Current in superfluid Fermi liquids and the structure of vortex cores

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It is shown that the mass current in superfluid Fermi liquids with arbitrary Cooper pairing is determined by the coordinate- and momentum-dependent phase $\Phi(\mathbf{k},\mathbf{r})$ of the spin matrix determinant of the order parameter $\Delta_{\alpha\beta}$. In those superfluid liquids in which the $\Phi(\mathbf{k},\mathbf{r})$ field has vortex singularities, e.g., in He^3-A, the mass current contains, along with the traditional term $\rho\mathbf{v}_s$, an additional term contributed by these vortex singularities. For all superfluid Fermi systems, including He^3-B, the vortex in coordinate space may "flow out" into k space by moving in six-dimensional $(\mathbf{k},\mathbf{r})$ space; as a result, the core singularity of the vortex is removed. In all Fermi superfluids, phase slippage is realized by motion of the vortices in $(\mathbf{k},\mathbf{r})$ space, including in particular the motion of vortices in ordinary space and the motion of point vortices (boojums) over the Fermi surface.

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1. INTRODUCTION

The expression for the current in the A-phase of superfluid He^3 has the following form at $T=0$:[1]

$$\mathbf{j}=\rho\mathbf{v}_s+\frac{1}{2}\operatorname{rot}\frac{\hbar}{2m}\rho\mathbf{l}-\frac{\hbar}{2m}C_0\mathbf{l}(\mathbf{l}\operatorname{rot}\mathbf{l}). \tag{1.1}$$

In addition to the usual term, which is proportional to the superfluid velocity $\mathbf{v}_s$, and the solenoidal contribution which is natural for a fluid possessing an intrinsic angular momentum $\hbar\mathbf{l}/2$ per atom, this expression contains an additional term, in which the coefficient C_0 is equal to the fluid density ρ (in the presence of symmetry between particles and holes near the Fermi surface). Many phenomena and hitherto unsolved problems are associated with the existence of this third term. For example, because of this term, only the variations of the local intrinsic angular momentum of the liquid turns out to be defined.[2] In spite of numerous efforts (see, for example, Ref. 3), because of the term with C_0, it has not yet been possible to obtain the equations of dynamics of the A-phase from the Lagrangian or Hamiltonian formalisms, while for the other superfluid liquids (He^3-B, He II), these methods have been applied successfully.[3–6] The very existence of the term with C_0 in (1.1) is connected with singularities in the wave function of the pair at $\mathbf{k}\|\mathbf{l}$, namely it is expressed in terms of the phase of the order parameter $\Delta_{\alpha\beta}=\Delta$,

$$\hat\Delta=i(\hat\sigma\mathbf{V})\hat\sigma_y\Delta_\mathbf{k},\quad \Delta_\mathbf{k}=\Delta_0(\mathbf{k}/k,\ \Delta'+i\Delta'')=|\Delta_\mathbf{k}|e^{i\Phi_\mathbf{k}} \tag{1.2}$$

in the following fashion:

$$-\frac{1}{2}C_0\mathbf{l}(\mathbf{l}\operatorname{rot}\mathbf{l})=\frac{1}{2}\sum_\mathbf{k}\mathbf{k}n_\mathbf{k}\left(\frac{\partial}{\partial\mathbf{k}}\frac{\partial}{\partial\mathbf{r}}-\frac{\partial}{\partial\mathbf{r}}\frac{\partial}{\partial\mathbf{k}}\right)\Phi_\mathbf{k}, \tag{1.3}$$

where $n_\mathbf{k}$ is the quasiparticle distribution function and is equal to

$$n_\mathbf{k}=2\theta(k_F-k) \tag{1.4}$$

in the first-order approximation. The difference of the term with C_0 from zero is due to the fact that the difference of the mixed derivatives of Φ_k is not equal to zero but has a delta-function singularity

$$\left(\frac{\partial}{\partial\mathbf{k}}\frac{\partial}{\partial\mathbf{r}}-\frac{\partial}{\partial\mathbf{r}}\frac{\partial}{\partial\mathbf{k}}\right)\Phi_\mathbf{k}=-2\pi(\mathbf{l}\operatorname{rot}\mathbf{l})(\mathbf{k}\mathbf{l})\delta(\mathbf{k}_\perp),\quad \mathbf{k}_\perp=\mathbf{k}-\mathbf{l}(\mathbf{k}\mathbf{l}). \tag{1.5}$$

In some studies, the singularities in the wave function of a Fermi liquid of the A-phase type are ignored, and, the third term of the current (1.1) does not appear at all (see, for example, Ref. 17). Frequently, opinions are expressed that these singularities are either a consequence of an incorrect transition to the limit of infinite volume, or are connected with the fact that BCS theory operates with plane waves and not with the eigenfunctions of the angular momentum,[8] or else that a weak departure from the A-phase, brought about, for instance, by fluctuations or by an inhomogeneity, can remove these singularities, so that the inconvenient third term drops out and the unsolved problems disappear along with it.

To determine the stability of the third term relative to small changes in the order parameter near the A-phase, we derive an expression for the current in the case of an arbitrary phase in a Fermi liquid with an arbitrary type of pairing, i.e., for any matrix of the order parameter $\Delta_{\alpha\beta}(\mathbf{k},\mathbf{r})$.

We shall show that the current is uniquely determined by the phase $\Phi(\mathbf{k},\mathbf{r})$ of the determinant of this matrix, while the terms of third order in (1.1) appear only if the six-dimensional $(\mathbf{k},\mathbf{r})$ space contains a vortex, in the circuit around which Φ changes by $2\pi N$, where N is an integer. The A-phase is one of the Fermi liquids in which the phase Φ has vortices in $\mathbf{k}$ space even in the homogeneous state. The maximum number of possible vortices in $\mathbf{k}$ space depends on the type of pairing and is equal to 0,2,4,6 ... for $s,p,d,f,$... pairing, respectively. the A-phase has in $\mathbf{k}$ space one vortex with topological charge $N=2$. Inasmuch as the topological charge of the vortex cannot be removed by small changes in the order parameter, the vortex cannot be annihilated by any small deformations of the A-phase, i.e., the inconvenient term in the current (1.1) is stable. This term is removed only by large order-parameter deformations in which the superfluid state is moved far from the A-phase. For this it is necessary that the vortex with $N=2$ be split into two vortices with charges

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$N = 1$, which are then turned antiparallel to one another and annihilate. The A-phase is transformed into a vortex-free state of a planar type or into a B-phase with current $\mathbf{j} = \rho \mathbf{v}_s$. The expression for the terms of the third type in the current in arbitrary phase depends only on the points of intersection of the Fermi surface with the vortices. We shall call these points boojums, in analogy with the point singularities on a surface in coordinate space.[9] Phases having an identical arrangement of boojums have an identical expression for the current.

