Stability of current-carrying states in narrow finite-length superconducting channels

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The stability of the normal state of a narrow current-carrying superconducting channel with ends connected to normal conductors is considered. It is shown that at currents lower than some critical value (which is in turn smaller than the Ginzburg-Landau critical current) the normal state of the channel is absolutely unstable near the *SN* boundary with respect to formation and growth of the superconducting nucleus. The latter expands in such a way that its boundary moves into the superconductor, and after a sufficiently long time the entire channel becomes superconducting.

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1. INTRODUCTION

We consider a narrow superconducting channel with transverse dimensions smaller than the depth of the penetration of the magnetic field and the coherence length $\xi(T)$. The channel length is assumed to be quite large. The Ginzburg-Landau theory predicts that below the superconducting-transition temperature such a channel can be either in the superconducting or in the normal state, depending on the current flowing through the channel. At low currents the channel is superconducting, and when the current density exceeds the so-called Ginzburg-Landau critical current j_{GL} the homogeneous superconducting state vanishes and the channel should go over into the normal state. Such a picture is known to be a simplification of the real situation in at least two respects. First, as shown by experiment,[1–3] the superconducting state does not vanish completely above the Ginzburg-Landau critical current, but goes over into the so-called resistive state,[1,4–7] in which the superconductivity in the sample as a whole is preserved, with the exception of small regions, called phase-slip centers, where the macroscopic phase coherence is lost and the order parameter oscillates with time. In such a resistive state the sample is under a finite potential difference. For more details on the properties of the resistive state see the reviews by Skocpol[8] and by Galaiko and Kopnin.[9]

Second, the Ginzburg-Landau theory does not consider the mechanism of the transition from the normal to the superconducting state when the current drops below j_{GL}. Gor'kov[10] and Kulik[11] have shown that the normal state of such a channel, of infinite length, is stable to infinitely small perturbations at an arbitrarily weak field (see also Ref. 6). The explanation is that the Cooper pair produced by fluctuation against the background of the normal state is accelerated by the electric field present in the sample unitl it acquires a sufficient velocity and disintegrates. This conclusion, however, cannot be extended to finite fluctuations. Since the electric field penetrates a finite depth l_E into the superconducting region, a critical nucleus of the superconducting phase can be produced in the normal phase. A feature of such a critical nucleus is that the electric field in it is quite suppressed and it cannot counteract the Cooper instability of the normal state. Nuclei larger than critical increase in the course of time and fill the entire sample, which thus goes over from the normal to the superconducting state. This transition process was investigated by Watts-Tobin *et al.*,[12] who used a numerical solution of the dynamic equations of superconductivity.

The transition from the normal to the superconducting state in a narrow channel of infinite length, in the presence of current, is thus in essence a first-order transition. The critical-nucleus size depends on the current and increases with the latter. At a current above a certain value j_2 the existence of the critical nucleus becomes impossible. Estimates yield for the current j_2 a value[12,5,7]

$$j_2 \sim j_{\mathrm{GL}}\, l_E^2/\xi^2. \tag{1}$$

The condition (1) determines the limit above which the superconducting region arising against the background of the normal state can no longer grow. Since the electric-field penetration depth l_E is usually much larger than the coherence length ξ, the current j_2 exceeds j_{GL} noticeably. The question of the onset and growth of an above-critical nucleus has not yet been sufficiently well studied. Nor is it quite clear whether the entire region between the currents j_{GL} and j_2 is occupied by the resistive state discussed above.

The transition accompanying the formation of the critical nucleus is connected with surmounting an energy barrier that is exceptionally high because of the macroscopic dimensions of the sample (its width and thickness exceed the atomic dimensions by hundreds or thousands of times). The probability of formation of a critical nucleus should therefore, generally speaking, be very small.

We have dealt so far with an infinitely long superconducting channel. In analogy with first-order transitions, it is natural to look into the role played by the ends of the superconducting channel on going from the normal to the superconducting state in the presence of current. Inasmuch as in a real experimental situation any superconducting sample is connected to measuring instruments through normal contacts, we consider a narrow superconducting channel whose

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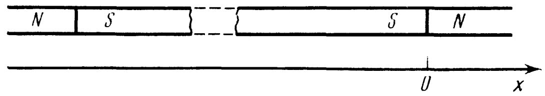

FIG. 1. Illustrating the formulation of the problem. The ends of the narrow superconducting channel (S) are connected to normal conductors (N).

ends are connected to normal conductors, so that SN boundaries are produced at the connection points (Fig. 1). We investigate in the present paper the influence of the SN boundary on the stability of the normal state of the current-carrying channel. We show that near the SN boundary, at a current lower than a certain value j_1 (which is lower than the critical Ginzburg-Landau current j_{GL}), there exists against the background of the normal state an instability that leads to the onset of an infinitely small superconducting nucleus, which then increases with time and broadens in such a way that its boundary penetrates into the channel so that after the lapse of a sufficiently long time the entire channel is in the superconducting state. This process is not connected with formation of a critical nucleus and proceeds without activation. In other words, at a current $j<j_1$ the normal state is absolutely unstable near the SN boundary. If, however, the current flowing in the channel exceeds j_1, the formation of an expanding superconducting region calls for a nucleus that exceeds a critical size that increase with increasing deviation from j_1. In this case the situation becomes similar to that discussed above for an infinitely long sample.

