Contribution to the Dicke superradiance theory. Exact solution of the quasi-one-dimensional quantum model

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The quantum inverse-scattering method problem is used to show that the Dicke quasi-one-dimensional model of superradiance theory is completely integrable. Commutation relations are obtained for the action-angle variables, and the eigenstates and eigenvalues of the commuting integrals of motion of the model are found. It is shown that the eigenstates of the system include, besides "continuous" spectrum states corresponding to usual spontaneous emission, also bound complexes of quasiparticles (quantum solitons) that correspond to the Dicke superradiance.

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§1. INTRODUCTION

Dicke[1] predicted in 1954 collective spontaneous emission from a system of two-level atoms that interact via a transverse electromagnetic field—the Dicke superradiance (SR). Whereas in the case of ordinary spontaneous radiation the individual atoms decay independently of one another with a characteristic decay time T_1, in the case of the Dicke effect the time τ_c required for the system of excited atoms to reach the ground state is inversely proportional to their number N_0, i.e., $\tau_c \propto N_0^{-1}$, and consequently the SR intensity is proportional to the square of the number of emitters, N_0^2.

Interest in the investigation of the Dicke effect increased strongly after the first observation of SR in the optical band,[2] and by now several methods of describing the phenomenon have been published. We shall not present here the results in any detail, since the present status of the experimental and theoretical research into this problem is treated in a rather exhausting review.[3] We note only that in contrast to our present paper, the approaches used up to now are approximate, since all involve some method of decoupling the correlators in the averaging of the Heisenberg equations of motion. At the same time, as shown in earlier brief communications,[4,5] the quasi-one-dimensional quantum model of SR theory, a detailed description of which is given in the second section of this paper, belongs to the class of completely integrable models of quantum field theory (see Refs. 6 and 7). We use here the method of the quantum inverse problem of scattering, developed in Refs. 6–9 for a number of exactly solvable field-theory models.

The quantum inverse-problem method makes it possible to transform from a description of a system in terms of local fields to a description in terms of variables such as action and angle, which are the scattering data for a certain auxiliary quantum spectral problem (see § 3 of the present paper). The transition to the description of the system in terms of variables such as action and angle, and the simple commutation relations for these variables, make it possible in turn to determine the eigenstates and the eigenvalues of the commuting integrals of motion of the investigated model. It was found that the eigenstates of the system include, besides the single-particle states of the "continuous" spec-

trum that correspond to the ordinary spontaneous radiation, also bound states of m basic quasiparticles of the system.

These bound m-particle complexes, which in analogy with the classical inverse-problem method[10] can be called quantum solitions, have a spatial dimension r_0 inversely proportional to the number of particles contained in them, $r_0 \propto m^{-1}$. It is therefore just these eigenstates of the model that should be set in correspondence with the Dicke SR pulses.

§2. DESCRIPTION OF MODEL AND BASIC EQUATIONS

A gas of two-level atoms that interact with a transverse electromagnetic field is described in the dipole approximation by the Hamiltonian

$$H = \frac{\omega_{12}}{2} \int d^3r \left[n(\mathbf{r}) + \frac{N_0}{V} \right] + \sum_q \omega_q a_q^+ a_q$$
$$- d \int d^3r [E^+(\mathbf{r}) P(\mathbf{r}) + P^+(\mathbf{r}) E(\mathbf{r})], \quad (1)$$

where ω_{12} is the frequency and d the dipole moment of the transition of the two-level atom. The operators $P(\mathbf{r})$ and $n(\mathbf{r})$, which satisfy the commutation relations

$$[P^+(\mathbf{r}), P(\mathbf{r}')] = n(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}'),$$
$$[P(\mathbf{r}), n(\mathbf{r}')] = 2P(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}'), \quad (2)$$
$$[n(\mathbf{r}), P^+(\mathbf{r}')] = 2P^+(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}'),$$

are the operators of the polarization and population per unit volume of the medium and are connected with the population and polarization operators n_i and p_i of a single atom by the relations

$$P(\mathbf{r}) = \sum_{i=1}^{N_0} p_i \delta(\mathbf{r}-\mathbf{r}_i), \quad n(\mathbf{r}) = \sum_{i=1}^{N_0} n_i \delta(\mathbf{r}-\mathbf{r}_i), \quad (3)$$

where N_0 is the total number of atoms in the system. The electric-field intensity atom can be written in the form of an expansion in the photon operators

$$E(\mathbf{r}) = i \sum_q (2\pi\omega_q/V)^{1/2} a_q e^{i\mathbf{q}\mathbf{r}}, \quad (4)$$

where $\omega_q = c|\mathbf{q}|$ is the photon frequency and V is the volume of the system.