Also connected with the properties of the phase Φ is the question of the structure of the vortex cores in the superfluid liquids. This question has now become important for B-phase He^3 because of the experimental discovery of a phase transition in the rotating vessel. This transition is presumedly brought about by rearrangement of the structure of the vortex core.[10] We shall show that in He^3-B a vortex in coordinate space can, passing over the six-dimensional $(\mathbf{k},\mathbf{r})$ space, transform into two vortices in $\mathbf{k}$ space, as a result of which the singularity inside (the core) is removed. The vortex core acquires a continuous complex structure, which is characterized by a topologically invariant degree of mapping (the number of times the boojum traverses the whole Fermi surface). It will also be shown that all phase-slippage processes in superfluid Fermi liquids, if only they are not connected with disruption of the order parameter, constitute motion of vortices in the six-dimensional $(\mathbf{k},\mathbf{r})$ space. Particular cases are the motion of vortices in ordinary space and the motion of boojums on the Fermi surface.

2. CURRENT IN A SUPERFLUID FERMI LIQUID

Let us consider a superfluid Fermi liquid consisting of atoms with spin $\frac{1}{2}$, in which Cooper pairing of the ordinary form takes place; these are combination of pairing with angular momenta $l = 0,1,2,\ldots$ The matrix of the order parameter has the following form:

$$\hat{\Delta}(\mathbf{n}) = i\hat{\sigma}_y d_0(\mathbf{n}) + i\hat{\boldsymbol{\sigma}}\mathbf{d}(\mathbf{n})\hat{\sigma}_y, \quad \mathbf{n} = \mathbf{k}/k, \tag{2.1}$$

where $d_0(\mathbf{n})$ is an even, and $\mathbf{d}(\mathbf{n})$ an odd, function of the momentum $\mathbf{k}$ corresponding to pairing with spins 0 and 1. The scalar function d_0 in the states s, d and g takes the respective forms A, $A_{ik}n_i n_k$, $A_{iklm}\, n_i n_k n_l n_m$ and so on, while the spin vector d_α in the states p and f takes the forms $A_{\alpha i}n_i$, $A_{\alpha ijk}n_i n_j n_k$, etc.

For calculation of the current in a system with arbitrary Cooper pairing, we used the well-known expression for the current in terms of the Green's function

$$\mathbf{j}(\mathbf{r}) = \mathrm{Sp}\sum_k \mathbf{k}T\sum_n G_{\alpha\beta}(\mathbf{k},\mathbf{r},\omega_n). \tag{2.2}$$

Here

$$G_{\alpha\beta}(\mathbf{k},\mathbf{r},\omega_n) = \int d^3\rho\, G_{\alpha\beta}(\mathbf{r},\boldsymbol{\rho},\omega_n)e^{-i\mathbf{k}\boldsymbol{\rho}},$$

and in the Green's function $G_{\alpha\beta}(\boldsymbol{\rho},\mathbf{r},\omega_n) = G_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2,\omega_n)$ we separated in standard fashion the dependence on the "fast" coordinates $\boldsymbol{\rho} = \mathbf{r}_1 - \mathbf{r}_2$ of the relative motion and the "slow" coordinates $\mathbf{r} = (\mathbf{r}_1 + \mathbf{r}_2)/2$ of the mass center.

The Green's function is found from the Gor'kov equation, which is conveniently written in integral form:

$$G_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2) = S(\mathbf{r}_1,\mathbf{r}_2)\delta_{\alpha\beta} + \int d^3r_3 d^3r_4 d^3r_5 d^3r_6 S(\mathbf{r}_1,\mathbf{r}_3)\Delta_{\alpha\tau}(\mathbf{r}_3,\mathbf{r}_4) \times S^*(\mathbf{r}_4,\mathbf{r}_5)\Delta^*_{\tau\gamma}(\mathbf{r}_5,\mathbf{r}_6)G_{\gamma\beta}(\mathbf{r}_6,\mathbf{r}_2), \tag{2.3}$$

where $S(\mathbf{r}_1,\mathbf{r}_2)$ is the Green's function of the normal metal and satisfies the equation

$$i\omega_n S(\mathbf{r}_1,\mathbf{r}_2) - \int d^3r_3 \varepsilon(\mathbf{r}_1,\mathbf{r}_3)S(\mathbf{r}_3,\mathbf{r}_2) = \delta(\mathbf{r}_1-\mathbf{r}_2), \tag{2.4}$$

the dependence of $\hat{G}$ and S on the frequency $\omega_n = (2n+1)\pi T$ in formulas (2.3) and (2.4) is omitted for brevity and ε in the $(\mathbf{k},\mathbf{r})$ representation has the form

$$\varepsilon(\mathbf{k},\mathbf{r}) = k^2/2 - \mu + U(\mathbf{r}),$$

$\Delta_{\alpha\beta}$ is given by the expression (2.1).

By assuming that $U(r)$ and $\Delta_{\alpha\beta}(\mathbf{k},\mathbf{r})$ are slowly varying functions of $\mathbf{r}$, we expand (2.3) and (2.4) in terms of the small gradients $\partial/\partial\mathbf{r}$ (see Refs. 11 and 12). Here it is convenient to use the following relations:

$$\int d^3(\mathbf{r}_1-\mathbf{r}_2)e^{-i\mathbf{k}(\mathbf{r}_1-\mathbf{r}_2)}\int d^3r_3 f(\mathbf{r}_1,\mathbf{r}_3)g(\mathbf{r}_3,\mathbf{r}_2) = f(\mathbf{k},\mathbf{r})g(\mathbf{k},\mathbf{r}) + \tfrac{1}{2}i[f,g],$$

