbation theory, we have that at finite $|x' - x| \sim |y' - y| \ll |x - y| \to \infty$, the correlation

$$\begin{aligned}\langle j_\alpha(x) j_\beta(x) j_\alpha(y) j_\beta(y)\rangle &\sim \varphi_0^2 \langle \varphi_x^{1\ldots121\ldots121\ldots1} \varphi_y^{1\ldots121\ldots121\ldots1}\rangle \\ &\sim \varphi_0^2/|\mathbf{x} - \mathbf{y}|.\end{aligned} \tag{2.18}$$

It is easy to verify that the long-range character is retained also in any order of perturbation theory.

Anticipating, we note that, as always, in the scaling

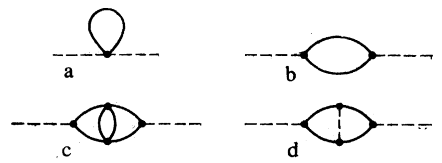

FIG. 1.

mode

$$\varphi_0^2 \sim (-\tau)^{2\beta},$$

and merging of x' with x and of y' with y means that

$$|x' - x| \sim |y - y'| < r_c \sim |\tau|^{-\nu},$$

where β and ν are the exponents of the model (2.7) (the Ising model when $n = 0$).

We shall show in our case, for $n \to 0$ the renormalized charge e_r, corresponding to effective temperature

$$T_{eff} = 1/e_r^2, \tag{2.19}$$

is small near the transition point. We recall that the free charge, on the contrary, is large at low temperatures. Therefore the assertion of the smallness of the renormalized charge is equivalent to the assertion that even at very low temperatures, the system is effectively hot; and consequently the interaction between frustrations is insignificant, and everything is basically determined by the statistics of the free frustration loops (2.7).

We shall denote the Green functions of the field φ by solid lines and the photon propagators A by broken (Fig. 1). It is clear that in the limit $n \to 0$, the additions to a photon propagator containing internal photon lines (radiation corrections) vanish (diagrams of the type 1(d). Therefore Dyson's equation gives an exact expression for the renormalized charge in terms of the correlators of the currents, averaged over the distribution of free frustration currents (2.7):

$$\frac{1}{e_r^2}\Delta = \frac{1}{e^2}\Delta + \frac{1}{6}\int \langle \mathbf{j}_\alpha(x) \mathbf{j}_\alpha(y)\rangle d^3y,$$

where Δ is the Laplacian operator. In the scaling region, this relation is rewritten

$$1/e_r^2 = 1/e^2 + \text{const} \int \langle \varphi^2(0) \varphi^2(x)\rangle d^3x.$$

The integral on the right side has the dimensions "heat capacity"[1)]; therefore

$$1/e_r^2 = T_{eff} = T + \text{const}/|\tau|^\alpha, \tag{2.20}$$

where α is the index of the Ising model.

At low temperatures, near the topological transition,

$$1/e_r^2 = T_{eff} \sim |\tau|^{-\alpha}. \tag{2.21}$$

The topological transition with respect to τ remains continuous at $n = 0$ and $T = 0$. This is again due to the absence at $n = 0$ of radiation corrections, which in principle made possible a first-order transition.[13]

From the fact that the effective temperature of the system is large, it follows that there are no power singularities with respect to a small physical temperature T. This fact can be verified, if one wishes, by direct calculation according to formula (2.14).

3. EFFECT OF THERMALLY EXCITED VORTICES

At finite temperatures, it is necessary to allow for thermally excited currents (1.1) with integral strength. If we neglect magnetic interaction of the currents, then we can use for the description a formula analogous to (2.2). We introduce an n-component complex field ψ_α. Then it is easy to understand that

$$\mathfrak{Z}_n^{term}=\int D\psi_\alpha^*(\mathbf{r}_a)D\psi_\alpha(\mathbf{r}_a)\prod_{a\alpha}(1+|\psi_\alpha(\mathbf{r}_a)|^2)\times\exp\left\{-\frac{1}{2V}\sum_{\langle ab\rangle\alpha}\psi_\alpha^*(\mathbf{r}_a)\psi_\alpha(\mathbf{r}_b)\right\}. \quad (3.1)$$

As before,

$$\mathfrak{Z}_n^{term}=\left[\sum_{LM}C_{LM}V^L2^M\right]^n, \quad (3.2)$$

but now V depends substantially on the temperature T. Specifically, since (3.2) is the Gibbs partition function, therefore

$$V\doteq e^{-I/T}, \quad (3.3)$$

where I is the energy of the vortex line per segment. In order of magnitude, I coincides with the Curie temperature T_c. In the continuous limit, we have

$$\mathfrak{Z}_n^{term}=\int D\psi D\psi^*\exp\left\{-\frac{1}{2}\int d^3x\sum_\alpha\left[\mu|\psi_\alpha|^2+|\nabla\psi_\alpha|^2+\frac{1}{2}v|\psi_\alpha|^4\right]\right\} \quad (3.4)$$

after we carry out, as in the derivation of (2.7), a scale transformation

$$\psi\to(Vd)^{1/2}\psi, \quad \mu=(1-2V)/d^2, \quad v=2V^2/d. \quad (3.5)$$

At low temperatures,

$$\mu\approx d^2, \quad v\sim e^{-2I/T},$$

therefore the self-action of the field ψ (the term ψ^4) may in fact be neglected with exponential accuracy. It is necessary to take into account the magnetic interaction of thermally excited vortices with each other and with frustration currents. We shall therefore suppose that the field ψ_α interacts with the photons $\mathbf{A}_\alpha$ of (2.15) and (2.16) in a gauge manner:

$$\mathscr{H}_n^{term}=\frac{1}{2}\sum_\alpha\int d^3x\left[\mu|\psi_\alpha|^4+\frac{1}{2}v|\psi_\alpha|^4+|(\nabla-ie\mathbf{A}_\alpha)\psi_\alpha|^2\right], \quad (3.6)$$

$$\psi_\alpha\to\exp(ie\chi_\alpha)\psi_\alpha, \quad \mathbf{A}_\alpha\to\mathbf{A}_\alpha+\nabla\chi_\alpha. \quad (3.7)$$

