Intense wave beams in smoothly inhomogeneous nonlinear media

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A method is proposed for a self-consistent description of the propagation of intense wave beams in smoothly inhomogeneous nonlinear media. A system of equations is derived and it includes an equation for a reference ray which is the energy center of a beam, as well as an equation for the field in a coordinate system linked to this ray. An analytic study is reported of the beam path in the strongly nonlinear case when a beam splits into separate waveguide channels. It is shown that the paths of wave beams in a plasma with a focusing nonlinearity mechanism become "rectified" so that the beams penetrate denser plasma layers.

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One of the main difficulties encountered in the description of the propagation of intense wave beams in inhomogeneous nonlinear media arises from the need to allow simultaneously for the influence of diffraction, nonlinearity, and inhomogeneity of the medium. Propagation of wave beams in homogeneous nonlinear media has been studied sufficiently thoroughly (for reviews see, for example, Refs. 1 and 2). However, if the medium is inhomogeneous, the problem becomes greatly complicated primarily because of the displacement of a beam from a linear path. This displacement is estimated in Ref. 3 using perturbation theory.

We shall propose a method for a self-consistent description of the propagation of wave beams in smoothly inhomogeneous nonlinear media and we shall allow for the diffraction effects as well as for the refraction due to the inhomogeneity and nonlinearity of the medium. This description presupposes, first, a weak angular divergence (a quasiplanar phase front) of a wave beam and, second, smoothness of the propagation path and a smooth variation of the inhomogeneity of the medium within the beam width. For the sake of simplicity, the treatment is carried out within the framework of the scalar Helmholtz equation.

The main idea behind this treatment is an intuitive assumption that a wave beam with a weak angular divergence may remain localized near a certain curve (a reference ray) over extended paths. We shall select the reference ray to be the geometric locus of the centers of gravity of the energy flux in the wave beam $\mathbf{R}$:

$$\mathbf{R}=\int_S \mathbf{r}\Pi\, dS \Big/ \int_S \Pi\, dS,$$

where S is a surface orthogonal to the field of the vectors of the energy density flux $\mathbf{\Pi}$ ($|\mathbf{\Pi}| = \Pi$); $\mathbf{r}$ is the radius vector of a point lying on the surface S; dS is a differentially small element of the surface area.

This definition of the reference beam allows us to use the field equation in a description of the beam dynamics in terms of integral characteristics of the beam calculated on a constant-phase surface (see Sec. 1). On the other hand, near the reference ray we can use the hypothesis of a weak angular divergence of a wave beam and simplify the initial Helmholtz equation going over to an equation of the parabolic type (Sec. 2). It is thus possible to derive a self-consistent system of equations for the reference ray path and for the structure of the field of the wave beam propagating along it. One should point out that although this system is simpler than the initial equation, concrete results can be obtained with its aid either using a computer or subject to further simplifications (for example, in the aberration-free approximation). The procedure for obtaining such concrete results is outside the scope of the present paper.

However, there is an important limiting case when the structure of a beam and the reference ray path can be calculated analytically. It corresponds to the case of a strong nonlinearity when in the initial part of the propagation path a wave beam splits into separate self-maintained waveguide channels (solitons) in accordance with laws governing a homogeneous medium,[4] and then the propagation of such waveguide channels occurs independently of their "neighbors." The path of each solitary channel in a plane-layer medium can be found in quadratures (Sec. 3) and, consequently, it is possible to identify some characteristic laws governing nonlinear penetration of a beam into a dense plasma.

Our treatment will be concluded (Sec. 4) by very simple estimates and a list of possible ways of generalizing the theory developed here.

1. In the case of wave beams described by the linear scalar Helmholtz equation

$$\Delta E+k_0^2\varepsilon(\mathbf{r}, |E|^2)E=0 \qquad (1.1)$$

[in the case of electromagnetic waves the quantity $\varepsilon(\mathbf{r},|E|^2)$ represents the permittivity of the medium and k_0 is the wave number in vacuum] the energy flux density vector is $\mathbf{\Pi} = \nabla\varphi A^2$ and the path of the reference beam is given by

$$\mathbf{R}=\int_S \mathbf{r}A^2|\nabla\varphi|\,dS \Big/ \int_S A^2|\nabla\varphi|\,dS, \qquad (1.2)$$

where A and φ are the real amplitude and phase of the field $E = Ae^{ik_0\varphi}$. The surface S in Eq. (1.2) is identical with the constant-phase surface. We shall rewrite Eq. (1.1) in the form of a system of equations for A and φ:

$$\mathrm{div}(A^2\nabla\varphi)=0, \qquad (1.3)$$

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$$(\Delta\varphi)^2=\tilde{\varepsilon}\equiv\varepsilon(\mathbf{r}, A^2)+\Delta A/k_0^2A, \quad (1.4)$$

and we shall derive certain general relationships satisfied by the reference ray path.