$$\int d^3(\mathbf{r}_1-\mathbf{r}_2)e^{-i\mathbf{k}(\mathbf{r}_1-\mathbf{r}_2)}\int d^3r_3 d^3r_4 d^3r_5 d^3r_6 f_1(\mathbf{r}_1,\mathbf{r}_3) \times f_2(\mathbf{r}_3,\mathbf{r}_4)f_3(\mathbf{r}_4,\mathbf{r}_5)f_4(\mathbf{r}_5,\mathbf{r}_6)f_5(\mathbf{r}_6,\mathbf{r}_2) = \prod_i f_i(\mathbf{k},\mathbf{r})$$
$$+\tfrac{1}{2}i\Big\{\Big[\prod_{i=1}^{4} f_i(\mathbf{k},\mathbf{r}), f_5(\mathbf{k},\mathbf{r})\Big] + \Big[\prod_{i=1}^{3} f_i, f_4\Big]f_5 + [f_1f_2,f_3]f_4f_5 + [f_1,f_2]f_3f_4f_5\Big\},$$

where $[f,g]$ is a Poisson bracket:

$$[f(\mathbf{k},\mathbf{r}),g(\mathbf{k},\mathbf{r})] = \frac{\partial f}{\partial\mathbf{r}}\frac{\partial g}{\partial\mathbf{k}} - \frac{\partial f}{\partial\mathbf{k}}\frac{\partial g}{\partial\mathbf{r}}. \tag{2.5}$$

We get, in zeroth order,

$$S^{(0)}(\mathbf{k},\mathbf{r}) = S = 1/(i\omega-\varepsilon), \quad G^{(0)}_{\alpha\beta}(\mathbf{k},\mathbf{r}) = f_{\alpha\beta}^{-1}S,$$
$$f_{\alpha\beta} = \delta_{\alpha\beta} + |S|^2\Delta_{\alpha\gamma}\Delta^+_{\gamma\beta}. \tag{2.6}$$

Here we have used the fact that $\Delta^*_{\alpha\beta} = -\Delta^+_{\alpha\beta}$. In first order,

$$S^{(1)}(\mathbf{k},\mathbf{r}) = 0,$$
$$\hat{G}^{(1)}(\mathbf{k},\mathbf{r}) = -\tfrac{1}{2}i\hat{G}^{(0)}\{[\hat{f},\hat{f}^{-1}](\hat{G}^{(0)})^{-1} + S^*[\hat{\Delta},\hat{\Delta}^+] + [\hat{\Delta},S^*]\hat{\Delta}^+ + \hat{\Delta}[S^*,\hat{\Delta}^+]\}\hat{G}^{(0)}. \tag{2.7}$$

Only the last two terms in the curly brackets of (2.7) are significant in the calculation of the current (2.2), because the first two terms consist of components that are odd either in the frequency ω_n or in ε and vanish on summation over ω_n and integration over ε (the latter by virtue of the symmetry between particles and holes).

Thus, the expression for the current can be written in the form

$$\mathbf{j}(\mathbf{r}) = \mathrm{Sp}\sum_{\mathbf{k}}\mathbf{k}T\sum_n\left(-\frac{i}{2}\right)\hat{G}^{(0)}\{\hat{\Delta}[S^*,\hat{\Delta}^+] + [\hat{\Delta},S^*]\hat{\Delta}^+\}\hat{G}^{(0)},$$

or

$$\mathbf{j}(\mathbf{r})=\mathrm{Sp}\sum_{\mathbf{k}}\mathbf{k}T\sum_{n}\left(-\frac{i}{2}\right)\hat{\varphi}^{-1}\hat{A}\hat{\varphi}^{-1}, \tag{2.8}$$

where

$$\hat{\varphi}=\hat{f}\,|^{-2}S\,|^{2}=\varepsilon^{2}+\omega^{2}+\hat{\Delta}\hat{\Delta}^{+},\quad \hat{A}=[\hat{\Delta},\varepsilon]\,\hat{\Delta}^{+}+\hat{\Delta}\,[\varepsilon,\hat{\Delta}^{+}].$$

We now sum over the frequencies. With this aim, we transform $\hat{\varphi}$ to diagonal form by means of the unitary transformation $\hat{U}$:

$$\hat{\hat{\varphi}}=\hat{U}^{-1}\hat{\varphi}\hat{U}.$$

Using the invariance of the trace, we have

$$\mathbf{j}(\mathbf{r})=-\frac{i}{2}\mathrm{Sp}\sum_{\mathbf{k}}\mathbf{k}T\sum_{n}\hat{\hat{A}}\hat{\hat{\varphi}}^{-2},$$

$$\hat{\hat{\varphi}}=\begin{pmatrix}\omega^{2}+E_{1}^{2} & 0\\ 0 & \omega^{2}+E_{2}^{2}\end{pmatrix},\quad \hat{\hat{A}}=\hat{U}^{-1}\hat{A}\hat{U}, \tag{2.9}$$

where $E_1^2=\varepsilon^2+\Delta_1^2$, $E_2^2=\varepsilon^2+\Delta_2^2$ are the energies of the single-particle excitations of the liquid, i.e., the eigenvalues of the matrix $\hat{\varphi}$ ($\omega=0$). It is easy to reason that the unitary matrix $\hat{U}$ does not depend on the frequency; therefore summation over the frequencies is easily carried out, and at $T=0$, we have

$$\mathbf{j}(\mathbf{r})=-\frac{i}{8}\mathrm{Sp}\sum_{\mathbf{k}}\mathbf{k}\hat{\hat{A}}\hat{\hat{E}}^{-3},$$

$$\hat{\hat{E}}^{-3}=\begin{pmatrix}E_2^{-3} & 0\\ 0 & E_1^{-3}\end{pmatrix}=-\begin{pmatrix}\dfrac{1}{\Delta_1^{2}}\dfrac{\partial n(E_1)}{\partial\varepsilon} & 0\\ 0 & \dfrac{1}{\Delta_2^{2}}\dfrac{\partial n(E_2)}{\partial\varepsilon}\end{pmatrix}. \tag{2.10}$$

Here $n(E_i)=1-\varepsilon/E_i$, $i=1,2$ is the energy distribution function of the particles in a superfluid Fermi liquid, with excitation energy E_i at $T=0$.