As has already been mentioned, problems of the type (3.6) were considered by Halperin and others[13,14] in a description of a phase transition in superconductors. In their case $\mu\to0$. Here we are interested in the case of limitingly low temperatures T, when μ is finite but the charge $v\to0$. In such a situation, closed thermal currents do not play an important role, because their length is small. The essentially new physical feature of the problem is the possibility of decay of a thermal current into two frustrational [see Fig. 2, where the thermal current is denoted by a wavy line and the strengths 1/2 and 1 of (1.1) are marked]. The law of conservation of strength in Fig. 2 requires that finite open thermal currents either connect two infinite frustration currents [Fig. 3(a)] above the topological transition point, $c>c_{top}$, or form closed or infinite chains of the type of Fig. 3(b), c below the topological transition, $c<c_{top}$.

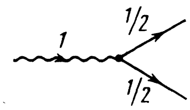

FIG. 2.

The presence of finite thermal currents of the type of Fig. 3(a) disturbs the long-range action (2.18) that is present in the purely frustrational model of Sec. 2, above the topological transition point. It is easy to understand that with allowance for processes of the form of Fig. 3(b), the direction of the current $\mathbf{j}_\alpha$ and also of $\mathbf{j}_\beta$ is no longer conserved along a whole infinite line, and at large distances it is random; therefore the product $j_\alpha j_\beta$ is also not conserved along an infinite furstration line, as was the case in (2.18).

Technically, this situation is described by addition to $\mathscr{H}_n^{fr}$ and $\mathscr{H}_n^{therm}$ of an interaction describing the decay of Fig. 2:

$$\mathscr{H}_n^{f-t}=-\lambda\sum_\alpha\int d^3x\sum_{e\neq e_\alpha}\{\psi_\alpha[\varphi^{e_1\ldots1\ldots e_n}+i\varphi^{e_1\ldots2\ldots e_n}]^2+\text{c. c.}\}. \quad (3.8)$$

Here it is very important that, in accordance with the scale transformation (3.5), the constant λ is exponentially small at low temperature:

$$\lambda\sim V^{1/2}\sim e^{-I/2T}.$$

The whole partition function now has the form

$$\mathfrak{Z}_n=\int D\varphi D\psi D\psi^*DA\exp\{-\mathscr{H}_n^{fr}-\mathscr{H}_n^{term}-\mathscr{H}_n^{f-t}\}. \quad (3.9)$$

In the molecular-field approximation, we vary the sum of the Hamiltonians in (3.9) with respect to (ψ_α). We get

$$\mu\psi_0\sim\lambda\varphi_0^2.$$

Thus if in the system of frustrations there are infinite open lines, i.e., there is a nonzero $\langle\varphi\rangle$, then there necessarily occurs also a $\langle\psi\rangle$. With allowance for a nonzero ψ_0, one can separate out in (3.8) the terms that give a finite correlation radius of Goldstone bosons:

$$-2\lambda\sum_\alpha\sum_{e\neq e_\alpha}\int d^3x\psi_0[(\varphi^{e_1\ldots1\ldots e_n})^2-(\varphi^{e_1\ldots2\ldots e_n})^2].$$

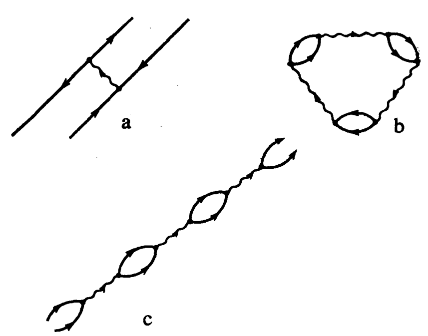

FIG. 3.

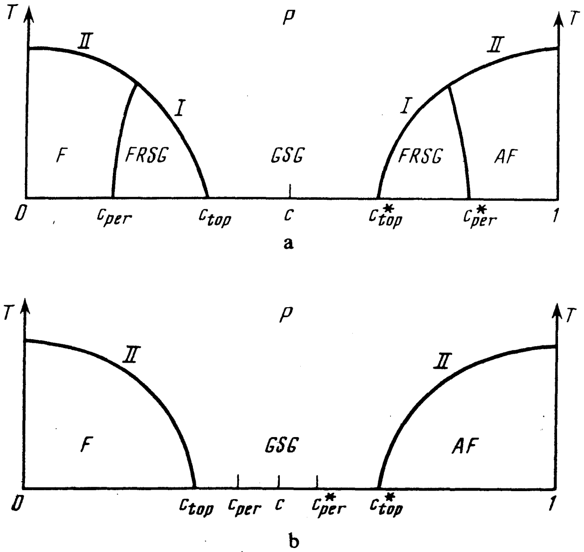

FIG. 4.

In this sum, a term of the form $(\varphi^{\varepsilon_1\ldots\varepsilon_n})^2$ containing m twos and $n-m$ ones among the indices $\varepsilon_1 \ldots \varepsilon_n$ is encountered m times with a plus sign and n–m times with a minus sign. Thus in the limit $n\to 0$, all Goldstone oscillations have the mass

$$M_G^2 \sim \lambda^2\varphi_0^2/\mu.$$

Thereby there appears in the system a new correlation distance R_c that becomes infinite at $T=0$. In the scaling range with respect to frustrations. ($r_c \sim |\tau|^{-\nu}$), we can write for R_c

$$R_c \sim r_c e^{1/2T}. \tag{3.10}$$

The exponential temperature dependence in (3.10) can also be obtained by simple physical considerations, connected with estimation of the probability of appearance of segments of thermal currents of finite length.