A direct consequence of Eq. (1.3) is the law of conservation of the total energy flux in a beam:

$$\mathscr{P}=\int_S A^2|\nabla\varphi|dS=\int_S \tilde{\varepsilon}^{1/2}A^2\,dS=\text{const}. \quad (1.5)$$

Using Eq. (1.3) and the formula known from vector analysis

$$\mathbf{r}\,\mathrm{div}(A^2\nabla\varphi)=\mathbf{x}_j^0\,\mathrm{div}(x_jA^2\nabla\varphi)-A^2\nabla\varphi \quad (1.6)$$

[$\mathbf{x}_j^0 (j=1,2,3)$ are unit vectors in a Cartesian coordinate system $x=x_1$, $y=x_2$, $z=x_3$], we find that after integration of Eq. (1.6) in an arbitrary volume V bounded by a smooth surface S_V, we have

$$\int_{S_V}\mathbf{r}A^2(\nabla\varphi\cdot d\mathbf{S}_V)=\int_V A^2\nabla\varphi\,dx\,dy\,dz. \quad (1.7)$$

We shall assume that V is the volume between two infinitesimally close phase fronts $\varphi=\varphi_1$ and $\varphi=\varphi_2$ $(d\varphi=\varphi_2-\varphi_1)$ and we shall assume that the distance along the normal between them is $dl=d\varphi/\tilde{\varepsilon}^{1/2}$. We then readily obtain from Eq. (1.7) the following expression for $d\mathbf{R}/d\varphi$:

$$\frac{d\mathbf{R}}{d\varphi}=\frac{1}{\mathscr{P}}\int_S\nabla\varphi\frac{A^2\,dS}{\tilde{\varepsilon}^{1/2}}. \quad (1.8)$$

Replacing in Eq. (1.8) the variable φ with a variable s representing the reference ray path length $\mathbf{R}(\varphi)$, and bearing in mind that

$$d\mathbf{R}/ds=\mathbf{s}_0 \quad (1.9)$$

[$\mathbf{s}_0$ is a unit vector of a tangent to the $\mathbf{R}(s)$ curve], we find the phase of the field in the reference ray:

$$\varphi_0=\int_0^s\varepsilon_0^{1/2}\,ds,\quad \varepsilon_0(s)=\left(\frac{1}{\mathscr{P}}\int_S\nabla\varphi\frac{A^2\,dS}{\tilde{\varepsilon}^{1/2}}\right)^{-2}. \quad (1.10)$$

We shall assume that Eq. (1.4) is subjected to the action of an operator $\nabla=\mathbf{x}_j^0\,\partial/\partial x_j$ and we shall multiply the resultant expression by $A^2/\tilde{\varepsilon}$. Then, simple transformations give

$$2\mathbf{x}_j^0\,\mathrm{div}\left(\frac{\partial\varphi}{\partial x_j}\frac{A^2\nabla\varphi}{\tilde{\varepsilon}}\right)+2\nabla\varphi A^2\frac{\nabla\varphi\nabla\tilde{\varepsilon}}{\tilde{\varepsilon}^2}=\frac{A^2\nabla\tilde{\varepsilon}}{\tilde{\varepsilon}}. \quad (1.11)$$

Integrating Eq. (1.11) in the volume between two infinitesimally close phase fronts, exactly as has been done in the derivation of Eq. (1.8), we obtain

$$\frac{d^2\mathbf{R}}{d\varphi^2}=\frac{1}{2\mathscr{P}}\int_S\left(\nabla\tilde{\varepsilon}-2\frac{\nabla\varphi(\nabla\varphi\nabla\tilde{\varepsilon})}{\tilde{\varepsilon}}\right)\frac{A^2\,dS}{\tilde{\varepsilon}^{3/2}}. \quad (1.12)$$

Going over in Eq. (1.12) from the variable φ to the variable s [see Eq. (1.9)], we transform Eq. (1.12) to

$$\frac{d\mathbf{s}_0}{ds}=\frac{\mathbf{n}_0}{\rho}=\frac{\varepsilon_0}{2\mathscr{P}}\times\int_S A^2\left(\frac{\nabla\tilde{\varepsilon}}{\tilde{\varepsilon}}+\frac{(d\varepsilon_0/ds)\nabla\varphi}{\varepsilon_0^{3/2}}-2\frac{\nabla\varphi(\nabla\varphi\nabla\tilde{\varepsilon})}{\tilde{\varepsilon}^2}\right)\frac{dS}{\tilde{\varepsilon}^{1/2}}. \quad (1.13)$$

Here, $\mathbf{n}_0(s)$ is a unit vector along the normal to the reference ray $\mathbf{R}(s)$ and ρ is the radius of curvature of the reference ray.[1)]

The relationships (1.9) and (1.13) represent a system of equations for the reference ray path $\mathbf{R}(s)$. However, this system is not closed, since the right-hand side of Eq. (1.13) depends on the integral characteristics of the wave beam. Therefore, it is necessary to supplement Eqs. (1.9) and (1.13) with equations describing the amplitude and phase structures of the field in a coordinate system linked to the reference ray.