Thus, the boundary promotes the transition from the normal to the superconducting state in the presence of current. A possible explanation is that near the SN boundary practically the entire current in the superconductor is transported by normal excitations, so that the current of the superconducting electrons (i.e., the velocity of the Cooper pairs) is small near the boundary. The influence of the electric field manifests itself then only at a certain distance from the boundary. As a result, the conditions near the SN boundary are favorable for the onset of superconductivity in the presence of a current.

2. BASIC EQUATIONS

The problem of the behavior of a superconductor in the presence of an electric field is essentially nonstationary. Unfortunately, in the general case the system of nonstationary equations for superconductors is extremely complicated and does not reduce to local differential equations for the superconducting parameters Δ, $\mathbf{Q}$, and Φ. Here Δ is the modulus of the order parameter,

$$\mathbf{Q}=\mathbf{A}-\frac{c}{2e}\nabla\chi,\quad \Phi=\varphi+\frac{1}{2e}\frac{\partial\chi}{\partial t}$$

are the gauge-invariant vector and scalar potentials ($\mathbf{A}$ and φ are the usual electromagnetic potentials), and χ is the phase of the order parameter. We consider therefore below temperatures close to critical, and assume in addition that the characteristic spatial and temporal scales of all the quantities are quite large: ω, $Dk^2\ll\tau_{ph}^{-1}$, where D is the diffusion coefficient and τ_{ph} is the time of inelastic electron-phonon relaxation. These conditions are always satisfied in a sufficiently narrow vicinity near T_c:

$$1-T/T_c\ll(\tau_{ph}T_c)^{-1}.$$

The dynamic equations were obtained under these assumptions by a number of workers.[12–15] They take the form

$$-\frac{\pi}{8T_c}(4\tau_{ph}^2\Delta^2+1)^{1/2}\frac{\partial\Delta}{\partial t}$$

$$+\frac{\pi D}{8T_c}\nabla^2\Delta+\frac{T_c-T}{T_c}\Delta-\frac{7\zeta(3)\Delta^3}{8\pi^2T^2}-\frac{\pi De^2}{2Tc^2}\mathbf{Q}^2\Delta=0, \tag{2}$$

$$\frac{\Delta^2\Phi}{(4\tau_{ph}^2\Delta^2+1)^{1/2}}+\frac{D}{c}\operatorname{div}(\Delta^2\mathbf{Q})=0, \tag{3}$$

$$\mathbf{j}=\sigma\mathbf{E}-\frac{\pi\sigma\Delta^2\mathbf{Q}}{2cT};\quad \mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{Q}}{\partial t}-\nabla\Phi. \tag{4}$$

Here

$$\tau_{ph}^{-1}=7\zeta(3)\pi\nu(0)g^2T^3/(sp_F)^2.$$

On the basis of the electroneutrality condition

$$\operatorname{div}\mathbf{j}=0, \tag{5}$$

we can obtain from (3) and (4)

$$D\nabla^2\Phi+\frac{D}{c}\operatorname{div}\frac{\partial\mathbf{Q}}{\partial t}=\frac{\pi}{2T}\frac{\Delta^2\Phi}{(4\tau_{ph}^2\Delta^2+1)^{1/2}}. \tag{6}$$

Equation (6) determines the penetration depth l_E of the constant ($\mathbf{E}=-\nabla\Phi$) electric field into the superconductor.

We shall find it convenient to transform to dimensionless variables. To this end, we choose the respective length and time scales

$$\xi(T)=[\pi D/8(T_c-T)]^{1/2} \text{ and } \tau_{GL}=2T/\pi\Delta_{GL}^2,$$

we measure the order parameter in units normalized to its equilibrium value:

$$\Delta_{GL}^2=8\pi^2T(T_c-T)/7\zeta(3),$$

and the current in units of $\pi\sigma\Delta_{GL}^2/4eT\xi$. In these units, the critical Ginzburg-Landau current is

$$j_{GL}=2/3\sqrt{3}\approx0.385.$$

Equations (2)–(6) take the form

$$-u\left(\frac{\Delta^2}{\Gamma^2}+1\right)^{1/2}\frac{\partial\Delta}{\partial t}+\frac{\partial^2\Delta}{\partial x^2}+(1-\Delta^2-Q^2)\Delta=0, \tag{7}$$

$$u\Delta^2\left(\frac{\Delta^2}{\Gamma^2}+1\right)^{-1/2}\Phi+\frac{\partial(\Delta^2Q)}{\partial x}=0, \tag{8}$$

$$j=-\frac{\partial Q}{\partial t}-\frac{\partial\Phi}{\partial x}-\Delta^2Q, \tag{9}$$