The finiteness of the correlation distance R_c of (3.10) means that the phase considered in the Introduction and in Sec. 2, "genuine spin glass" (GSG; see Fig. 4), is actually paramagnetic (see, however, the Conclusion). Strictly speaking, this state is a so-called mictomagnetic state, i.e., a phase with magnetic clusters, whose size increases exponentially with lowering of the temperature to zero.

In the mictomagnetic phase, as in any paramagnet, the spin stiffness ρ_s is zero. In our model, this fact is easy to understand from the following considerations. Above the topological transition, when all frustrational currents are closed, in accordance with the magnetic analog, the "susceptibility" inverse to the rigidity ρ_s is finite, while the "magnetic field" **A** has no mass. After the topological transition, frustrational currents of infinite length appear, and in consequence (we recall Ampere's old picture!) the "susceptibility" tends to infinity and the stiffness ρ_s to zero. The "magnetic field" meanwhile acquires mass.

We finally consider the region immediately before the topological phase transition, $c < c_{\text{top}}$ $(0<\tau\ll 1)$, where with increase of the temperature T there should occur a transition to the paramagnetic phase. This transition is again topological: closed chains of thermal currents, of the type of Fig. 3(b), split, and infinite thermal chains, of the type of Fig. 3(c), appear. Here a cardinal fact is that the thermal segments (the wavy lines) in a chain remain short, because the probability of their originating is exponentially small with respect to temperature [see (3.3)]. A large (or infinite) length of the chains is attained at low temperatures because for $\tau\to 0$ the lengths of the frustrational currents [the loops in Fig. 3(b) and (c)] tend to infinity. The topological transition with respect to temperature means the appearance of a nonzero mean field $\langle\psi\rangle$ in the high-temperature phase.

In order to treat the transition, we integrate over the fields φ in (3.9). A renormalized Hamiltonian for ψ and the photons A emerges:

$$\mathcal{H}_n{}^r = \frac{1}{2}\sum_\alpha \int d^3x \left\{\mu_r|\psi_\alpha|^2 + \frac{1}{2}v_r|\psi_\alpha|^4 + \rho_r|(\nabla - ie_r\mathbf{A}_\alpha)\psi_\alpha|^2 + (\operatorname{rot}\mathbf{A}_\alpha)^2\right\}. \tag{3.11}$$

Here the renormalized charge e_r in the scaling region with respect to τ is given by formula (2.21), and

$$\mu_r = \mu - \lambda^2 I_1, \quad \rho_r = 1+\lambda^2 I_2, \quad v_r = v+\lambda^4 I_3. \tag{3.12}$$

In the scaling region with respect to τ, the integrals I_1, I_2, and I_3 are given by the relations [average over the distribution of free frustrations (2.7)]

$$I_1 \sim \int d^3x\langle\varphi^2(0)\varphi^2(x)\rangle \sim \tau^{-\alpha},$$

$$I_2 \sim \int d^3x x^2\langle\varphi^2(0)\varphi^2(x)\rangle \sim \tau^{-\alpha-2\nu},$$

$$I_3 \sim \int d^3x_1 d^3x_2 d^3x_3\langle\varphi^2(0)\varphi^2(x_1)\varphi^2(x_2)\varphi^2(x_3)\rangle \sim \tau^{-2-\alpha}.$$

In the derivation of (3.12), all diagrams have been discarded that vanish when the number of replicas n tends to zero. We recall once more that α and ν are the indices of the Ising model.

Recalling now that λ and v are exponentially small at low temperatures, we rewrite (3.12) (to within unimportant constant factors) in the form

$$\mu_r \sim 1-\tau^{-\alpha}e^{-1/T}, \quad \rho_r \sim 1+\tau^{-\alpha-2\nu}e^{-1/T},$$
$$v_r \sim e^{-1/T}(1+\tau^{2-\alpha}e^{-1/T}), \quad e_r^2 \sim \tau^\alpha. \tag{3.13}$$

The critical properties of the system (3.11) have so far not been definitively explained. Whereas in the first papers of Halperin and others[13] it was asserted that the transition to the low-temperature phase is a transition of first order, in a recent paper of Dasgupta and Halperin[14] it was asserted that for large charges e_r the transition is continuous. In the opinion of the authors of Ref. 14, it can become a transition of first order only for very small e_r. These conclusions are drawn on the basis of analysis of a Hamiltonian of the form (3.11), but not in the continuous limit, as in our case, but in the lattice variant of this theory. Here the characteristic fluctuations of the field ψ are from the very beginning assumed not to be small.

Thus two situations are possible: *a) The transition* may be described within the framework of the theory of the self-consistent field, when the characteristic fluctuations of the field ψ before and after the transition may be neglected.[13]

This is a first-order transition. b) If the fluctuations of the field ψ are important, then the transition is apparently continuous.