2. We shall now investigate propagation of wave beams with a weak angular divergence or, in other words, with a quasiplanar phase front. In the case of such beams the total phase of the field can be represented by a sum $\varphi=\varphi_0(s)+\tilde{\varphi}$, where $\varphi_0(s)$ is the phase advance in the reference ray and $\tilde{\varphi}$ represents phase distortions within the beam localization region; it is also assumed that

$$|\nabla\tilde{\varphi}|\ll|\partial\varphi_0/\partial s|=[\varepsilon_0(s)]^{1/2}. \quad (2.1)$$

Moreover, we shall assume that the wave beam is narrow on the scale of radii of curvature (ρ) and torsion (T) of the reference ray and on the scale of the characteristic lines of the inhomogeneity of the medium L $(L=|\varepsilon/\{\partial\varepsilon[\mathbf{r},A^2(\mathbf{r})]/\partial\mathbf{r}\}_{A^2=\text{const}}|)$:[2)]

$$\Lambda_\perp\ll\min\{\rho, T, L\} \quad (2.2)$$

($\Lambda_\perp$ is the beam width).

The restrictions imposed by Eqs. (2.1) and (2.2) allow us to simplify greatly the initial Eq. (1.1). With this in mind, we shall go over in Eq. (1.1) to orthogonal curvilinear coordinates (s,ξ,η) related to the Cartesian coordinates $\mathbf{r}=\{x,y,z\}$ by

$$\mathbf{r}=\mathbf{R}(s)+\xi\mathbf{a}(s)+\eta\mathbf{b}(s)=\mathbf{R}(s)+\mathbf{r}_\perp, \quad (2.3)$$

where

$$\mathbf{a}(s)=\mathbf{n}_0(s)\cos\theta(s)-\mathbf{m}_0(s)\sin\theta(s),$$

$$\mathbf{b}(s)=\mathbf{n}_0(s)\sin\theta(s)+\mathbf{m}_0(s)\cos\theta(s)$$

is the orthogonal basis in a cross section transverse to the reference ray $\mathbf{R}(s)$; this basis rotates relative to the normal [$\mathbf{n}_0(s)$ is a unit vector of the normal] and relative to the binormal [$\mathbf{m}_0(s)$ is the unit vector of the binormal] at an angular velocity $d\theta/ds=1/T$. The Lamé coefficients of the new coordinate system are

$$h_s=[1-(\xi\cos\theta+\eta\sin\theta)/\rho],\quad h_\xi=h_\eta=1, \quad (2.4)$$

and Eqs. (1.3) and (1.4), subject to the conditions (2.1) and (2.2), become

$$\frac{\partial}{\partial s}(\varepsilon_0^{1/2}A^2)+\frac{\partial}{\partial\xi}\left(\frac{\partial\tilde{\varphi}}{\partial\xi}A^2\right)+\frac{\partial}{\partial\eta}\left(\frac{\partial\tilde{\varphi}}{\partial\eta}A^2\right)=0, \quad (2.5)$$

$$2\varepsilon_0^{1/2}\frac{\partial\tilde{\varphi}}{\partial s}+\left(\frac{\partial\tilde{\varphi}}{\partial\xi}\right)^2+\left(\frac{\partial\tilde{\varphi}}{\partial\eta}\right)^2=\left(\varepsilon(\mathbf{r},A^2)-\frac{\varepsilon_0}{h_s^2}\right)+\frac{A_{\xi\xi}+A_{\eta\eta}-\rho^{-1}(\cos\theta A_\xi+\sin\theta A_\eta)}{k_0^2A}. \quad (2.6)$$

It should be noted that Eq. (2.6) is derived by dropping, on the basis of Eq. (2.1), terms of the order of $1/k_0^2 \min\{\rho^2, T^2, L^2\}$; however, terms linear in respect of the small parameters $\nu = |\nabla\tilde{\varphi}|/\varepsilon_0^{1/2}$ and $\mu = A_\perp/\min\{\rho, T, L\}$ are included in Eq. (2.6). We shall introduce a complex field amplitude along the coordinate s:

$$\mathscr{E} = \varepsilon_0^{1/4} A e^{ik_0\tilde{\varphi}}. \tag{2.7}$$

Then, Eqs. (2.5) and (2.6) are readily transformed to the same parabolic equation

$$2ik_0\varepsilon_0^{1/2}\frac{\partial\mathscr{E}}{\partial s} + \frac{\partial^2\mathscr{E}}{\partial\xi^2} + \frac{\partial^2\mathscr{E}}{\partial\eta^2} + k_0^2\left(\varepsilon\left(\mathbf{r}, \frac{|\mathscr{E}|^2}{\varepsilon_0^{1/2}}\right) - \frac{\varepsilon_0}{h_s^2}\right)\mathscr{E} - \frac{1}{\rho}\left(\cos\theta\frac{\partial|\mathscr{E}|}{\partial\xi} + \sin\theta\frac{\partial|\mathscr{E}|}{\partial\eta}\right) = 0. \tag{2.8}$$