Since $\hat{U}$ does not depend on ε, and

$$\int\frac{\partial n(E_i)}{\partial\varepsilon}d\varepsilon=-2,$$

i.e., it does not depend on i, we can assume the quantity n to be the distribution function particle energy (1.4) in the normal Fermi liquid. Then,

$$\mathbf{j}(\mathbf{r})=\frac{i}{8}\sum_{\mathbf{k}}\mathbf{k}\frac{\partial n}{\partial\varepsilon}\mathrm{Sp}\,(\hat{\hat{A}}\hat{\hat{\Delta}}^{-2}),\quad \hat{\hat{\Delta}}=\begin{pmatrix}\Delta_1 & 0\\ 0 & \Delta_2\end{pmatrix}.$$

Carrying out the inverse unitary transformation $\hat{U}^{-1}$ under the sign of the trace, we have

$$\mathbf{j}(\mathbf{r})=\frac{i}{8}\sum_{\mathbf{k}}\mathbf{k}\frac{\partial n}{\partial\varepsilon}\mathrm{Sp}\,(\hat{A}\,(\hat{\Delta}\hat{\Delta}^{+})^{-1})=\frac{i}{8}\sum_{\mathbf{k}}\mathbf{k}\,\mathrm{Sp}\,\{[\hat{\Delta},n]\,\hat{\Delta}^{+}-\hat{\Delta}\,[\hat{\Delta}^{+},n])\,(\hat{\Delta}\hat{\Delta}^{+})^{-1}\}=\frac{i}{8}\sum_{\mathbf{k}}\mathbf{k}\left[\ln\frac{\det\hat{\Delta}}{\det\hat{\Delta}^{+}},n\right]. \tag{2.11}$$

Finally, introducing the phase of the determinant of the matrix $\hat{\Delta}$:

$$\det\hat{\Delta}=d_0^{2}-\mathbf{d}^{2}=|d_0^{2}-\mathbf{d}^{2}|\,e^{i\Phi(\mathbf{k},\mathbf{r})}, \tag{2.12}$$

we write down $\mathbf{j}$ in the final form

$$\mathbf{j}=-\frac{1}{4}\sum_{\mathbf{k}}\mathbf{k}[\Phi,n]. \tag{2.13}$$

Thus, the current depends on the order parameter only through the phase of the determinant of the matrix $\hat{\Delta}$. This general expression is identical with the expression obtained earlier[13] for the case of the A-phase by a semiphenomenological method. It must be kept in mind below only that Φ in formula (2.13) is the phase of the determinant of the matrix (2.1), i.e., it is twice as great as the $\Phi_{\mathbf{k}}$ used in Ref. 13 and in formula (1.2). In the following, we shall take Φ to mean the phase of det $\hat{\Delta}$, so that we shall write the current in the form

$$\mathbf{j}=\frac{1}{4}\sum_{\mathbf{k}}\mathbf{k}\left(\frac{\partial\Phi}{\partial\mathbf{k}}\frac{\partial n}{\partial\mathbf{r}}-\frac{\partial n}{\partial\mathbf{k}}\frac{\partial\Phi}{\partial\mathbf{r}}\right)=\frac{1}{4}\sum_{\mathbf{k}}n\nabla\Phi+\frac{1}{4}\nabla_i\left(\sum_{\mathbf{k}}\mathbf{k}n\frac{\partial\Phi}{\partial k_i}\right)-\frac{1}{4}\sum_{\mathbf{k}}\mathbf{k}n\left(\frac{\partial}{\partial\mathbf{r}}\frac{\partial}{\partial\mathbf{k}}-\frac{\partial}{\partial\mathbf{k}}\frac{\partial}{\partial\mathbf{r}}\right)\Phi. \tag{2.14}$$

3. CURRENT AND BOOJUMS ON THE FERMI SURFACE

Thus, for an arbitrary order parameter, the mass current at $T=0$ is determined exclusively by the coordinate- and momentum-dependent phase Φ (**k**,**r**) of the determinant of the matrix of the order parameter. In this section, we shall show that the unusual terms arise in the current only when there are singularities in the field of the phase Φ. Since the phase Φ varies along a circle, the topologically stable singularities are vortices. Usually we are dealing with vortices in r-space, when Φ does not depend on **k**. These are singular lines, circling around which changes the phase by $2\pi N$ and on which the phase is not defined, while the order parameter modulus goes to zero. In the case of the A phase, we have encountered vortex located in **k** space. Actually, the distribution of the phase Φ in the order parameter of the A phase [see (1.2)]

$$\Delta_{\mathbf{k}}=|\Delta_{\mathbf{k}}|\,e^{i\Phi_{\mathbf{k}}/2} \tag{3.1}$$

is a rectilinear vortex with topological charge $N=2$, passing through the origin $\mathbf{k}=0$ and directed along the axis circling around the axis, the phase of Φ changes by 4π, and $\Delta_{\mathbf{k}}=0$ on the axis of the vortex ($\mathbf{k}_\perp=0$).

It is easy to see that the vortex in **r**-space and the vortex in **k**-space are particular cases of the general case in which the phase depends both on **k** and on **r**, and the vortex is defined in the six-dimensional space (**k**,**r**). In six-dimensional space, the axis of the vortex is itself a four-dimensional manifold on which the order parameter modulus vanished and the phase Φ (**k**,**r**) changes by $2\pi N$ when it traces a closed contour around it. The vortices considered above differ in the location of this axis and they can transform into one another by continuous motion of the axis of the vortex over the six-dimensional space (**k**,**r**). In the first case, the vortex is parallel to **k** space, while in the second case, it is parallel to the coordinate space. In the next section, we shall see that the unusual continuous relaxation of the superflow in the A-phase due just to the motion of the vortices in the six-dimensional (**k**,**r**) space.

The fact that the third term in the current (1.1) in A-phase is connected just with a vortex in (**k**,**r**) space is already seen from Eq. (1.4) since the difference from zero of the mixed derivatives of Φ (**k**,**r**) is a consequence of the vortex singularity in the phase Φ. We shall consider the effect of the vortices on the current for an arbitrary Fermi system. The antisymmetric combination of the derivatives of Φ is expressed in terms of an integral along the axis of the vortex, and just as in the three-dimensional case we need only take into account the four-dimensionality of the axis. Let $x^i = (\mathbf{r},\mathbf{k})$ be the coordinate in six-dimensional space. Then the following formula is valid:

$$\left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^k}-\frac{\partial}{\partial x^k}\frac{\partial}{\partial x^i}\right)\Phi=2\pi N e_{iklmnp}$$

$$\times\int dS^{lmnp}\delta(x-x(\eta^1,\eta^2,\eta^3,\eta^4)). \quad (3.2)$$

Here dS^{lmnp} is the element of volume of the four-dimensional hyperspace occupied by the axis of the vortex, $x^i = x^i(\eta^\alpha)$ are the points of this hyperspace, N is the topological charge of the vortex, equal to two for the vortex in the A-phase, e_{iklmnp} is an absolute antisymmetric object in six-dimensional space. The validity of the formula (3.2) is easily confirmed by integrating both parts of Eq. (3.2) over an arbitrary two-dimensional surface intersecting the axis of the vortex. Actually, let dS^{ik} be an element of this two-dimensional surface; then the left-hand side of the equation is transformed by the Stokes formula into an integral over the line dS^i encircling the vortex:

$$\int dS^{ik}\left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^k}-\frac{\partial}{\partial x^k}\frac{\partial}{\partial x^i}\right)\Phi=\oint dS^i\frac{\partial}{\partial x^i}\Phi=2\pi N,$$

and the integral of the right-hand side reduces to an integral over the entire volume of six-dimensional space of the δ function, and is equal to $2\pi N$.