In the high-temperature phase, the nonvanishing $\langle \psi_\alpha \rangle = \psi_0$ is found from the obvious condition

$$\frac{\partial \langle \mathcal{H}_n{}^r \rangle}{\partial \langle \psi_\alpha \rangle} = 0, \tag{3.14}$$

$$\langle \mathcal{H}_n{}^r \rangle = \frac{1}{2} \sum_\alpha \left\{ \mu_r \langle \psi_\alpha \rangle^2 + \rho_r e_r{}^2 \langle \mathbf{A}_\alpha{}^2 \rangle \langle \psi_\alpha \rangle^2 + \frac{1}{2} v_r \langle \psi_\alpha \rangle^4 \right\}.$$

The value of $\langle \mathbf{A}_\alpha^2 \rangle$ in the lowest approximation can be calculated from Dyson's equation for a photon,

$$D^{-1} = D_0^{-1} + \tag{3.15}$$

where the lightning signs denote the mean field ψ_0. For $D_0^{-1}(k)$ it is necessary to substitute k^2. We have for $\langle \mathbf{A}_\alpha^2 \rangle$

$$\langle \mathbf{A}_\alpha{}^2 \rangle \sim \int \frac{d^3k}{k^2 + \rho_r e_r{}^2 \psi_0{}^2}.$$

This integral (again except for unimportant constants) is

$$\langle \mathbf{A}_\alpha{}^2 \rangle \sim 1 - e_r \rho_r^{1/2} |\psi_0|. \tag{3.16}$$

Thus the energy determining ψ_0 has the form

$$\langle \mathcal{H}_n{}^r \rangle = \frac{1}{2} n \psi_0{}^2 \left\{ \mu_r - \rho_r e_r{}^2 - e_r{}^3 \rho_r{}^{3/2} |\psi_0| + \frac{1}{2} v_r \psi_0{}^2 \right\}. \tag{3.17}$$

Equation (3.17) describes a first-order transition. In order to determine whether fluctuations of the field ψ may be neglected, we compare the characteristic distance l_m of screening of the currents below the transition point with the scale l_ψ on which fluctuations of the field ψ become important:

$$l_m \sim v_r / e_r{}^4 \rho_r{}^2 \sim \max \{ \tau^{-2\alpha} e^{-I/T},\ \tau^{-2+4v-\alpha} \},$$

$$l_\psi \sim \rho_r{}^2 / v_r \sim \max \{ e^{I/T},\ \tau^{2-4v-\alpha} \}.$$

Since $l_m \ll l_\psi$, the approximation in which (3.17) was written is justified.

We consider the case in which $\tau \sim 1$. This means that the mean length of the frustration loops is small; that is, their size is comparable with the size of an elementary cell of the lattice. If we return to the original model of a ferromagnet, in which certain bonds have been replaced by antiferromagnetic ones, this means that there are few such "spoiled" bonds. Then both lengths are of the same order of magnitude, and the transition is continuous (case b).

The first-order transition line on the T, c diagram near the topological transition, $\tau \ll 1$, has the form

$$T \sim -1/\ln \tau. \tag{3.18}$$

On the other hand we know, of course, that when c is not close to c_{top}, *there is the usual* second-order phase transition to the paramagnetic state, at the Curie temperature $T_C(c)$. Therefore the state diagram has the form of Fig. 4(a). It seems reasonable that when a first-order transition is present, the spin glass with finite rigidity mentioned above, also exists [the phase FRSG, finite-rigidity spin glass, in Fig. 4(a)]. The diagrams of Fig. 4(a) and (b) then correspond to the cases $c_{\text{per}} < c_{\text{top}}$ and $c_{\text{per}} > c_{\text{top}}$ respectively. The possibility may of course also not be excluded that the line separating the phase F from the phase GSG in Fig. 4(b) has a tricritical point, i.e., consists of two pieces—transitions of first and second order.

We note that a spin glass with finite rigidity was observed in experiments of Maletta and others,[16] although in other physical systems.

Within the framework of the Villain model, it is also possible to calculate the correlation function of the spins in phase GSG. Here, if the phase FRSG is absent [Fig. 4(b)], this will be the ferromagnetic short-range order; if it is present, the correlation of the corresponding quantity in the FRSG (see Refs. 3–6). For definiteness, we shall speak of the ferromagnetic case, Fig. 4(b).

In the Villain model, it is necessary to add to the Hamiltonian in (3.9) a spin-wave part (1.3) in each replica:

$$\mathcal{H}_n^{s.w} = \frac{1}{2} \rho_s \sum_\alpha \int (\nabla \chi_\alpha)^2 d^3x,$$

where χ_α coincides with the gauge in (2.13). The correlation of the spins in a single replica is

$$\Phi(R) = \langle \exp \{ i(\chi_\alpha(0) - \chi_\alpha(\mathbf{R})) \} \rangle. \tag{3.19}$$

In the phase GSG, where the spin-wave part of the fluctuations is insignificant, $\Phi(R)$ is determined by the gauge-invariant part of (3.19). There is a standard procedure for this. We write (3.19) in the form

$$\Phi(R) = \left\langle \exp \left\{ i \int_0^{\mathbf{R}} \operatorname{grad} \chi_\alpha dl \right\} \right\rangle,$$

where the integral is taken along an arbitrary path between the points 0 and $\mathbf{R}$. The gauge-invariant part is the mean of the integral along a closed contour C_{0R} passing through 0 and $\mathbf{R}$:

$$\Phi(R) \sim \left\langle \exp \left\{ i \oint_{C_{0R}} \operatorname{grad} \chi_\alpha dl \right\} \right\rangle.$$

By Stokes's theorem,

$$\Phi(R) \sim \left\langle \exp \left\{ i \int_{S_{0R}} \operatorname{rot grad} \chi_\alpha d\mathbf{S} \right\} \right\rangle,$$

where S_{0R} is the surface resting on the contour C_{0R}; and by virtue of the definition of the currents $\mathbf{j}_\alpha$ as densities of disclinations in the field $\cos \chi_\alpha$, $\sin \chi_\alpha$

$$\Phi(R) \sim \left\langle \exp \left\{ i \int_{S_{0R}} \mathbf{j}_\alpha d\mathbf{S} \right\} \right\rangle = \langle \exp (iJ_{\alpha 0R}) \rangle, \tag{3.20}$$

where J_{0R} is the total current through the surface S_{0R}.