Using Eqs. (2.1) and (2.2), we can simplify also the equations deduced earlier for the reference beam path [Eqs. (1.9)] and [(1.13)]. Firstly, the assumption (2.1) of a quasiplanar phase front makes it possible to replace integration over the surface area S by integration over a cross section orthogonal to the reference ray, i.e., integration with respect to ξ and η. Secondly, it follows from Eqs. (1.4), (2.1), and (2.2) that

$$\bar{\varepsilon} = \varepsilon(\mathbf{R}, A^2) + \frac{A_{\xi\xi} + A_{\eta\eta}}{k_0^2 A} + \left(\frac{\partial\varepsilon(\mathbf{R}+\mathbf{r}_\perp, A^2)}{\partial\mathbf{r}_\perp}\right)_{A^2=\text{const},\ \mathbf{r}_\perp=0}\mathbf{r}_\perp - \frac{A_\xi\cos\theta + A_\eta\sin\theta}{\rho k_0^2 A} + O(\mu^2, \mu\nu, \nu^2)$$

$$\approx \varepsilon_0 + 2\frac{\varepsilon_0}{\rho}(\xi\cos\theta + \eta\sin\theta), \tag{2.9}$$

where $O(\mu^2, \nu\mu, \mu^2)$ are terms which are quadratic in respect of small parameters ν and μ.

We shall multiply Eq. (2.9) by $(\xi A_\xi A + \eta A_\eta A)$ and average over ξ and η. Bearing in mind that in view of the definition of the reference ray (1.2)

$$\iint_{-\infty}^{\infty} A^2(\mathbf{a}\xi + \mathbf{b}\eta)\,d\xi\,d\eta \approx 0,$$

we find after integration by parts that

$$\varepsilon_0(\mathbf{R}) = \frac{\iint_{-\infty}^{\infty} d\xi\,d\eta\,(U(\mathbf{R}, A^2(\xi,\eta,s)) + k_0^{-2}(A_\xi^2 + A_\eta^2))}{\iint_{-\infty}^{\infty} A^2(\xi,\eta,s)\,d\xi\,d\eta} + O(\mu^2\nu^2), \tag{2.10}$$

where[3)]

$$U(\mathbf{r}, A^2) = \int_0^{A^2}\varepsilon(\mathbf{r}, A^2)\,dA^2. \tag{2.11}$$

We shall now substitute Eqs. (2.9) and (2.10) into Eq. (1.13). Standard but fairly cumbersome transformations (in particular, integration by parts) yield, if we neglect terms quadratic in ν and μ, the following approximate equations for the reference ray path:

$$\frac{d\mathbf{R}}{ds} = \mathbf{s}_0, \quad \frac{d\mathbf{s}_0}{ds} = \left(\frac{\mathbf{a}(s)\,\partial U_{\text{eff}}/\partial\xi + \mathbf{b}(s)\,\partial U_{\text{eff}}/\partial\eta}{2U_{\text{eff}}}\right)_{\mathbf{r}=\mathbf{R}} = \frac{1}{2}\left(\frac{\partial U_{\text{eff}}/\partial\mathbf{r}_\perp}{U_{\text{eff}}}\right)_{\mathbf{r}=\mathbf{R}}, \tag{2.12}$$

where

$$U_{\text{eff}}(\mathbf{r}, \mathbf{R}) = \frac{\iint_{-\infty}^{\infty} d\xi_1\,d\eta_1\,U(\mathbf{r}, A^2(\xi_1, \eta_1, \mathbf{R}))}{\iint_{-\infty}^{\infty} A^2(\xi_1, \eta_1, \mathbf{R})\,d\xi_1\,d\eta_1}; \tag{2.13}$$

It is natural to call U_{eff} the effective potential energy of a beam.

When the system (2.12) is used, we must remember that this system is valid only on condition $\varepsilon_0 > 0$, i.e., when the phase of the field in a reference ray (and, generally, in a beam) is a real quantity.