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2Q}{\partial x\,\partial t}=u\Delta^2\left(\frac{\Delta^2}{\Gamma^2}+1\right)^{-1/2}\Phi. \tag{10}$$

Here Γ is the depairing factor

$$\Gamma=\frac{1}{2\tau_{ph}\Delta_{GL}}=\frac{\pi}{8u^{1/2}(\tau_{ph}T_c)(1-T/T_c)^{1/2}}, \tag{11}$$

and the numerical parameter $u=\pi^4/14\zeta(3)=5.79$. The gauge-invariant potentials take in these units the form

$$\Phi=\varphi+\partial\chi/\partial t,\quad Q=A-\partial\chi/\partial x.$$

We neglect below the magnetic field and assume that

$Q = -\partial\chi/\partial x$.

In the gapless case $\Delta \ll \Gamma$ Eqs. (7)–(10) go over into the time-dependent Ginzburg-Landau equations. The depairing factor Γ depends on the temperature. In a very narrow region near $T_c 1 - T/T_c \ll (\tau_{\text{ph}} T_c)^{-2}$, the factor Γ is large, $\Gamma \gg 1$, and the gapless situation obtains always. At the usual experimental temperatures, however, the factor Γ is as a rule considerably less than unity, since the product $\tau_{\text{ph}}\ T_c$ for real materials is quite large. Thus, the product $\tau_{\text{ph}}\ T_c$ for real materials is quite large. Thus, the product $\tau_{\text{ph}}\ T_c$ is of the order of $(2-4)\times 10^1$ for lead and indium and 10^2 for tin. Therefore in situations when Δ is of the order of its equilibrium value ($\Delta = 1$ in our units), the inequality $\Delta \gg \Gamma$ usually holds, corresponding to the presence of a gap in the energy spectrum. In this case we find from (19), in particular, that the depth of penetration of the electric field, $l_E \sim (u\Gamma)^{-1/2}$, is considerably larger than $\xi(T)$ ($\xi = 1$ in our units).

Equations (7)–(9) can also be written in complex form, by introducing the complex order parameter $\psi = \Delta\ \exp i\chi$:

$$-\frac{u}{(|\psi|^2/\Gamma^2+1)^{1/2}}\left[\frac{\partial\psi}{\partial t}+i\varphi\psi+\frac{1}{2\Gamma^2}\psi\frac{\partial|\psi|^2}{\partial t}\right]$$

$$+\frac{\partial^2\psi}{\partial x^2}+\psi-|\psi|^2\psi=0, \qquad (12)$$

$$j=-\frac{\partial\varphi}{\partial x}+\frac{1}{2i}\left(\psi^*\frac{\partial\psi}{\partial x}-\psi\frac{\partial\psi^*}{\partial x}\right). \qquad (13)$$

3. BEHAVIOR OF AN INFINITELY SMALL SUPERCONDUCTING NUCLEUS NEAR THE *SN* BOUNDARY

We shall consider below a semi-infinite narrow superconducting channel occupying the region $x<0$ (see Fig. 1). On the boundary with the normal metal ($x=0$) we assume $\Delta = 0$, which holds true if the contact is a "good" normal metal, i.e., if the order parameter in it decreases over distances smaller than $\xi(T)$ in the superconductor.

We seek the solution for an infinitely small superconducting nucleus, and linearize for this purpose Eqs. (12) and (13) at $|\psi| \ll \Gamma$ and $|\psi| \ll 1$. In this case $\varphi = -jx$ and

$$-u\left(\frac{\partial\psi}{\partial t}-ijx\psi\right)+\psi+\frac{\partial^2\psi}{\partial x^2}=0. \qquad (14)$$

We put

$$\psi=e^{-i\omega t}f(x). \qquad (15)$$

We obtain for $f(x)$ the equation

$$[1+iu(\omega+jx)]f+\partial^2 f/\partial x^2=0, \qquad (16)$$

the solution of which can be expressed in terms of Bessel functions of order 1/3:

$$f(x)=[1+iu(\omega+jx)]^{1/2}Z_{1/3}(z),$$

where $Z_{1/3}$ is one of the solutions of the Bessel equation, and

$$z=(2i/3uj)[1+iu(\omega+jx)]^{3/2}.$$

The region of variation of z in the complex plane as x changes from $-\infty$ to $+\infty$ is shown in Fig. 2. The solution that decreases as $x\to-\infty$ is obtained by choosing $Z_{1/3} = H^{(2)}_{1/3}$,

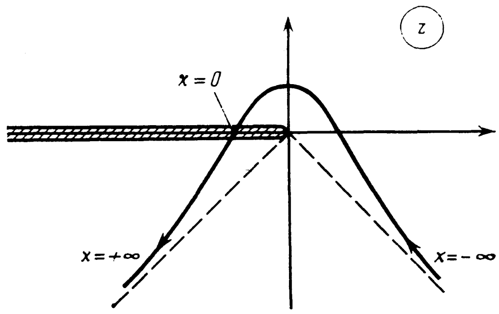

FIG. 2. Range of variation of the argument of the Hankel function $H^{(2)}_{1/3}(z)$ in the complex plane as x changes from $-\infty$ to $+\infty$. The cut corresponds to the definition of the principal branch of the function $H^{(2)}_{1/3}(z)$.