The following considerations, which are correct in the scaling regime, are very simple. Since $\langle J_{0R} \rangle$ is obviously zero at large R,

$$\langle \exp (iJ_{\alpha 0R}) \rangle \sim \exp (-{}^1/_2 \langle J_{0R}{}^2 \rangle).$$

For $R \gg r_c$, only currents of infinite length are important; therefore

$$\langle J_{\alpha 0R}^2 \rangle \sim S_{0R} \alpha_\infty r_c$$

with α_∞ from (A1.10). The correlation function (3.20) is determined by the minimum value of $S_{0R}\sim Rr_c$, whence

$$\Phi(R)\sim\exp(-R/R_\chi),\quad R_\chi\sim 1/\alpha_\infty r_c^2\sim(-\tau)^{-2+\alpha+2\nu}\sim(-\tau)^{-\nu}\sim r_c. \tag{3.21}$$

(The scaling relation $\alpha = 2 - d\nu$ for space dimensionality $d = 3$ has been used.) Thus for ferromagnetic order in the phase GSG dies out at a distance of the same radius r_c.

Naive scaling considerations give a $1/T$ law for the magnetic susceptibility χ. We have by definition

$$\chi=\frac{1}{T}\int\Phi(R)\,d^3R.$$

Substituting Φ from (3.21) here, we find

$$\chi\sim r_c^3/T\sim(-\tau)^{-3\nu}/T. \tag{3.22}$$

Of course the dependence on τ and T in (3.22) is not too certain. The angles of χ_α are not fundamental in the phase transition. Therefore the considerations leading to (3.21) correctly determine only the exponents. But the degrees of the dependence on R and T may arise also from non-scaling and gauge-noninvariant corrections. It can hardly be doubted, however, that the susceptibility is infinite at $T = 0$, while the number of "effective spins"—r_c^3 in formula (3.22)—becomes infinite at the topological transition point.

A dependence $\chi\sim 1/T$ in the Ising model with random bonds was indicated, for example, in a paper of Lyuksyutov,[17] who found a formula of the form $\chi\sim c^2/T$ for small concentrations of wrong bonds c.

4. HEISENBERG SPIN GLASS

In the case of Heisenberg spins, the transition from a microscopic lattice picture to a continuous is not so lucid and simple as in the Villain model for an XY spin glass. We shall limit ourselves to a discussion of the consequences of a naive generalization of the continuous gauge theory developed above to the non-Abelian gauge fields of the group SO(3), corresponding to Heisenberg spins.[3,4]

A generalization of the partition for frustration lines is carried out without difficulty. Instead of (2.14) we now have

$$\mathcal{Z}_n^{fr}=\int D\varphi D\hat{A}\exp\Big\{-\frac{1}{2}\int d^3x\Big[\tau\bar\varphi\varphi+\frac{1}{2}u(\bar\varphi\varphi)^2 +\bar\varphi\Big(\nabla-\sum_\alpha\hat{A}_\alpha\Big)^2\varphi+T\sum_{\alpha kl}\mathrm{Sp}\,\hat F_{\alpha kl}\hat F_{\alpha kl}\Big]\Big\}. \tag{4.1}$$

Here φ is a tensor of rank n in the three-dimensional space of the group SO(3):

$$\varphi^{e_1\ldots e_n},\quad e_\alpha=1,2,3,$$

$\hat A_{k\alpha}$ is the Yang-Mills field of the group SO(3); $k = x, y, z$; and $\hat F_{\alpha kl}$ is the intensity of the Yang-Mills fields (the curvature; see, for example, Ref. 3 and 4),

$$\hat F_{\alpha kl}=\frac{\partial\hat A_{\alpha l}}{\partial x_k}-\frac{\partial\hat A_{\alpha k}}{\partial x_l}-[\hat A_{\alpha k},\hat A_{\alpha l}].$$

In rotations of the group SO(3), the fields φ, A, and F transform as

$$\varphi\to\hat G\varphi,\quad \hat F_\alpha\to\hat G_\alpha\hat F_\alpha\hat G_\alpha^{-1},$$

$$\hat A_{\alpha k}\to\hat G_\alpha\hat A_{\alpha k}\hat G_\alpha^{-1}+\hat G_\alpha\frac{\partial}{\partial x_k}\hat G_\alpha^{-1}.$$

Here

$$\hat G=\hat O_1\times\ldots\times\hat O_n,\quad \hat G_\alpha=1\times\ldots\times1\times\hat O_\alpha\times1\times\ldots\times1,$$

where $\hat O_\alpha$ is a three-row orthogonal matrix of the group SO(3).

Qualitatively, all the conclusions of the theory (4.1) are the same as those of the theory of Sec. 2. Below the topological transition point, the bosons φ are massive, while the photons A are massless. On appearance of infinite frustration lines, $\langle\varphi^{1\ldots1}\rangle$ is nonzero. All the photons acquire mass, but in return there appear again a certain number N_G of Goldstones (see Ref. 15):

$$N_G=3^n-2n-1.$$

Thereby there appear in the system long-range correlations of currents in different replicas, of the type (2.18).