Equation (1) is based on a number of assumptions. First, we assume that the transition ω_{12} of a two-level atom is the same for all atoms that make up the gas. It will be shown below (§5) that this simplification is of no principal significance and the inhomogeneous width of the transition line can be taken into account within the formalism used in this paper.

Second, we have left out of the interaction of the atoms with the electric field terms of the type $E^+P^+ + $ H.c., which describe simultaneous creation (annihilation) of a photon and excitation in the atom. This approximation is widely used to describe resonant interaction of radiation with matter, and is frequently called the rotating-wave approximation. Its physical justification is the prominent role of the photons whose frequency is close to the transition frequency ω_{12} of the two-level system. This circumstance allows us to retain in the expansion of the operator $E(\mathbf{r})$ (4) only the contribution of the resonant photons:

$$E(\mathbf{r}, t) \approx i \left(\frac{2\pi\omega_{12}}{V} \right)^{1/2} e^{i(\mathbf{q}_0\mathbf{r} - \omega_{12}t)} \sum_{\mathbf{k}} a_{\mathbf{k}}(t) e^{i\mathbf{k}\mathbf{r}} \equiv e^{i(\mathbf{q}_0\mathbf{r} - \omega_{12}t)} \tilde{E}(\mathbf{r}, t),$$

(5)

where $|\mathbf{q}_0| = \omega_{12}/c$, and $\mathbf{k} = \mathbf{q} - \mathbf{q}_0$.

Expression (5) is the quantum analog of the transition to "slow" variables, which is used in the description of resonant interaction of a classical electromagnetic field with matter. In the quantum case, and important consequence of the changeover to the slow variables $\tilde{E}(\mathbf{r})$ and $\tilde{E}^+(\mathbf{r})$ is that these variables are the canonical variables of the system:

$$[\tilde{E}(\mathbf{r}), \tilde{E}^+(\mathbf{r}')] = 2\pi\omega_{12}\delta(\mathbf{r} - \mathbf{r}').$$

(6)

Separation of the rapidly oscillating part must be carried out also in the polarization operator $P(\mathbf{r}, t)$:

$$P(\mathbf{r}, t) = p(\mathbf{r}, t) e^{i(\mathbf{q}_0\mathbf{r} - \omega_{12}t)}.$$

The commutation relations (2) do not change form in this case.

In the Hamiltonian (1) we take into account interaction with one of the two possible polarizations of the transverse field. Strictly speaking, this is justified either in the case of an oriented gas of anisotropic two-level systems, or in the case of an isotropic transition under the assumption that the initial state of the system is specially prepared. For example, if the excited state of the two-level atom corresponds to a total angular momentum $J = 1$, and the ground state to $J = 0$, it can be assumed that all the atoms have in the initial estate state identical projections of the total angular momentum on the quantization axis. Allowance for interaction with the second possible polarization makes it necessary to go outside the framework of the two-level approximation and is undoubtedly vital.

It is known (see, e.g., Ref. 3) that an important role is played in the Dicke effect by the geometry of the volume filled with the atoms. We shall assume here that the sample length l greatly exceeds its transverse dimension $\sqrt{S}$, i.e., $l \gg \sqrt{S}$, where S is the cross-section area. In addition, we confine ourselves here to the geometry of a sample with large Fresnel numbers $F = S/\lambda_0 l \gg 1$, where λ_0 is the characteristic wavelength of the radiation. All this allows us to take into account the interaction of the atoms with only those photons

whose wave vector is directed along the principal axis of the sample, the x axis, and also expand, as in Eq. (4), the electric-field operator in plane waves. The independence of the operators $\tilde{E}(x)$ and $\tilde{E}^+(x)$ in the Hamiltonian (1) of the transverse coordinates enables us to integrate with respect to these coordinates in (1) and thus make the problem one-dimensional. Integration over the transverse coordinate can be carried out also in the commutation relations (2) and (6) without altering their form.