Let the four-dimensional set of points of the vortex axis is given in the form $\mathbf{k} = \mathbf{k}(l,\mathbf{r})$, where l is the coordinate along the vortex in **k** space. By setting $\eta^\alpha = (l,\mathbf{r})$ and taking it into account that

$$dS^{lmnp}=e^{\alpha\beta\gamma\delta}\frac{\partial x^l}{\partial\eta^\alpha}\frac{\partial x^m}{\partial\eta^\beta}\frac{\partial x^n}{\partial\eta^\gamma}\frac{\partial x^p}{\partial\eta^\delta}d^3r\,dl,$$

We obtain from (3.2)

$$\left(\frac{\partial}{\partial\mathbf{r}}\frac{\partial}{\partial\mathbf{k}}-\frac{\partial}{\partial\mathbf{k}}\frac{\partial}{\partial\mathbf{r}}\right)\Phi(\mathbf{k},\mathbf{r})=2\pi N\int dl\,\delta(\mathbf{k}-\mathbf{k}(l,\mathbf{r}))\left(\frac{\partial\mathbf{k}}{\partial l}\operatorname{rot}\mathbf{k}\right) \quad (3.3)$$

and the third term in the current (2.14) is equal to

$$-\frac{N}{16\pi^2}\int dl\,\mathbf{k}n\left(\frac{\partial\mathbf{k}}{\partial l},\operatorname{rot}\mathbf{k}\right). \quad (3.4)$$

After simple transformations of Eq. (3.4), we reduce the expression for the current (2.14) to the following form:

$$\mathbf{j}=\frac{1}{4}\sum_{\mathbf{k}}n\nabla\Phi-\frac{N}{48\pi^2}\int dl\,n e_{imn}k_n\frac{\partial k_m}{\partial l}\nabla k_i+\frac{1}{2}\operatorname{rot}\mathbf{L}$$

$$+\nabla_i\lambda_i+\frac{N}{48\pi^2}\int dl\,\mathbf{k}\frac{\partial n}{\partial l}(\mathbf{k}\operatorname{rot}\mathbf{k}). \quad (3.5)$$

Here

$$\mathbf{L}=\frac{1}{4}\sum_{\mathbf{k}}n\left[\mathbf{k}\times\frac{\partial\Phi}{\partial\mathbf{k}}\right]+\frac{N}{48\pi^2}\int dl\,n\left[\mathbf{k}\times\left[\frac{\partial\mathbf{k}}{\partial l}\times\mathbf{k}\right]\right], \quad (3.6)$$

and $\lambda_i = \lambda_{ij}$ is a symmetric tensor:

$$\lambda_{ij}=\frac{1}{12}\delta_{ij}\sum_{\mathbf{k}}n\mathbf{k}\frac{\partial\Phi}{\partial\mathbf{k}}-\frac{1}{24}\sum_{\mathbf{k}}\left\{k_i\left[\left[\frac{\partial n}{\partial\mathbf{k}}\times\frac{\partial\Phi}{\partial\mathbf{k}}\right]\times\mathbf{k}\right]_j\right.$$

$$\left.+k_j\left[\left[\frac{\partial n}{\partial\mathbf{k}}\times\frac{\partial\Phi}{\partial\mathbf{k}}\right]\times\mathbf{k}\right]_i\right\}. \quad (3.7)$$

The expression (3.5) is valid for any change of Δ (**k**,**r**) including such that the state of the system is, transformed from one phase into another.

We now calculate the expression for the current in the case in which we find ourselves in some definite phase, but the change of Δ (**k**,**r**) is connected only with motion over the degeneracy space of the state, i.e., it is connected only with the rotations of the order parameter through an angle $\delta\theta$ (**r**) in orbital space and with a change in the total phase of the order parameter φ(**r**) in the considered approximation, in which the order parameter does not depend on the time (rotations of the spin space do not affect the mass current). In this case we have

$$\mathbf{j}=-L_i\nabla\delta\theta_i+{}^1/_2\rho\nabla\varphi+{}^1/_2\operatorname{rot}\mathbf{L}+\nabla_i\lambda_i+\tilde{\mathbf{j}}, \quad (3.8)$$

where $\tilde{\mathbf{j}}$ is the last term in the current (3.5) and **L** is given by the previous expression (3.6).

The first three terms in (3.8) are typical of the mass current in an arbitrary superfluid liquid, including Bose liquids, which possess an intrinsic angular momentum **L**(**r**). The first two terms in (3.8) are always obtained from the Lagrangian formalism (see, for example, Ref. 3), since **L**, δ, $\boldsymbol{\theta}$ and $-\rho$, $\frac{1}{2}\varphi$ are pairwise canonically conjugate variables. From the viewpoint of the Lagrangian approach, the canonically conjugate variables P_a and Q_a are such that

$$\{P_a(\mathbf{r}),\,Q_{a'}(\mathbf{r}')\}=\delta_{aa'}\delta(\mathbf{r}-\mathbf{r}'),$$

always enter into the expression for the momentum density in the form

$$-\sum_a P_a\nabla Q_a. \quad (3.9)$$

We note that in the general case of an arbitrary change of Δ (**k**,**r**), too, the first two terms in the current (3.5) can also be obtained from the Lagrangian formalism. The first term is obtained if we assume that $-n$ and $\frac{1}{4}\Phi$ are canonically conjugate variables. The second term is derived if we assume that the components of the displacement of the vortex in **k**-space commute exactly as they do for a vortex in ordinary space (see, for example, Ref. 14). To be precise, if the vortex is directed along the k_z axis, then

$$\{k_x(l,\mathbf{r}),k_y(l',\mathbf{r}')\}=\frac{48\pi^2}{n(l,\mathbf{r})N}\delta(l-l')\delta(\mathbf{r}-\mathbf{r}'). \quad (3.10)$$

The fourth term in the current (3.8) generally does not contradict the Lagrangian formalism, since the addition of the symmetric tensor λ_{ij} to the current leads only to a rearrangement of the symmetric the momentum flux tensor

$$\pi_{ij}\to\pi_{ij}+\partial\lambda_{ij}/\partial t. \qquad (3.11)$$

Thus, the last term in both (3.8) and (3.5) is the only term in the current which distinguishes it from the expression dictated by the Lagrangian formalism. We call attention to the important fact that this term depends only on those points **k** at which the vortex intersects the Fermi surface, since $\partial n/\partial l$ is a δ function differing from zero only on the Fermi surface. In what follows, we shall call the points of intersection of the vortices with the Fermi surface "boojums," by analogy with the point vortices on a surface in ordinary space. This analogy is justified by the fact that, as we shall see, the boojums on the Fermi surface play the same role in the process of relaxation of superfluids that boojums play in ordinary space.