We note that the description of the system of frustration lines in a Heisenberg magnet by means of (4.1) has a topological basis. The frustration lines in this case are, as is well known,[3,4] disclinations in the field of a magnetic moment **m** with strength 1/2 (the same as in a nematic liquid crystal). Topologically they are equivalent to disclinations in the field of orthogonal matrices SO(3), a fact that was used earlier in a derivation of (4.1) by Volovik and one of the authors[3,4] for description of a Heisenberg spin glass. Here it is very important that although disclinations with strengths $\pm 1/2$ are topologically equivalent, nevertheless for a topologically continuous transition from disclinations with strength $-1/2$ to strength $+1/2$ it is necessary to surmount a large energy barrier. The interaction of disclinations with each other, as is again well known from the theory of liquid crystals, depends on the product of their strengths and is described by the formula of the type (1.2).

The situation changes importantly for thermal disclinations with integral strengths. All such disclinations are topologically equivalent to a uniform state; usually they are not even local minima of the energy functional. Therefore there are no infinite or closed thermal disclinations. What is still more important, there are also no finite segments of thermal disclinations. The decays of Fig. 2 are now forbidden. One easily shows this by noticing that the only candidate for a decay interaction of the type (3.8) that is permitted by the symmetry SO(3) is

$$\sum_{abc}\sum_{\alpha}\sum_{\{e\}\neq e_\alpha}\varepsilon_{abc}\psi_\alpha^a\varphi^{e_1\ldots b\ldots e_n}\varphi^{e_1\ldots c\ldots e_n}, \tag{4.2}$$

where ε_{abc} is the antisymmetric unit tensor, and where the vector ψ_α, as in Sec. 2, describes thermal disclinations. The expression (4.2) is, in an obvious way, equal to zero.

Without thermal disclinations, the genuine spin glass phase GSG will exist also at finite temperature.

5. CONCLUSION

The results of Secs. 3 and 4 contradict computer calculations and theoretical considerations[19,20] that show that

$d = 4$ is the lowest marginal dimensionality for spin glasses with short-range action. The contradiction is especially bad for the Heisenberg model. It indicates that a disclination model that takes no account of the spin-wave degrees of freedom is not adequate for the problem. Spin wave fluctuations destroy the long-range order in the phase GSG (Sect. 4) and make the correlation radius R_c finite, although it still becomes infinite according to a power law with lowering of the temperature. It can only be hoped that our principal qualitative conclusion is preserved, that in the three-dimensional case the phase transition with repsect to concentration from the ferromagnetic phase (or a spin glass with finite rigidity) to the paramagnetic state at low temperatures is a topological transition.

The situation with XY spins is more delicate. We solved the Villain model, in which the spin waves are free, with asymptotic accuracy at low temperatures and small τ. Furthermore, we obtained a correct conclusion about the absence of long-range order in the phase GSG at finite temperatures. But formula (3.10) for the correlation radius indicates that $d = 3$ and not $d = 4$ is the marginal dimensionality for the Villain model. It is noteworthy that on this point there is no direct contradiction between the topological model, which takes account only of disclinations, and computer calculations[18] that include the interaction of spin waves. The correct transition from $d = 3$ to $d = 4$ in the topological model is given not by the field model (2.15) for $d = 4$, but by something quite different. A disclination in the field of XY spins, corresponding to the homotopic group $\Pi_1(S_1)$ in four-dimensional space, is not a line but a two-dimensional plane (loops for $d = 3$ correspond to closed two-dimensional surfaces for $d = 4$). Therefore the corresponding *field* theory must operate not with particles but with strings. As is well known, only the first steps have so far been taken in this direction,[21] and it is premature to speak of a transfer of the theory of strings to the theory of spin waves. But from Polyakov's results[21] one can carry away the impression that in four-dimensional space a topological transition connected with two-dimensional surfaces always exists.

We recall that another possible candidate for a topological transition with two-dimensional surfaces is an Ising spin glass in three-dimensional space. In four-dimensional space, the topology of the Ising model is determined by three-dimensional hypersurfaces.

In conclusion, we note as a curiosity that the marginal dimensionality $d = 3$ is possessed not only by the XY spin-glass model, but also, according to arguments of Volovik and one of the authors,[3] by a dilute magnetic alloy with RKKY interaction. We express our thanks to M. V. Feĭgel'man for valuable counsel.

APPENDIX 1

Below, we present some formulas that relate the concentration of the segments that make up the frustration lines (i.e. the total length of them in unit volume, L/V), the concentration of segments belonging to lines of infinite length, etc. to the parameter W of the distribution (2.2).

The polymer partition function $\mathfrak{Z}_n$ of (2.3) can be generalized so as to include finite sections of polymers. This is accomplished[9,10] by introduction of a "magnetic field" h in such a way that

$$\mathfrak{Z}_n(h) = \sum h^{2N_f} U^L N^M C(N_f, L, M), \qquad \text{(A1.1)}$$

where N_f is the number of open polymers, and where $C(...)$ is the number of different configurations with total length L and with M closed polymers; $N = 2^n$.

The total concentration of monomers is

$$\alpha_m = \frac{\langle L\rangle}{V} = \frac{1}{V} U \frac{\partial \ln \mathfrak{Z}_n}{\partial U}, \qquad \text{(A1.2)}$$

the number of closed contours is

$$\langle M\rangle = N \frac{\partial \ln \mathfrak{Z}_n}{\partial N}, \qquad \text{(A1.3)}$$

the number of open contours is

$$\langle N_f\rangle = \frac{1}{2} h \frac{\partial \ln \mathfrak{Z}_n}{\partial h}. \qquad \text{(A1.4)}$$

We recall that we are interested in the case $h \to 0$, $n \to 0$, $N \to 1$.