We shall operate hereafter in the system of units $\hbar = c = 1/N_0 = 1$. In dimensionless units, with account taken of the remarks made, the Hamiltonian of the quasi-one-dimensional quantum model of the Dicke superradiance takes then the form

$$H = -i \int_{-\infty}^{\infty} dx\, \varepsilon^+(x)\, \partial_x \varepsilon(x) - \sqrt{\varkappa} \int_{-\infty}^{\infty} dx\, [\varepsilon^+(x)\, p(x) + p^+(x)\, \varepsilon(x)],$$

(7)

where the interaction constant $\varkappa = 2\pi\omega_{12}d^2/S$, and the dimensionless field operators $\varepsilon(x), p(x)$, and $n(x)$ satisfy the commutation relations

$$[\varepsilon(x), \varepsilon^+(y)] = \delta(x-y), \quad [p(x), n(y)] = 2p(x)\delta(x-y).$$

$$[p^+(x), p(y)] = n(x)\delta(x-y),$$

$$[n(x), p^+(y)] = 2p^+(x)\delta(x-y).$$

(8)

The first term in (7), corresponding to the photon kinetic energy leads to the spectral dependence $\omega = k$ in lieu of the initial spectral dependence $\omega = |k|$. Such an approximation was already used by Wiegmann in an exact solution of the Kondo problem.[11] In our case, just as in Ref. 11, this circumstance is of no importance, since the applicability of the model is restricted to a parallelogram with sides $\Delta\omega \ll \omega_{12}$ and $\Delta k \ll q_0 = \omega_{12}/c$, and with a center at the point of intersection of the spectra of the free fields $\omega = ck$ and $\omega = \omega_{12}$.

We have left out from the Hamiltonian (7) a term

$$\omega_{12} \int_{-\infty}^{\infty} dx\, \{\varepsilon^+(x)\, \varepsilon(x) + \tfrac{1}{2}[n(x) + 1]\} \equiv \omega_{12}N,$$

(9)

which commutes with the Hamiltonian, since such an operation is simply equivalent to a change of the energy and momentum reference points. Namely, the origin of the (ω, k) coordinate system is now located at the point of the intersection of the spectra of the free fields.

When (7) and (8) are used, the Heisenberg equations for the operators $\varepsilon(x)$, $p(x)$, and $n(x)$ take on the form of the Bloch-Maxwell equations:

$$i(\varepsilon_t + \varepsilon_x) = -\varkappa^{1/2}p, \quad ip_t = \varkappa^{1/2}\varepsilon n,$$

$$in_t = 2\varkappa^{1/2}(\varepsilon^+p - p^+\varepsilon).$$

(10)

The classical analog of the system (10) is extensively used in the theory of effects of resonant coherent interaction of radiation with matter, such as self-induced transparency, photon echo, and others. In these problems, the individual radiators are also in phase and the radiation intensity is also proportional to N_0^2, but this phasing is due to the external coherent field. As applied to the SR problem, where

allowance for quantum effects plays a principal role, the classical analog of Eqs. (10) was used in a number of papers (see Ref. 3 and the literature cited there), with a fluctuation source ε_p added into the equation for the function $p(x,t)$. Such a model was therefore named in the SR theory "quasi-classical," while by "quantum model" is meant one in which the uncoupled correlators are not binary but ternary.

The inverse quantum problem method makes it possible to analyze the Bloch-Maxwell equations in their operator from (10), without the need for uncoupling the correlators in any way. It is therefore natural to define as the quasi-one-dimensional model of the Dicke SR the model with Hamiltonian (7), commutation relations (8), and the ensuing evolution equations (10).