We denote by $\mathbf{k}_a(\mathbf{r})$ the positions of the boojums on the Fermi surface and write out the current for the general case of an arbitrary number of boojums:

$$\tilde{\mathbf{j}}=-\frac{1}{8}\sum_a C_aN_a\mathbf{n}_a(\mathbf{n}_a\operatorname{rot}\mathbf{n}_a), \quad \mathbf{n}_a=\frac{\mathbf{k}_a}{k_a}, \qquad (3.12)$$

where the coefficient C_a is given by the expression

$$C_a=k_a^3/3\pi^2, \qquad (3.13)$$

and for a spherical Fermi surface all

$$C_a=C_0=k_F^3/3\pi^2=\rho.$$

The quantities N_a are the topological charges of the boojums. The quantity $2\pi N_a$ shows by how much the phase Φ changes on circling around the boojum on the Fermi surface if the boojum is viewed from a region outside the Fermi surface.

We also call attention to the fact that the angular momentum $\mathbf{L}(\mathbf{r})$ that enters into the current (3.5) is determined by the location of the boojums. Actually, let us consider the expression (3.6) for **L**. It consists of two parts, each of which has a simple physical meaning if we regard **k**-space as the same as ordinary three-dimensional space **R**.

We consider the flow of liquid in a drop of radius $R_0=k_F$ with density

$$\rho(\mathbf{R})=\frac{1}{4(2\pi)^3}n(\mathbf{k}\to\mathbf{R})$$

and with a velocity potential $\varphi(\mathbf{R})=\Phi(\mathbf{k}\to\mathbf{R})$. Then the first term in (3.6) is the angular momentum of the liquid expressed in terms of the velocity $\partial\varphi/\partial\mathbf{R}$:

$$\mathbf{L}_1=\int d^3R\rho(\mathbf{R})\left[\mathbf{R}\times\frac{\partial\varphi}{\partial\mathbf{R}}\right].$$

The second term is connected with the vortex in the liquid. If the vortex lies inside the drop the second term is none other than the angular momentum of the liquid expressed in terms of the vorticity and taken with opposite sign:

$$L_2=\frac{2\pi N}{3}\int dl\rho(\mathbf{R}(l))\left[\mathbf{R}\times\left[\frac{\partial\mathbf{R}}{\partial l}\times\mathbf{R}\right]\right]=-L_1.$$

Thus, if the vortex lies inside the drop and boojums are absent, then there is no angular momentum either. If the vortex emerges on the surface of the drop, no contraction takes place and, with the help of simple transformations, we obtain **L** in terms of the coordinates of the boojums:

$$\mathbf{L}(\mathbf{r})=\frac{1}{8}C_0\sum_a N_a\frac{\mathbf{k}_a(\mathbf{r})}{k_F}. \qquad (3.14)$$

Thus, the difference of the current form the expression dictated by the Lagrangian formalism is governed by the boojums. Evidently, the study of the dynamics of boojums on the Fermi surface allows us to include boojums in the general scheme of the Lagrangian or Hamiltonian formalism. Leaving this question for future investigation, we proceed to consideration of the effect of these boojums on the superfluid property of the Fermi liquid.

4. BOOJUMS ON THE FERMI SURFACE AND THE STRUCTURE OF THE VORTEX CORE

The number of boojums on the Fermi surface depends on the type of pairing and on the state. In p-pairing in the A phase in **k** space there is a rectilinear vortex with charge $N=2$, which corresponds to a pair of diametrically opposing boojums with charges $N_1=2$, $N_2=-2$. In the general case of p-pairing, when the system leaves the state of the A phase, the vortex can split into two with charges $N=1$, which corresponds to two pairs of diametrically opposed boojums with charges $N_1=-N_2=N_3=N_4=1$. These boojums can be annihilated, transforming into a state without vortices, in which the current has the form $\frac{1}{2}\rho\nabla\varphi$. As an example we can cite the transformation of the A phase into the B phase. No matter what path is followed by this transformation in phase space, it is inevitably accompanied by splitting and annihilation of the vortices. For example, let us consider a single-parameter transformation in which the matrix $\Delta_{\alpha\beta}$ depends in the following fashion on the one parameter $0\leqslant a\leqslant 1$:

$$\hat{\Delta}=\begin{pmatrix}-n_x+(2a-1)in_y & an_z\\ an_z & n_x+in_y\end{pmatrix}, \quad \mathbf{n}=\frac{\mathbf{k}}{k_F}; \qquad (4.1)$$

at $a=0$, this is the A phase, and at $a=1$, the B phase. The location of the boojums is determined by the zeros of the determinant of the matrix and is given by the formula

$$n_x=0, \quad n_y=\pm\frac{a}{(1-2a)^{1/2}}n_z, \quad a\leqslant\frac{1}{2}.$$

At $a=0$ there is one pair of boojums, and $a\neq 0$ the boojums are split and are annihilated at $a=\frac{1}{2}$. At $1\geqslant a\geqslant\frac{1}{2}$, the boojums are absent.

The topology of the transition from A to B phase through the planar phase proposed by Cross[15] for the computation of the surface energy of the boundary separating the A and B phases is similar.

In the case of other pairings, the number of boojums changes. Thus, in pairing with orbital angular momentum l, the maximum number of pairs of diametrically opposed boojums is equal to $2l$, since this number is determined by the possible number of zeros of the corresponding Legendre polynomials.