The expression (A1.1) can also be written as a functional integral over the N-component field φ_γ:

$$\mathfrak{Z}_n = \int D\varphi \prod_a \Big[1 + \sum_\gamma \varphi_\gamma^2(\mathbf{r}_a) + h\varphi_1(\mathbf{r}_a) \Big] \times \exp\Big\{ -\frac{1}{2U} \sum_{\langle ab\rangle \gamma} \varphi_\gamma(\mathbf{r}_a)\varphi_\gamma(\mathbf{r}_b) \Big\}. \qquad \text{(A1.5)}$$

Here φ_1 is the first component of the field φ_γ ($\varphi^{1\cdots 1}$ in the spinor representation).

On passing to the continuous limit, we get near the topological transition, in analogy with (2.7),

$$\mathfrak{Z}_n = \int D\varphi \exp\Big\{ -\int d^3x \Big[\frac{1}{2} \sum_\gamma (\tau\varphi_\gamma^2 + (\nabla\varphi_\gamma)^2) + \frac{1}{4} u \Big(\sum_\gamma \varphi_\gamma^2 \Big)^2 - \tilde{h}\varphi_1 \Big] \Big\}, \quad \tilde{h} = (Ud)^{1/2} h. \qquad \text{(A1.6)}$$

In the scaling range, the singular part of α_m is $\langle \varphi^2(0)\rangle$; i.e., it has an entropy index:

$$\alpha_m = \alpha_0 + |\tau|^{1-\alpha}. \qquad \text{(A1.7)}$$

The singular part of $\langle M\rangle$ is $\partial\mathfrak{Z}_n/\partial n$ for $n \to 0$, i.e., the free energy

$$\langle M\rangle = M_0 + \tau + |\tau|^{2-\alpha}. \qquad \text{(A1.8)}$$

It is possible to calculate the concentration of monomers belonging to open polymers, α_f, and thereby the concentration of infinite polymers

$$\alpha_\infty = \lim_{\tilde{h} \to 0} \alpha_f$$

in the scaling region from the following considerations. Before the topological transition, for $\tilde{h} \to 0$, according to (A1.4) and (A1.6)

$$\langle N_f\rangle = {}^1/_2 \tilde{h} \langle \varphi_1\rangle.$$

By eliminating $\tilde{h}$ by means of the relation $\langle \varphi_1\rangle = \chi \tilde{h}$, where χ is the magnetic susceptibility of the model (A1.6), we find

$$\langle N_f\rangle=\langle\varphi_1\rangle^2/2\chi. \tag{A1.9}$$

Again, before the topological transition the concentration of monomers in closed contours is

$$\alpha_c=\alpha_m\frac{\langle M\rangle}{\langle N_f\rangle+\langle M\rangle},$$

and the concentration of monomers in open contours is

$$\alpha_f=\alpha_m\frac{\langle N_f\rangle}{\langle N_f\rangle+\langle M\rangle}.$$

These formulas retain meaning also after the transition. For $\tilde h\equiv 0$, we have in the scaling region, in accordance with (A1.7)–(A1.9),

$$\alpha_c\approx\alpha_m,\quad \alpha_\infty\sim(-\tau)^{2\beta+\gamma}. \tag{A1.10}$$

In the range where scaling considerations are inapplicable, expressions for α_m and α_c can be obtained by the method of the self-consistent field.[10] The situation is somewhat more complicated with the calculation of α_∞. Below, we set forth a method by which this can be done. We consider a partition function in which we introduce two different fields, φ and $\tilde\varphi$, for separate description of finite contours and of infinite lines:

$$\mathfrak{Z}=\int' D\varphi D\tilde\varphi\prod_a(1+\varphi^2+\tilde\varphi^2+h\tilde\varphi_1)$$

$$\times\exp\left\{-\frac{1}{2}\sum_{\langle ab\rangle}\left[\frac{1}{U}\varphi(a)\varphi(b)+\frac{1}{\tilde U}\tilde\varphi(a)\tilde\varphi(b)\right]\right\}. \tag{A1.11}$$

Here

$$\tilde\varphi=(\tilde\varphi_1\ldots\tilde\varphi_Q),\ \varphi=(\varphi_1\ldots\varphi_N),\ h\to 0,\ Q\to 0,\ N\to 1. \tag{A1.12}$$

In the limit $\tilde U\to U$, the partition function (A1.11) coincides with (A1.6). Here we have introduced two different potentials, U and $\tilde U$, for "monomers" belonging to ring and to open lines, respectively. The configurations with closed lines, obtained an integration over the fields $\tilde\varphi$, drop out by virtue of the condition $Q=0$. The partition function (A1.11) can be put into the form

$$\mathfrak{Z}=\sum U^{L_c}\tilde U^{L_f}h^{2N_f}N^M C(L_c,L_f,N_f,M), \tag{A1.13}$$

where $C(\ldots)$ is the number of different configurations containing L_c monomers in closed lines and L_f in open; $2N_f$ is the number of ends of open lines. In analogy with (A1.2) and (A1.3),

$$\alpha_\infty=\frac{1}{V}\tilde U\frac{\partial\ln\mathfrak{Z}}{\partial\tilde U},\quad h\to 0, \tag{A1.14}$$

$$\alpha_c=\frac{1}{V}U\frac{\partial\ln\mathfrak{Z}}{\partial U}. \tag{A1.15}$$

In the region where the fluctuations are small, the partition function $\mathfrak{Z}$ can be calculated by the method of steepest descents,[10] and after the differentiation in (A1.14) one can set $\tilde U=U$.