We shall now introduce in Eq. (2.12) a variable τ which is related to s by $d\tau = (ds/U_{\text{eff}}^{1/2})_{\mathbf{r}=\mathbf{R}}$. Then, instead of the system (2.12), we obtain

$$\frac{d^2\mathbf{R}}{d\tau^2} = \frac{1}{2}(\nabla U_{\text{eff}}(\mathbf{r}, \mathbf{R}))_{\mathbf{r}=\mathbf{R}}, \quad d\tau = \left(\frac{ds}{U_{\text{eff}}^{1/2}}\right)_{\mathbf{r}=\mathbf{R}},$$

$$(\nabla\mathscr{F}(\mathbf{r}, \mathbf{R}(s)))_{\mathbf{r}=\mathbf{R}} = \left(\frac{\partial\mathscr{F}}{\partial\mathbf{r}}\right)_{\mathbf{r}=\mathbf{R}} + \frac{d\mathbf{R}}{ds}\left(\frac{d\mathbf{R}}{ds}\frac{\partial\mathscr{F}}{\partial\mathbf{R}}\right)_{\mathbf{r}=\mathbf{R}}. \tag{2.14}$$

It is interesting to note that, in contrast to a linear medium when the condition $\varepsilon_0 = \varepsilon(\mathbf{R}) > 0$ implies also $U_{\text{eff}} = \epsilon(\mathbf{R}) > 0$, in a nonlinear problem we encounter a situation when $\varepsilon_0 > 0$ and $U_{\text{eff}} < 0$ (for an example see Sec. 3). The validity of the system (2.12) is self-evident. However, the system (2.14) can be used if $U_{\text{eff}} < 0$, but now $d\tau = i\,d\tau_H$ is a purely imaginary quantity.

Two more or less self-evident consequences follow directly from the system (2.14): firstly, in the case of homogeneous nonlinear media a wave beam satisfying the conditions (2.1) and (2.2) propagates along a straight path and, secondly, in linear media where $U_{\text{eff}} = \varepsilon(\mathbf{r})$ the center of gravity of the energy flux in a beam is concentrated near the reference ray and it lies on an ordinary geometric locus. If the medium is both nonlinear and inhomogeneous, then the system (2.14) must be solved together with Eq. (2.8).

We shall now give the expression for U_{eff} applicable to a medium with a cubic nonlinearity [$\varepsilon = \varepsilon_1(\mathbf{r}) + \varepsilon_2(\mathbf{r})A^2$]:

$$U_{\text{eff}} = \varepsilon_1(\mathbf{r}) + \frac{\varepsilon_2(\mathbf{r})}{2}\iint_{-\infty}^{\infty} A^4\,d\xi\,d\eta \Big/ \iint_{-\infty}^{\infty} A^2\,d\xi\,d\eta. \tag{2.15}$$

In the case of a plasma in which the concentration gradient ∇N is antiparallel to $\nabla\varepsilon_1$ but parallel to $\nabla\varepsilon_2$ (striction nonlinearity), we can substitute Eqs. (2.15) and (2.12) to demonstrate that the radius of curvature of the path of the center of gravity of the energy flux in a wave beam increases compared with the linear case. In other words, a beam which displaces a plasma by its own field tends to "rectify" its propagation path and thus penetrate "deeper" into denser plasma layers.

3. We shall analyze the problem further by considering the example of two-dimensional $\mathbf{r} = \{x,z\}$ wave beams propagating in a smoothly inhomogeneous medium with a cubic nonlinearity. In this case we have to modify Eqs. (2.10) and (2.15) by replacing double integration with respect to ξ and η with single integration with respect to ξ [$\mathbf{r} = \mathbf{R}(s) + \xi\mathbf{n}_0(s)$] and rewriting Eq. (2.8) bearing in mind that $\mathscr{E}$ and ε are independent of η_τ:

$$2\varepsilon_0^{1/2}k_0\frac{\partial\mathscr{E}}{\partial s}+\frac{\partial^2\mathscr{E}}{\partial\xi^2}+k_0^2\left[\varepsilon_1(\mathbf{r})+\varepsilon_2(\mathbf{r})\frac{|\mathscr{E}|^2}{\varepsilon_0^{1/2}}-\frac{\varepsilon_0}{(1-\xi/\rho)^2}\right]\times\mathscr{E}-\frac{1}{\rho}\frac{\partial|\mathscr{E}|}{\partial\xi}=0. \tag{3.1}$$

We can identify two limiting situations:

a) a wave beam in a "weakly nonlinear" medium when, in addition to Eqs. (2.1) and (2.2), we can assume that the following condition is satisfied:

$$\left|\varepsilon_2(\mathbf{R})\frac{|\mathscr{E}|^2}{\varepsilon_0^{1/2}}\right|\ll\varepsilon_1(\mathbf{R}); \tag{3.2}$$

b) a wave beam in a "strongly nonlinear" medium, when

$$\varepsilon_2(\mathbf{R})\frac{|\mathscr{E}|^2}{\varepsilon_0^{1/2}}\gtrsim|\varepsilon_1(\mathbf{R})|. \tag{3.3}$$