where $H^{(2)}_{1/3}$ is a Hankel function of the second kind. Thus,

$$f(x)=[1+iu(\omega+jx)]^{1/2}H^{(2)}_{1/3}(z). \qquad (17)$$

Figure 2 shows a cut along the negative real axis z, corresponding to the determination of the principal branch of the function $H^{(2)}$. The argument z of the Hankel function goes out of the definition region of the principal branch of the function $H^{(2)}$ as x moves from $-\infty$. Analytic continuation under the cut yields

$$H^{(2)}_{1/3}(z)=H^{(2)}_{1/3}(z')+e^{i\pi/3}H^{(1)}_{1/3}(z')$$
$$=2e^{i\pi/6}[J_{1/3}(z')\cos(\pi/6)-Y_{1/3}(z')\sin(\pi/6)], \qquad (18)$$

where $z = e^{i\pi}z'$, while $J_{1/3}$ and $Y_{1/3}$ are respectively a Bessel and a Neumann function. As seen from (18), at $x\to+\infty$ the function $H^{(2)}_{1/3}(z)$ would increase exponentially, corresponding to the absence of an infinitely small stationary solution of Eqs. (12) and (13) in a superconducting channel of infinite length. In our case, however, the superconducting channel is confined to the region $x<0$ and we must stipulate $H^{(2)}_{1/3}(z) = 0$ at $x=0$. The roots of the cylindrical function

$$J_{1/3}(z')\cos(\pi/6)-Y_{1/3}(z')\sin(\pi/6)$$

are located on the positive real z' axis (see Ref. 16). We need the smallest positive root, which corresponds to[17] $z'+s_1 = 2.383$. Equating $z = s_1 \exp i\pi$ at $x=0$, we obtain the condition for the frequency ω:

$$-iu\omega=\left[1-\left(\frac{3}{2}s_1uj\right)^{2/3}\cos\frac{\pi}{3}\right]-i\left(\frac{3}{2}s_1uj\right)^{2/3}\sin\frac{\pi}{3}$$
$$=\left[1-\left(\frac{j}{j_1}\right)^{2/3}\right]-i\sqrt{3}\left(\frac{j}{j_1}\right)^{2/3}, \qquad (19)$$

where the critical current is determined from

$$uj_1=4\sqrt{2}/3s_1\approx 0.791. \qquad (20)$$

With allowance for the numerical value of the parameter $u = 5.79$ we have

$$j_1=0.137=0.356j_{\text{GL}}.$$

It can be seen from (19) that at a current $j<j_1$ we have $\text{Re}(-i\omega)>0$, and an infinitely small solution will increase with time. At $j>j_1$ the infinitely small solution attenuates. To ascertain the subsequent fate of the infinitely small nucleus at $j<j_1$ we consider Eqs. (12) and (13) in the region

$|j_1-j|\ll j_1$ and take into account in them the principal nonlinear terms. We have

$$\varphi=-jx-\varphi_s,\quad \varphi_s=\int_x^0 j_s\,dx,$$

where

$$j_s=\frac{1}{2i}\left(\psi^*\frac{\partial\psi}{\partial x}-\psi\frac{\partial\psi^*}{\partial x}\right).$$

Expressing as before ψ in the form (15), we have at $|\mathrm{Re}(-i\omega)|\ll 1$

$$[1+iu(\omega+jx)]f+\frac{\partial^2 f}{\partial x^2}+iu\varphi_s f-\frac{iu}{2\Gamma^2}(\omega+jx)|f|^2 f-|f|^2 f=0. \tag{21}$$

We put

$$f=Cf_0(x)+f_1(x),$$

where C is a positive constant, $C\ll 1$, and $f_0(x)$ is given by (17). The small correction $f_1\sim C^3$ also satisfies the conditions $f_1(x)=0$ at $x=0$ and $f_1(x)\to 0$ as $x\to-\infty$. With the aid of the orthogonality condition, the solutions of the homogeneous linear Eq. (18) relative to the nonlinear part of (21) we obtain for the frequency

$$-iu\omega=\left[\frac{2}{3}\,\frac{j_1-j}{j_1}-\alpha'\right]-i\left(\sqrt{3}+\frac{2}{\sqrt{3}}\,\frac{j-j_1}{j_1}+\alpha''\right), \tag{22}$$

where α' and α'' are defined as

$$\alpha'=C^2\frac{uj}{|f(0)|'^2}\left\{\int_{-\infty}^0 |f_0|^4\sin[2(\theta_0-\theta)]dx+\right.$$
$$+u\int_{-\infty}^0 \varphi_s|f_0|^2\cos[2(\theta_0-\theta)]dx$$
$$\left.+\frac{1}{2\Gamma^2}\int_{-\infty}^0\frac{\partial j_s}{\partial x}|f_0|^2\cos[2(\theta_0-\theta)]dx\right\}, \tag{23}$$

$$\alpha''=C^2\frac{uj}{|f(0)|'^2}\left\{\int_{-\infty}^0 |f_0|^4\cos[2(\theta_0-\theta)]dx\right.$$
$$-u\int_{-\infty}^0 \varphi_s|f_0|^2\sin[2(\theta_0-\theta)]dx$$
$$\left.-\frac{1}{2\Gamma^2}\int_{-\infty}^0\frac{\partial j_s}{\partial x}|f_0|^2\sin[2(\theta_0-\theta)]dx\right\}. \tag{24}$$