APPENDIX 2

We shall now describe the lattice variant of the partition function (2.14). In gauge theories on lattices, instead of a vector potential A_α one introduces a gauge field $A_\alpha(\mathbf r_a,\mathbf r_b)$ defined on the edges of the lattice (the segments of the polymer). The gauge transformation (2.13) in the discrete variant has the form

$$\varphi_a^{\varepsilon_1\ldots1\ldots\varepsilon_n}-i\varphi_a^{\varepsilon_1\ldots2\ldots\varepsilon_n}$$

$$\to\exp\left(\tfrac{1}{2}i\chi_\alpha(\mathbf r_a)\right)\left(\varphi_a^{\varepsilon_1\ldots1\ldots\varepsilon_n}-i\varphi_a^{\varepsilon_1\ldots2\ldots\varepsilon_n}\right),$$

$$A_\alpha(\mathbf r_a,\mathbf r_b)\to A_\alpha(\mathbf r_a,\mathbf r_b)+\chi_\alpha(\mathbf r_a)-\chi_\alpha(\mathbf r_b). \tag{A2.1}$$

The gauge-invariant Hamiltonian on a lattice has the form

$$\mathscr{H}_n=\frac{1}{2U}\sum_{\substack{\langle ab\rangle\\ \{\varepsilon,\varepsilon'\}}}\varphi_a^{\varepsilon_1\ldots\varepsilon_n}\exp\left\{\frac{1}{2}i\sum_\alpha(-)^{\varepsilon_\alpha-\varepsilon_\alpha'}A_\alpha(\mathbf r_a,\mathbf r_b)\right.$$

$$\left.+\frac{1}{2}\pi i\sum_\alpha(\varepsilon_\alpha-\varepsilon_\alpha')\right\}\varphi_b^{\varepsilon_1'\ldots\varepsilon_n'}+\frac{T}{2}\sum_{\langle abcd\rangle\alpha}[A_\alpha(\mathbf r_a,\mathbf r_b)$$

$$+A_\alpha(\mathbf r_b,\mathbf r_c)+A_\alpha(\mathbf r_c,\mathbf r_d)+A_\alpha(\mathbf r_d,\mathbf r_a)]^2,\quad (A(r_d,r_a)=-A(r_a,r_d)). \tag{A2.2}$$

Here the summation over $\langle abcd\rangle$ means sumamtion over all faces of the cubic lattice with vertices $abcd$. On passing to the continuous limit in (A2.1) and (A2.2), we get the Hamiltonian (2.14). We further calculate a mean of the form $\langle\varphi_a^2\cdots\varphi_g^2\rangle$ with the Hamiltonian (A2.2). After integration over φ, we get

$$\langle\varphi_a^2\ldots\varphi_g^2\rangle=U^L\int DA\sum_{\{\sigma\}=\pm}\sum_{\text{conf}}\exp\left\{\frac{1}{2}\sum_{\langle ab\rangle\alpha}\sigma_\alpha\right.$$

$$\times A_\alpha(\mathbf r_a,\mathbf r_b)+\frac{T}{2}\sum_{\langle abcd\rangle\alpha}[A_\alpha(\mathbf r_a,\mathbf r_b)+A_\alpha(\mathbf r_b,\mathbf r_c)$$

$$+A_\alpha(\mathbf r_c,\mathbf r_d)+A_\alpha(\mathbf r_d,\mathbf r_a)]^2. \tag{A2.3}$$

Here the sum over configurations means a sum over all possible subdivisions into products of pair means according to Wick's theorem, i.e. over all possible configurations of closed contours passing through the points at which the fields φ on the left side of (A2.3) are prescribed. The integral over A must be calculated, let us say, for a transverse lattice gauge.[12]

On each contour, 2^n different combinations of signs plus or minus are possible before A_α in the exponent; this corresponds to 2^n possible states of n currents on the contours. On calculating the integral over A, we get

$$\langle\varphi_a^2\ldots\varphi_g^2\rangle=U^L\sum_{(\pm)}\exp\left\{-\frac{1}{2T}\int\frac{d^3x_1\,d^3x_2\sum_\alpha \mathbf j_{\alpha1}\mathbf j_{\alpha2}}{R_{12}}\right\}, \tag{A2.4}$$

where the sum, as in (1.4), extends over the possible directions of the currents (1.1) passing through the points $a,\ldots,g$.

Finally, on calculating with the Hamiltonian (A2.2) the mean of

$$\prod_a\left(1+\sum_\gamma\varphi_\gamma^2(\mathbf r_a)\right), \tag{A2.5}$$

we get the expression (1.7). Thus we have shown that (1.7) agrees with (2.14).

Replacing (A2.5) by

$$\prod_a \left\{ \prod_\alpha (1+|\psi_\alpha(\mathbf{r}_a)|^2)+\bar\varphi(\mathbf{r}_a)\prod_\alpha [1+(1-i\hat\sigma_{1\alpha})\times\psi_\alpha(\mathbf{r}_a)+(1+i\hat\sigma_{1\alpha})\psi_\alpha^*(\mathbf{r}_a)]\varphi(\mathbf{r}_a)\right\}$$

(here $\hat\sigma_{1\alpha}$ are the $\hat\sigma_1$ Pauli matrices, acting on the α-th index of the spinor $\varphi^{\cdots}$) and introducing an integration over the fields ψ_α with weight $\exp(-\mathcal{H}_\alpha)$, where

$$\mathcal{H}_\alpha=\frac{1}{2V}\sum_{\langle ab\rangle}\psi_\alpha^*(\mathbf{r}_a)\exp\{iA_\alpha(\mathbf{r}_a,\mathbf{r}_b)\}\psi_\alpha(\mathbf{r}_b),$$

we obtain an expression for the partition function of immovable frustration lines and of movable thermal vortices, which can pass only through points not occupied by frustrations but can begin and end on frustrations, with conservation of the corresponding currents. In the continuous limit, we obtain (3.9) for this partition function.

[1)]We once more recall that "heat capacity" in our case is the second derivative with respect to the concentration of negative bonds c (with respect to τ).

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