We now consider how the singularity in the core of an arbitrary vortex in a Fermi liquid is removed because of the boojums, particularly in the B phase of He^3. Let the state of the vortex have the following form in an arbitrary phase:

$$\Phi(\mathbf{k}, \mathbf{r}) = 2\varphi(\mathbf{r}) + \Phi_0(\mathbf{k}), \tag{4.2}$$

where $\Phi_0(\mathbf{k})$ is the phase of the determinant of the order parameter in the homogeneous state, and $\varphi(\mathbf{r})$ the total condensate phase, which has a vortex singularity, for example, on the z axis around which the phase $\varphi(\mathbf{r})$ changes by $2\pi N$. This is the usual quantized vortex, similar to the vortex in a Bose liquid. On its axis, the entire order parameter vanishes. For example, in the case of p-pairing, $A_{\alpha i} = 0$ at $\rho \equiv (x,y) = 0$; if this is the B phase, the order parameter is

$$A_{\alpha i}(\mathbf{r}) = C(\rho) R_{\alpha i} e^{i\varphi N}, \tag{4.3}$$

where φ is the azimuthal angle of the cylindrical system of coordinates, $R_{\alpha i}$ is a constant of the rotational matrix, $C(\rho) \to 0$ as $\rho \to 0$.

In contrast with a Bose liquid, a singularity on the axis of a vortex can be removed continuously because of the strong coupling between vortices in $\mathbf{k}$ space and $\mathbf{r}$ space (these vortices can transform into one another); furthermore, the entire vortex in $\mathbf{r}$-space can also be removed under certain conditions. We shall show that the continuous vortex core that is formed as a result of the removal of the singularity possesses definite topological charges that are connected with mappings on the Fermi surface.

Let the condition (4.2) be satisfied at large distances, i.e., we have asymptotically the usual quantized vortex in $\mathbf{r}$ space at large distances. We now explain how $\Phi(\mathbf{k},\mathbf{r})$ should vary at small distances in order that there be no singularity. We have the following chain of equations:

$$\begin{aligned} 2\pi N &= \oint_C dr^i \frac{\partial \varphi}{\partial r^i} = \frac{1}{2} \oint_C dr^i \frac{\partial \Phi(\mathbf{k},\mathbf{r})}{\partial r^i} \\ &= \frac{1}{2} \int_\sigma dx\, dy \left(\frac{\partial}{\partial x}\frac{\partial}{\partial y} - \frac{\partial}{\partial y}\frac{\partial}{\partial x} \right) \Phi(\mathbf{k},\mathbf{r}). \end{aligned} \tag{4.4}$$

Here C is the contour circuiting the vortex in $\mathbf{r}$ space at large distances, where (4.2) is satisfied, and let σ be a surface that bears against this contour, i.e., intersects the line of the vortex. The first equality is the definition of the topological change of the vortex N, i.e., the phase φ changes by $2\pi N$ upon circling around the vortex. The second equation occurs because of (4.2). The last equality denotes the transition, by Stokes theorem, from an integral along the contour to an integral over the surface σ, which is chosen to be the plane (x,y). The condition (4.2) is no longer satisfied by $\Phi(\mathbf{k},\mathbf{r})$ on this surface. We now make use of the general formula (3.2), which expresses the antisymmetric combination of the derivatives of Φ in terms of the parameters of the vortex in six-dimensional $(\mathbf{k},\mathbf{r})$ space. Proceeding as in the derivation of the formula (3.3), and then transforming from vortices $\mathbf{k} = \mathbf{k}(l,\mathbf{r})$ in $\mathbf{k}$ space to boojums $\mathbf{k}_a(\mathbf{r})$, we obtain

$$\begin{aligned} &\int dx\, dy \left(\frac{\partial}{\partial x}\frac{\partial}{\partial y} - \frac{\partial}{\partial y}\frac{\partial}{\partial x} \right) \Phi(\mathbf{k},\mathbf{r}) \\ &= \frac{1}{2} \sum_a N_a \int dx\, dy\, \mathbf{n}_a \left[\frac{\partial \mathbf{n}_a}{\partial x} \frac{\partial \mathbf{n}_a}{\partial y} \right], \end{aligned} \tag{4.5}$$

where N_a and $\mathbf{n}_a$ are the topological charge of the a-th boojum and the unit vector in the direction of the a-th boojum. We further take it into account that the integral on the right side of (4.5)

$$\frac{1}{4\pi} \int dx\, dy\, \mathbf{n}_a \left[\frac{\partial \mathbf{n}_a}{\partial x} \frac{\partial \mathbf{n}_a}{\partial y} \right] = \nu_a \tag{4.6}$$

is an integer that shows how many times the boojum travels over the entire Fermi surface, when the coordinate $\rho = (x,y)$ runs over the surface σ that intersects the vortex in coordinate space. This is the topological invariant called the degree of reflection of a surface σ that, with account taken of the given distribution of Φ on the boundary, can be regarded as equivalent to a sphere on the Fermi surface.

Thus, we obtain the following formula for the connection of the charge of a vortex N quantized, in coordinate space and having a continuous core; with the topological charges of the boojums N_a and ν_a:

$$N = \frac{1}{2} \sum_a N_a \nu_a. \tag{4.7}$$

The coefficient $\frac{1}{2}$ should not disturb us, since the boojums occur as pairs with diametrically opposed locations and with opposite signs of N_a and ν_a. Therefore the charge of the vortex is always an integer.

We apply formula (4.7) to the A and B phases of He^3. We first consider a vortex with $N = 1$ in the B phase. At large distances from the vortex, in the pure B phase, there are no boojums. If there are also no boojums at small distances, then the vortex has the form (4.3) (with $N = 1$), and the modulus of the order parameter goes to zero on the axis of the vortex. Another alternative is the appearance of boojums on the Fermi surface, beginning with distances of the order of the coherence length ξ. Here only two pairs of boojums can be created. One of the possible combinations of topological charges N_a and ν_a, which ensures that the right side of (4.7) be equal to unity, is the following

$$N_1 = -N_2 = N_3 = -N_4 = 1, \quad \nu_1 = -\nu_2 = 1, \quad \nu_3 = -\nu_4 = 0. \tag{4.8}$$