Here θ is the phase of the function $f_0(x)$, and $\theta_0=\theta(x=0)$. In the derivation of (23) and (24) we used the relations

$$|f_0(0)|'^2\cos(2\theta_0)=-uj\int_{-\infty}^0|f_0|^2\sin(2\theta)\,dx,$$

$$|f_0(0)|'^2\sin(2\theta_0)=uj\int_{-\infty}^0|f_0|^2\cos(2\theta)\,dx,$$

which is obtained from (16) by multiplying it by $\partial f_0/\partial x$ and integrating with respect to dx from $-\infty$ to 0.

Calculation of the quantities α' and α'' should be carried out with the aid of the known function $f_0(x)$ of the form (17). Instead of tabulating the function (17), we used a different method of calculating the integrals in (23) and (24). The function ψ was obtained by direct numerical integration with a computer. By virtue of (15), the moduli of the functions ψ and f_0 are equal at $j=j_1$, $|\psi|=|f_0|$, but their phases differ by an amount $\chi-\theta=-\omega t$ that does not depend on x. This difference, however, disappears in the calculation of the difference $\theta_0-\theta=\chi_0-\chi$, in (23) and (24). The normalization of the function was chosen such that $\max|f_0(x)|=1$. Figure 3 shows the functions $|f_0(x)|$, $j_s(x)$ and $\varphi_s(x)$ obtained by numerical integration of Eq. (14) at $j=j_1$.

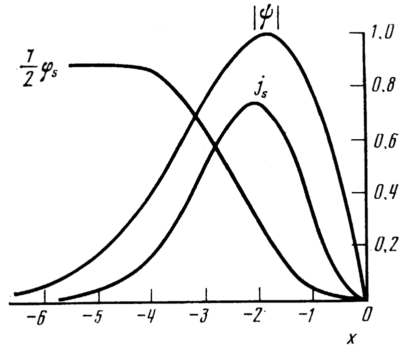

FIG. 3. Results of numerical integration of Eq. (14) at $j=j_1$. The function ψ is normalized such that $\max|\psi|=1$. To obtain the true values of Δ, φ_s, and j_s the result for $|\psi|$ must be multiplied by C, and the results for j_s and φ_s by C^2, where C is defined in (28).

To investigate the stability of the obtained small superconducting nucleus, we must know the value of α'. The integrals that determine it, calculated by the method described above, are

$$\gamma I_1=\int_{-\infty}^0|f_0|^4\sin[2(\theta_0-\theta)]dx=1.04,$$

$$-\gamma I_2=\int_{-\infty}^0\varphi_s|f_0|^2\cos[2(\theta_0-\theta)]dx=-0.525,$$

$$-\gamma I_3=\int_{-\infty}^0\frac{\partial j_s}{\partial x}|f_0|^2\cos[2(\theta_0-\theta)]dx=-0.394,$$

where $\gamma=|f(0)|'^2/uj=1.06$. We can thus write

$$\alpha'=C^2(I_1-I_2u-I_3/2\Gamma^2). \tag{25}$$

The approximate values of I_1, I_2, and I_3 are

$$I_1=0.981,\quad I_2=0.496,\quad I_3=0.372.$$

The stability of a nucleus with small amplitude depends on the sign of the expression in the parentheses in (15). We consider two cases.

1. If $\alpha'=aC^2$, where

$$a=(I_1-I_2u-I_3/2\Gamma^2)>0,$$

the growth rate of the solution (15)

$$\mathrm{Re}(-i\omega)=\frac{1}{u}\left(\frac{2}{3}\,\frac{j_1-j}{j_1}-aC^2\right) \tag{26}$$

vanishes when the renormalization constant of the function $f(x)=Cf_0(x)$ satisfies the equation

$$C=\left(\frac{2}{3a}\frac{j_1-j}{j_1}\right)^{1/2}.$$

It can be easily seen from (26) that such a solution is stable at $j<j_1$. At $j>j_1$ the normal state is stable.

The condition $a>0$ cannot be satisfied for Eqs. (2)–(4) or (7)–(9), since the parameter $u=5.79$. Nor is is satisfied for another known case of gapless superconductors, namely alloys with paramagnetic impurities,[18] for which $u=12$. This condition, however, can be satisfied for the time-dependent Ginzburg-Landau equations ($\Gamma=\infty$), in which the coefficient u is assumed to be a free parameter and satisfies the condition $u<I_1/I_2$. Similar equations were considered earlier.[6] We have here a manifestation of one of the most important differences between the dynamics of real superconductors and of systems described by the model time-dependent Ginzburg-Landau equations.