In case a) we find from Eq. (2.12) that the radius of curvature of the reference ray is close to the value predicted by the linear theory. Therefore, the displacement of a wave beam because of nonlinearity in a path of length of the same order as the characteristic inhomogeneity scale of the medium is small compared with the path length, but—in principle—it can be considerably greater than the transverse dimensions of the beam itself. Then, a "nonlinear" displacement of the reference ray path can be found by the method of successive approximations. We shall represent $\mathbf{R}$ in the form

$$\mathbf{R}=\mathbf{R}_1(t)+Q\mathbf{n}_1(t), \tag{3.4}$$

where $\mathbf{R}_1(t)$ is a linear geometric-optical beam ($d^2\mathbf{R}_1/dt^2 = \frac{1}{2}\nabla\varepsilon_1, dt = ds/\varepsilon_1^{1/2}$); $\mathbf{n}_1(t)$ is a unit vector along the normal to the curve $\mathbf{R}_1$; $Q(t)$ is a nonlinear displacement of the reference ray. We shall substitute Eq. (3.4) into Eq. (2.14). Then, in the first order of perturbation theory with respect to a small parameter $|Q|/\min\{\rho,L\}$, we find that $Q(t)$ obeys the equation

$$\frac{d^2Q}{dt^2}+\left(\frac{3}{4\varepsilon_1}\left(\frac{\partial\varepsilon_1}{\partial\xi}\right)^2-\frac{1}{2}\frac{\partial^2\varepsilon_1}{\partial\xi^2}\right)_{\mathbf{r}=\mathbf{R}_1}Q=\left(\frac{\dfrac{\partial\varepsilon_2}{\partial\xi}-\dfrac{\partial\varepsilon_1}{\partial\xi}\dfrac{\varepsilon_2}{\varepsilon_1}}{2\sqrt{\varepsilon_1}}\right)_{\mathbf{r}=\mathbf{R}_1}\int_{-\infty}^{\infty}|\mathscr{E}|^4d\xi\Big/\int_{-\infty}^{\infty}|\mathscr{E}|^2d\xi \tag{3.5}$$

and the intial conditions

$$Q(t=0)=0,\quad \frac{dQ}{dt}(t=0)=0.$$

We can estimate the displacement of a wave beam relative to the curve $\mathbf{R}_1(t)$ by replacing in Eq. (3.5)

$$\int_{-\infty}^{\infty}|\mathscr{E}|^4d\xi\Big/\int_{-\infty}^{\infty}|\mathscr{E}|^2d\xi\quad\text{by}\quad\mathscr{P}/\Lambda_\perp\sim|\mathscr{E}_{max}|^2$$

($\mathscr{P}$ is the total energy flux in the beam and $\mathscr{E}_{\max}$ is the maximum value of the field in the beam), and we can find the beam width $\Lambda_\perp$ using, for example, the aberration-free approximation.[5]

In case b), when the nonlinearity is sufficiently strong, it is meaningful to consider the reference ray path for a wave beam in which the field structure is close to the field of a stationary soliton. This is due to the fact that even over short paths when the inhomogeneity of the medium can be ignored a fairly wide wave beam rapidly splits into separate (soliton) channels.[4]

We shall therefore seek a solution in the form

$$\mathscr{E}=v_0+\bar{\mu}v_1+\ldots, \tag{3.6}$$

where

$$\bar{\mu}=\max\left\{\frac{\Lambda_\perp}{\rho},\ \left|\frac{d\Lambda_\perp}{ds}\right|,\left|\frac{d}{ds}\frac{1}{k_0\varepsilon_0^{1/2}}\right|\right\}\ll1 \tag{3.7}$$

($\bar{\mu}$ is a small parameter of the problem). In the zeroth approximation with respect to the parameter $\bar{\mu}$, we find that v_0 obeys an equation

$$\frac{d^2v_0}{d\xi^2}+k_0^2\left(\varepsilon_1+\varepsilon_2\frac{|v_0|^2}{\varepsilon_0^{1/2}}-\varepsilon_0\right)v_0=0 \tag{3.8}$$

whose localized (soliton) solution corresponding to real ε_1 and positive ε_2 ($\varepsilon_2>0$) is

$$v_0=\frac{A_0}{\mathrm{ch}(\xi/\Lambda_\perp)},\quad \Lambda_\perp=\left(\frac{1}{2}q\frac{k_0\varepsilon_2}{\varepsilon_0^{1/2}}\right)^{-1},$$
$$A_0=q\left(\frac{\varepsilon_2}{2\varepsilon_0^{1/2}}\right)^{1/2},\quad \varepsilon_0=\frac{1}{2}[\varepsilon_1+(\varepsilon_1^2+q^2\varepsilon_2^2)^{1/2}],$$
$$q=\frac{1}{2}k_0\int_{-\infty}^{\infty}v_0^2d\xi, \tag{3.9}$$

where $\Lambda_\perp$ is the channel width: A_0 is the maximum field amplitude in a channel; ε_0 is the square of the propagation constant (ε_0 is always greater than zero); q is a conserved quantity, proportional to the total energy flux in a channel.