This is the simplest combination and evidently corresponds to minimum energy of the vortex. Equation (4.8) means that one of the pairs of boojums circuits the entire Fermi sphere once, while the location of the other, for example, does not depend on the coordinates. How the real configuration corresponds to a minimum energy at a given choice of topological invariants (4.8) is a complicated problem, since it involves determination of the minimum in an at least 18-dimensional space of the order parameter $A_{\alpha i}$. We write down one of the possible arrangements of the boojums in $\mathbf{k}$ space as a function of the coordinates, satisfying (4.8):

$$\mathbf{n}_1 = -\mathbf{n}_2 = \hat{\mathbf{z}} \cos\theta(\rho) + \hat{\boldsymbol{\rho}} \sin\theta(\rho), \qquad \mathbf{n}_3 = -\mathbf{n}_4 = \hat{\mathbf{z}}, \tag{4.9}$$

where $\theta(\rho)$ is a continuous function, equal to 0 at $\rho = 0$ and π at $\rho \geqslant \rho_0 \sim \xi$. The order-parameter field which realizes such a distribution of the boojums, can be represented, for example, in the form of an A phase ($\rho = 0$) that transforms continuously into a planar phase ($\rho = \rho_0$) with increase in ρ, where the boojums are annihilated, and then into the B phase ($\rho \to \infty$). Thus, we see that the structure of the vortex core in the B phase is rather complicated. Different structures are possible, characterized either by different topological invariants or by different orientations of the boojums inside a given topological class, or even by different configurations of the fields $A_{\alpha i}$ at a given configuration of the boojums. Phase transitions are possible between the different structures of the core, and one of them has been observed experimentally.[10]

The vortex with $N = 1$ can be constructed in the A phase in similar fashion. Here a pair of boojums with $N_a = \pm 2$ is already present at infinity. Since a pair of boojums with double charge cannot yield unity on the right side of Eq. (4.7), the boojums must split up. One of the possible combinations of the invariants is again given here by Eq. (4.8), and the arrangement of the boojums is given by formula (4.9), where $\theta(\rho)$ is a continuous function, now equal to zero at ∞ and to π at $\rho = 0$. The construction of one of the realizations of removal of the singularity in the vortex core with $N = 1$ in the A phase belongs to Mermin.[8] In this construction, a planar phase is located in the center of the vortex ($\rho = 0$).

The singularity in the vortex with $N = 2$ in the A phase can be removed in accord with Eq. (4) and without splitting of the boojums. Therefore, the system is always in the A phase, and this means that the vortex has actually no core. Thus we have deduced that the vortices are stable in an A phase with even topological charge N. This conclusion is very well known.[16,17] However, we have arrived at it not by the way of investigation of the degeneracy space of the A phase, but only from the fact that the A phase contains in **k** space a pair of boojums with double charges.

5. BOOJUMS ON THE FERMI SURFACE AND PHASE SLIPPAGE

Just as vortices can shed their phase when moving in coordinate space, boojums can shed their phase while moving on the Fermi sphere. This is actually a special case of phase slippage, due to motion of the vortices in six dimensional (**k**,**r**) space. We consider the second case. Let a homogeneous state of a system with flow be given. We choose the z axis along the flow; then the function $\Phi(\mathbf{k},\mathbf{r})$ is given in the following form

$$\Phi(\mathbf{k}, \mathbf{r}) = 2\varphi(z) + \Phi_0(\mathbf{k}). \tag{5.1}$$

Here, as in (4.2), $\Phi_0(\mathbf{k})$ is independent of the coordinates of the phase of the order-parameter determinant in the ground state, while $\varphi(z)$ is the phase of the condensate and is of the form $2mv_s z/\hbar$, where $\mathbf{v}_s$ is the velocity of flow.

We are interested in the continuous process of relaxation of the flow, i.e., in the transition to the same homogeneous state but with another lesser flow. We assume that such a process exists in a finite region of z and t. This means that outside this region $\Phi(\mathbf{k},\mathbf{r})$ is given by the formula (5.1). We consider the change in φ in going around a contour circling the region of phase slippage in two-dimensional coordinate space (z,t). This change, which is equal to $2\pi N$, where N is an integer, can be expressed in terms of boojums if we assume that in the region of phase slippage, there is no singularity in coordinate space. Actually, using formulas (4.4)–(4.6), in which the coordinates x and y must be replaced by z and t, we obtain

$$N = \frac{1}{8\pi}\sum_a N_a \int dt\, dz\, \mathbf{n}_a \left[\frac{\partial \mathbf{n}_a}{\partial t} \times \frac{\partial \mathbf{n}_a}{\partial z}\right]. \tag{5.2}$$

It is seen from this formula that to shed a single quantum of phase, it is necessary that any pair of boojums sweep the entire Fermi surface in its motion. In the A phase, where the boojums are already present on the Fermi surface and possess a double charge, the continuous shedding of two quanta of the phase takes place without leaving the A phase. Therefore, such a phase slippage mechanism is preferred to the motion of singular vortices in coordinate space. In the B phase, the situation is reversed. For the existence of slippage of the type considered here, creation of boojums on the Fermi surface is required. For this it is necessaryto destroy the B phase in a large volume, to be precise, over the entire surface intersecting the channel (since there is no dependence of the order parameter on x and y). Therefore, in broad channels, phase slippage must take place at the expense of motion of vortices in the cross section of the channel. In narrower channels with a diameter of the order of the coherence length ξ, one must expect the appearance of the boojum mechanism of phase slippage. If this mechanism dominates, the equation of motion for the averaged superfluid flow velocity in the channel has the following form, in accord with (5.2):

$$\frac{\partial}{\partial t}\langle \mathbf{v}_s\rangle = -\nabla\mu + \frac{1}{8}\sum_a N_a e_{imn}\left\langle n_a{}^i \frac{\partial n_a{}^m}{\partial t}\nabla n_a{}^n\right\rangle. \tag{5.3}$$

Here μ is the chemical potential. In the A phase, setting $N_1 = -N_2 = 2$ and $n_1 = -n_2 = l$, we obtain the well known equation for $\mathbf{v}_s$.

6. CONCLUSION

The usual superfluid properties of Fermi systems, in which Cooper pairing is realized with nonvanishing orbital momentum, are due to boojums—point vortices—on the Fermi surface. The presence of boojums on the Fermi surface produces in the superfluid current, an additional term which does not arise in a Bose liquid. Thanks to the boojums, the vortices in Fermi systems can have a non-singular core, in which order parameter $A_{\alpha i}$ vanishes nowhere, i.e., the superfluid state is disrupted nowhere. The motion of the boojums is accompanied by a phase transition between different states.

In the future, it will be necessary to investigate the dynamics of boojums and to ascertain whether Hamiltonian and Lagrangian formalisms exist that describe their motion.

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