2. In real situations, at the parameter value $u=5.79$ (and in the more general case at $u>I_1/I_2$) we have $\alpha'=-bC^2$, where

$$b=I_2u-I_1+I_3/2\Gamma^2>0.$$

In this case the growth rate

$$\mathrm{Re}(-i\omega)=\frac{1}{u}\left[bC^2+\frac{2}{3}\frac{j_1-j}{j_1}\right] \tag{27}$$

is always positive at $j<j_1$ and an infinitely small solution will increase exponentially with time. If, however, $j>j_1$ the curve described by the formula

$$C(j)=\left(\frac{2}{3b}\frac{j-j_1}{j_1}\right)^{1/2}, \tag{28}$$

determines the stability limit of the normal state with respect to formation, near the *SN* boundary, of a superconducting nucleus with amplitude C (Fig. 4). In particular, at $\Gamma\ll 1$ we obtain for the stability limit

$$C(j)=\frac{2\Gamma}{(3I_3)^{1/2}}\left(\frac{j}{j_1}-1\right)^{1/2}.$$

Thus, in superconductors described by Eqs. (2)–(4) the normal state near the *SN* boundary is absolutely unstable with respect to formation a superconducting nucleus that increase with time at currents $j<j_1$, where

$$j_1=j_{\mathrm{GL}}\frac{2\sqrt{6}}{us_1}\approx\frac{2{,}06}{u}j_{\mathrm{GL}}.$$

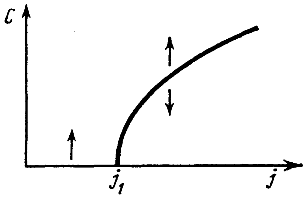

FIG. 4. Schematic dependence of the amplitude of the critical nucleus $C(j)$ on the current. The arrows show the direction of the change of the amplitude C of the nucleus in the different regions. In the region $j<j_1$ and above the $C(j)$ curve the amplitude of the nucleus increases. At $j>j_1$ under the curve, the amplitude of the nucleus decreases, i.e., in this region the normal state is stable to infinitely small perturbations, but unstable to perturbation of finite magnitude.

At currents $j<j_1$, the normal state near the *SN* boundary is stable to infinitely small perturbations but unstable to formation of a nucleus with amplitude exceeding $C(j)$ from (28).

4. BEHAVIOR OF SUPERCONDUCTING NUCLEUS OF FINITE AMPLITUDE

Thus, at currents $j<j_1$ an infinitely small nucleus is produced near the *SN* boundary and grows subsequently with time. The results of the preceding section pertain to small-amplitude nuclei, $C\ll\Gamma$ and $C\ll 1$. To ascertain the subsequent behavior of *a* the nucleus after it has acquired a finite amplitude, we must turn directly to Eqs. (7)–(9) or (12) and (13). Equations (12) and (13) were therefore solved numerically. The length of the superconducting channel was $2L=40$. On the ends of the channel was imposed the condition

$$\mathrm{Re}\,\psi=\mathrm{Im}\,\psi=0,$$

and at the midpoint of the channel, the symmetry condition

$$\partial|\psi|/\partial x=0,\quad \Phi=0.$$

The problem was solved for one half of the channel, $0\leqslant x\leqslant L$; an explicit scheme was used to integrate (12) and (13). We investigated the values of the parameters $u=5.79$; $\Gamma=\infty$, $\Gamma=1/3$, $\Gamma=1/10$. The initial perturbation was chosen to be the function

$$|\psi|=A(L-x)\exp(x-L)$$

with small amplitude A. The results were the following.

At currents $j>j_1$ the small perturbation attenuate with time. There exists, however, a critical amplitude $A_c(j)$ such that an initial perturbation with large amplitude $A>A_c(j)$ increases also in the case $j>j_1$. This behavior agrees with the results obtained in the preceding section. This situation calls for a more detailed investigation.

At currents $j<j_1$ the small perturbations grow, and the first to grow is the nucleus amplitude, until it reaches a value close to unity; the nucleus then begins to broaden, and its boundary moves into the interior of the superconductor. This behavior of the boundary is analogous to the expansion of a superconducting domain, investigated in Refs. 19 and 20 within the framework of the time-dependent Ginzburg-Landau equations. Figure 5 shows the results of the numerical calculations for the parameter values $\Gamma=\infty$, $\Gamma=1/3$, and $\Gamma=1/10$ at a current $j=0.1$. It can be seen from the results of the numerical calculations that at $j<j_1$ the growth of the small perturbation does not stop at finite amplitudes and continues until the nucleus spreads over the entire sample. The velocity of the nucleus boundary and its slope decrease with decreasing Γ at a given current. With decreasing current the velocity increases and the slope decreases.