We shall not analyze the equation for v_1 obtained in the next order with respect to $\bar{\mu}$, but simply note that the condition of absence in v_1 of secular terms is the selection of the reference ray path $\mathbf{R}(s)$ satisfying Eq. (2.14) (Ref. 6).

We shall now substitute Eq. (3.9) into Eq. (2.15). We then find that

$$U_{\text{eff}}=\varepsilon_1(\mathbf{r})+\frac{q^2}{6}\varepsilon_2(\mathbf{r})F(\mathbf{R}),$$
$$F(\mathbf{R})=\frac{2\varepsilon_2(\mathbf{R})}{\varepsilon_1(\mathbf{R})+[\varepsilon_1^2(\mathbf{R})+q^2\varepsilon_2^2(\mathbf{R})]^{1/2}}. \tag{3.10}$$

We thus find that Eq. (2.14) can be written in the form

$$\frac{d\mathbf{R}}{d\tau_H}=\mathbf{p},\quad \frac{d\mathbf{p}}{d\tau_H}=\pm\frac{1}{2}\left\{\nabla_{\mathbf{R}}\varepsilon_1(\mathbf{R})+\frac{q^2}{6}F(\mathbf{R})\nabla_{\mathbf{R}}\varepsilon_2(\mathbf{R})+\frac{q^2\varepsilon_2(\mathbf{R})}{6|U_{\text{eff}}|}\mathbf{p}(\mathbf{p}\nabla_{\mathbf{R}}F(\mathbf{R}))\right\},\quad d\tau_H=|d\tau|=ds/|U_{\text{eff}}|^{1/2}. \tag{3.11}$$

(the plus sign corresponds to $U_{\text{eff}}>0$ and the minus sign to $U_{\text{eff}}<0$).[4)]

In a plane-layer medium, when ε_1 and ε_2 depend only on one coordinate x [$\varepsilon_1(x)$, $\varepsilon_2(x)$], the system (3.11) simplifies. In addition to the integral of motion

$$p_x^2+p_z^2=|U_{\text{eff}}| \tag{3.12}$$

(p_x and p_z are the projections of the vector $\mathbf{p}$ on the x and z axes, respectively), we can obtain an explicit expression for p_z as a function of the coordinate x. In fact,

$$\frac{dp_z}{d\tau_H}=\pm p_z\frac{q^2\varepsilon_2(x)}{12|U_{\text{eff}}(x)|}\frac{dx}{d\tau_H}\frac{dF}{dx}.$$

Integrating the above relationship with respect to τ_H and bearing in mind that

$$F(x)=\varepsilon_2(x)/\varepsilon_0(x),\ \varepsilon_0(x)={}^1/_2[\varepsilon_1+(\varepsilon_1^2+q^2\varepsilon_2^2)^{1/2}],$$
$$U_{\text{eff}}(x)={}^1/_3[2\varepsilon_1+(\varepsilon_1^2+q^2\varepsilon_2^2)^{1/2}],$$

we obtain

$$p_z(x)=p_z(x_0)\left(\frac{\varepsilon_0(x)}{U_{\text{eff}}(x)}\bigg/\frac{\varepsilon_0(x_0)}{U_{\text{eff}}(x_0)}\right)^{1/2}, \tag{3.13}$$

where x_0 is the coordinate of a point at which the initial (corresponding to $\tau_H=0$) value p_z is specified:

$$p_z(x_0)=|U_{\text{eff}}(x_0)|^{1/2}\sin\alpha$$

(α is the angle between the initial direction of propagation of the wave beam and the x axis). We shall substitute Eq. (3.13) into the integral motion (3.12). Substituting in this expression $p_x=0$, we obtain an equation which defines the position of a turning point x_t ("refletion point"):

$$\pm\left(\frac{U_{\text{eff}}(x_t)}{U_{\text{eff}}(x_0)}\right)^2=\sin^2\alpha\frac{\varepsilon_0(x_t)}{\varepsilon_0(x_0)} \tag{3.14}$$

[the plus sign corresponds to $U_{\text{eff}}(x_t)/U_{\text{eff}}(x_0)>0$; and the minus sign to $U_{\text{eff}}(x_t)/U_{\text{eff}}(x_0)<0$].

The right-hand side of Eq. (3.14) contains a positive determinate quantity. Therefore, there is no solution of Eq. (3.14) with the minus sign, i.e., there are no waveguide channels which will penetrate from the region $U_{\text{eff}}>0$ to the region $U_{\text{eff}}<0$ or vice versa.[5)]

4. It follows that the propagation of intense wave beams satisfying the conditions (2.1) and (2.2) in smoothly inhomogeneous nonlinear media can be analyzed conveniently using a system of equations which includes Eq. (2.14) for the reference ray path (center of gravity of the energy flux in the beam) and Eq. (2.8) for the field in a coordinate system linked to the reference ray path. It should be stressed that the nonlinearity of the medium may have a considerable influence not only on the structure of the field in the beam, but also on its propagation path. For example, it follows from Eq. (3.5) that even a weak nonlinearity displaces a beam relative to the path predicted by the linear theory. Even the simplest estimate of the displacement over paths of length l, which gives

$$Q\sim\frac{\varepsilon_2|\mathscr{E}_{max}|^2}{L\varepsilon_1^{3/2}}l^2,$$

shows that if

$$\frac{|\mathscr{E}_{max}|^2}{\varepsilon_1^{1/2}}>\frac{\varepsilon_1\Lambda_\perp}{\varepsilon_2 l^2}L,$$

then Q is greater than the transverse dimensions of the beam $\Lambda_\perp$.

In the strongly nonlinear case when the propagation of the field in a beam is soliton-like it is possible to simplify most of the problem. Then, as shown in Sec. 3, the propagation path can be found by solving a closed system of ordinary differential equations. In this case the conditions of validity of Eqs. (2.1) and (2.2) are supplemented also by the requirement (3.7) of a sufficiently slow change in the width and length of a soliton wave. The restriction on the smoothness of the wavelength $\lambda=2\pi/k_0\varepsilon_0^{1/2}$ coincides with the condition of validity of linear geometrical optics; the only change is that the linear permittivity $\varepsilon_1(\mathbf{R})$ in Eq. (3.7) is replaced with $\varepsilon_0(\mathbf{R})$ representing the phase advance in the reference ray. It should be noted that an increase in the soliton power (parameter q) widens the region of space where the inequalities (3.7) are obeyed. Bearing in mind that in the case of a strong nonlinearity in the homogeneous medium any wave beam with a quasiplanar phase front splits into separate solitons[4] and, therefore, we can expect that in the case when the distance in which solitons are formed is short compared with the characteristic inhomogeneity scale of a medium the reference ray can also be found by solving a system of ordinary differential equations.

We have considered media with a local nonlinearity. However, this is not of fundamental importance. The above analysis can generally be applied without changes to media for which the permittivity is specifieid as some functional of the field (for example, in the case of a thermal nonlinearity).

We shall conclude by noting that in generalization of the results obtained to vector problems we may encounter new effects associated with the behavior of the field polarization. An analysis of these effects is outside the scope of the present paper.

[1)]It should be noted that Eqs. (1.5)–(1.13) are valid also in the case when S represents a region of the phase front surface ($\varphi=$ const) bounded by the same lines of the vector field $\nabla\varphi$. Only then one can speak of a path of the center of gravity of the energy flux penetrating a given area S and not of the total energy flux. Going over in Eq. (1.13) to a differentially small area $S=dS$, we obtain an equation for the line of force of the vector field $\nabla\varphi$ (i.e., an equation for a ray): $d^2\mathbf{R}/d\tau^2=\frac{1}{2}\nabla\bar{\varepsilon}$, which is a consequence of the transfer of energy along rays ($d\tau=ds/\bar{\varepsilon}^{1/2}$).

[2)]Here, $((\partial/\partial\mathbf{r})\mathscr{F}(\mathbf{r},A^2(\mathbf{r})))_{A^2=\text{const}}$ means that in finding the derivative we regard the quantity $A^2(\mathbf{r})$ as a parameter independent of $\mathbf{r}$.

[3)]Here, $U(\mathbf{r},A^2)$ is the potential energy in the expression for the density of the Lagrange function ($\mathscr{L}$) of the initial equation (1.1):

$$\mathscr{L}=\nabla E\nabla E^*-k_0^2U(\mathbf{r},EE^*). \tag{A}$$

[4)]We shall use the fact that

$$(\nabla F(\mathbf{R}(s)))_{\mathbf{r}=\mathbf{R}}=\frac{\partial F}{\partial s}\mathbf{s}_0=\frac{\mathbf{p}}{|U_{\text{eff}}|}(\mathbf{p}\nabla_{\mathbf{R}}F(\mathbf{R})),\qquad \nabla_{\mathbf{R}}=\frac{\partial}{\partial \mathbf{R}}.$$

[5)]This analysis ceases to be valid for small angles of incidence ("normal" incidence).

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