It can be assumed from the character of the boundary motion, shown in Fig. 5, that at sufficiently long times, when the nucleus size exceeds l_E substantially, the motion of the boundary is described by a self-similar solution of the type $\Delta(x,t)=\Delta(x+vt)$ (the boundary moves to the left). We shall investigate the motion of the boundary at $\Gamma\ll 1$. In this case Eqs. (7)–(9) near the boundary can be written in the form

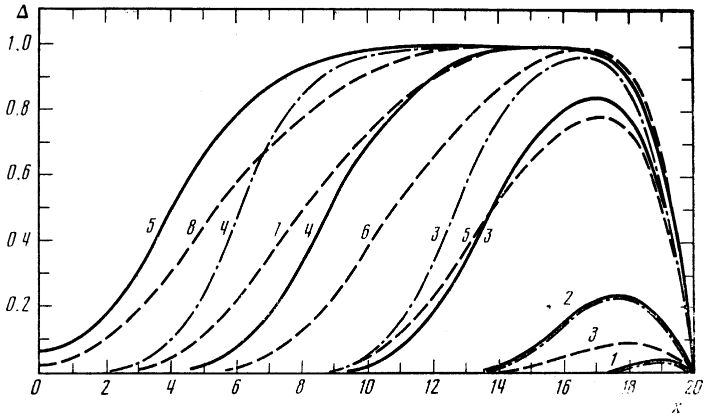

FIG. 5. Numerical-calculation results that describe the growth and spreading of a superconducting nucleus at a current $j = 0.1$ and at the parameter values $\Gamma = \infty$ (dash-dot), $\Gamma = 1/3$ (solid curves) and $\Gamma = 1/10$ (dashed curves). The curves numbered 1, 2, 3, ..., n correspond to successive instants of time $t_1 = 0$, $t_2 = 30$, $t_3 = 60$, etc.: $t_n = 30(n-1)$.

$$-uv\left(\frac{\Delta^2}{\Gamma^2}+1\right)^{1/2}\frac{\partial\Delta}{\partial x}+\frac{\partial^2\Delta}{\partial x^2}+\left(1-\Delta^2-\frac{j_s^2}{\Delta^4}\right)\Delta=0, \quad (29)$$

$$\frac{u\Delta^2\Phi}{(\Delta^2/\Gamma^2+1)^{1/2}}=\frac{\partial j_s}{\partial x}, \quad (30)$$

$$j=-\partial\Phi/\partial x+j_s, \quad (31)$$

where we have put $j_s = -\Delta^2 Q$ and used the assumption, corroborated below, that the velocity $v \ll 1$ at $\Gamma \ll 1$.

We consider first the case of small currents $\Gamma^{1/2} \ll j \ll 1$. In this case $\Delta \approx 1$ in the interior of the nucleus (far from its boundaries). From (30) and (31) we obtain at $\Delta = 1$

$$\Phi(j_s)=(u\Gamma)^{-1/2}(j-j_s).$$

Near the boundary of the nucleus this expression gives only the order of magnitude, namely $\Phi \sim (u\Gamma)^{-1/2} j$.

We consider first the region $x \to -\infty$, $\Delta \ll \Gamma$. Retaining the principal terms in (29), we obtain

$$\partial^2\Delta/\partial x^2 - j_s^2/\Delta^3=0. \quad (32)$$

We see that the characteristic scales of the lengths over which Δ varies in this region are much smaller than unity. The potential Φ hardly varies over such lengths:

$$\Phi=\Phi_0\sim(u\Gamma)^{-1/2}j.$$

From (30) we obtain

$$\partial j_s/\partial x=u\Delta^2\Phi_0. \quad (33)$$

Simultaneous solution of (32) and (33) yields

$$\Delta=ce^{x/\lambda};\quad j_s=\Delta^2/\lambda,$$

where $\lambda^2 = 2/u\Phi_0 \ll 1$.

We consider now the region $\Delta \ll \Gamma$. Equation (29) takes here the form (we recall that $j_s \ll 1$)

$$-\frac{uv}{\Gamma}\Delta\frac{\partial\Delta}{\partial x}+\frac{\partial^2\Delta}{\partial x^2}+\Delta-\Delta^3=0.$$

Its solution is

$$\Delta=\mathrm{th}[\beta(x-x_0)] \quad (34)$$

under the condition

$$-2\beta^2-uv\beta/\Gamma+1=0. \quad (35)$$

We now multiply Eq. (29) by $\partial\Delta/\partial x$ and integrate it with respect to dx from $-\infty$ to x:

$$-uv\int_{-\infty}^{x}\left(\frac{\partial\Delta}{\partial x}\right)^2\left(\frac{\Delta^2}{\Gamma^2}+1\right)^{1/2}dx+\frac{1}{2}\left(\frac{\partial\Delta}{\partial x}\right)^2+\frac{1}{2}\left(\Delta^2-\frac{1}{2}\Delta^4\right)-\int_{-\infty}^{x}\frac{j_s^2}{\Delta^3}\frac{\partial\Delta}{\partial x}dx=0. \quad (36)$$

Letting $x' = x - x_0$ go to $+\infty$ (into the interior of the nucleus), where $\partial\Delta/\partial x = 0$ and $\Delta = 1$, we obtain, taking $j_s \ll 1$ into account,

$$\frac{uv}{\Gamma}\int_0^\infty\left(\frac{\partial\Delta}{\partial x'}\right)^2\Delta\,dx'=\frac{1}{4}. \quad (37)$$

We have used here the fact that in the principal region of integration we have $\Delta \gg \Gamma$ and the solution takes the form (34). Substituting (34) in (37) we obtain

$$uv\beta/\Gamma=1. \quad (38)$$

Comparing this expression with (35) we conclude that β should be much less than unity.

To determine the velocity we must know the slope β of the boundary (36) in the region $\Gamma \ll \Delta \ll 1$. For this purpose we must find an exact solution of Eqs. (29)–(31) in this region. Since this is a complicated problem, we confine ourselves to estimates of β and of the velocity v of the boundary, so that the results that follow are only of the correct order of magnitude. To estimate β in (38), we consider Eq. (36) in the region $\Gamma \ll \Delta \ll 1$, were $\Delta \sim \beta x'$. The first term in this equation is of the order of $uv\beta^3x'^2/\Gamma \sim \beta^2x'^2$, the second of the order β^2, and the third of the order $\beta^2x'^2$. To estimate the last term we use the relation

$$\partial j_s/\partial x\sim u\Gamma\Delta\Phi\sim(u\Gamma)^{1/2}\Delta j,$$

form which we obtain

$$\int\frac{j_s^2\partial\Delta}{\Delta^3\partial x}dx\sim u\Gamma j^2x'^2.$$

It will be seen from the following that β must lie in the interval $\Gamma \ll \beta \ll 1$. In the region $\Gamma/\beta \ll x' \ll 1$ the principal terms in (36) are the second and fourth, while at $1/\beta \gg x' \gg 1$ the principal terms are the first, third, and fourth. At $x' \sim 1$ all the terms in (36) should be of the same order, whence

$$\beta\sim(u\Gamma)^{1/2}j.$$

Substituting this result in (38) we obtain

$$v=\frac{\Gamma}{u\beta}\sim\left(\frac{\Gamma}{u}\right)^{1/2}\frac{1}{uj}.$$

This estimate is valid at $\Gamma^{1/2}<j\lesssim j_{GL}$. The boundary velocity, given in the customary units, is quite high

$$v\sim\frac{\xi\Gamma^{1/2}}{\tau_{GL}}\frac{j_{GL}}{j}\sim v_F\frac{j_{GL}}{j}\left[\tau_{imp}T_c\Gamma\left(1-\frac{T}{T_c}\right)\right]^{1/2},$$

where τ_{imp} is the time of collision with the impurities.

It should be noted that once the superconducting nucleus is produced, it will expand also at currents exceeding j_{GL}, and in this case the channel will go over not into a homogeneous superconducting state, but into the resistive state. The order of magnitude of the upper current limit, above which the nucleus stops expanding, should be determined by condition (1), which yields in our units $j_2\sim\Gamma^{-1}$. An attempt can be made to estimate the boundary velocity at currents $j\lesssim j_2$ in analogy with the procedure used above. The difference is that now we cannot neglect at $\Delta\sim 1$ the term with j_s in (29), so that we do not have the exact solution (34). Equation (36) can be used as before, but (38) is accurate only in order of magnitude. Let us estimate the terms of (36) at $x'\sim 1$, The estimates for the first, second, and third terms do not change, while for the last we have

$$\int\frac{j_s^2}{\Delta^3}\frac{\partial\Delta}{\partial x}dx\sim\Gamma^2\Phi^2x'^2.$$

For the potential Φ at large currents $j\gg 1$ we can obtain $\Phi\sim(j/\Gamma)^{1/2}$ (see, e.g., Ref. 7), so that the last term in (36) is of the order of $\Gamma jx'^2$. Assuming that at $x'\sim 1$ all the terms in (36) are of the same order, we find that the slope of the boundary is $\beta\sim(\Gamma j)^{1/2}$, and for the boundary velocity we have from (38) $v\sim(\Gamma/j)^{1/2}$. At $j\sim j^2$ the velocity is of the order of $v\sim\Gamma$. At $j=j_2$ the velocity v of the boundary should, generally speaking vanish because the supercritical nucleus stops growing. This effect, of course, cannot be traced in such a rough estimate.

5. CONCLUSION

We have thus shown that at currents $j<j_1$ the normal state of a superconducting channel described by Eqs. (2)–(4) is absolutely unstable near the *SN* boundary with respect to formation and growth of a superconducting nucleus that expands in the course of its evolution in such a way that its boundary moves into the interior of the superconductors. For a growing nucleus to be produced at currents $j>j_1$ the amplitude of the nucleus must exceed a certain critical value that increases with increasing deviation from j_1.

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