§3. AUXILIARY QUANTUM SPECTRAL PROBLEM

Lamb[12] has shown that the classical analog of the system (10) admits of an exact solution of the Cauchy problem by the method of the inverse scattering problem.[10] We show here that in the quantum case, too, this model is exactly integrable, and obtain the commutation relations for the transition-matrix elements of the model.

We introduce a two-component field $\psi_\nu(x)$, $\nu = 1,2$, which satisfy the Fermi commutation relations

$$\{\psi_\nu(x), \psi_{\nu'}{}^+(y)\} = \delta_{\nu\nu'}\delta(x-y) \tag{11}$$

and is connected with the operators $p(x)$ and $n(x)$ by the relations

$$n(x) = \psi_2{}^+(x)\psi_2(x) - \psi_1{}^+(x)\psi_1(x), \quad p(x) = \psi_1{}^+(x)\psi_2(x). \tag{12}$$

If we impose on the operators ψ_ν the additional completeness condition

$$\psi_1{}^+(x)\psi_1(x) + \psi_2{}^+(x)\psi_2(x) = 1, \tag{13}$$

the operators $p(x)$ and $n(x)$ [Eq. (12)] satisfy, when (11) is taken into account, the commutation relations (8). In terms of the variables $\varepsilon(x)$ and $\psi_\nu(x)$ the operators N and H take the form

$$N = \int_{-\infty}^{\infty} dx[\varepsilon^+(x)\varepsilon(x) + \psi_2{}^+(x)\psi_2(x)], \tag{14}$$

$$H = -i\int_{-\infty}^{\infty} dx\,\varepsilon^+(x)\,\partial_x\varepsilon(x)$$
$$-\sqrt{\varkappa}\int_{-\infty}^{\infty} dx[\psi_1{}^+(x)\varepsilon^+(x)\psi_2(x) + \psi_2{}^+(x)\varepsilon(x)\psi_1(x)].$$

We consider the auxiliary quantum spectral problem on the finite interval $-L \leqslant x \leqslant L$:

$$\Phi_x(x, \lambda) =: Q(x, \lambda)\Phi(x, \lambda):, \tag{15}$$

where the matrix $Q(x,\lambda)$ is of the form

$$Q(x,\lambda) = i\begin{pmatrix} \frac{1}{2}(\lambda - \varkappa/\lambda) & \varkappa^{1/2}\varepsilon^+ \\ \varkappa^{1/2}\varepsilon & -\frac{1}{2}(\lambda - \varkappa/\lambda) \end{pmatrix}$$
$$+ i\frac{\varkappa}{\lambda}\begin{pmatrix} \psi_2{}^+\psi_2 & \psi_2{}^+\psi_1 \\ \psi_1{}^+\psi_2 & -\psi_2{}^+\psi_2 \end{pmatrix}, \tag{16}$$

and write the matrix of the solutions (15) in the form

$$\Phi(x,\lambda) = \begin{pmatrix} \overline{\varphi}^{(2)} & \varphi^{(1)} \\ -\overline{\varphi}^{(1)} & \varphi^{(2)} \end{pmatrix}. \tag{17}$$

The symbol :: denotes normal ordering of the operators in the spectral problem (15); in particular, the equations for $\varphi^{(1)}$ and $\varphi^{(2)}$ are

$$\varphi_x{}^{(1)} = \frac{i}{2}\left(\lambda - \frac{\varkappa}{\lambda}\right)\varphi^{(1)} + \frac{i\varkappa}{\lambda}\psi_2{}^+\varphi^{(1)}\psi_2 + i\varkappa^{1/2}\varepsilon^+\varphi^{(2)}$$
$$+ \frac{i\varkappa}{\lambda}\psi_2{}^+\varphi^{(2)}\psi_1, \tag{18}$$

$$\varphi_x{}^{(2)} = -\frac{i}{2}\left(\lambda - \frac{\varkappa}{\lambda}\right)\varphi^{(2)} - \frac{i\varkappa}{\lambda}\psi_2{}^+\varphi^{(2)}\psi_2 + i\varkappa^{1/2}\varphi^{(1)}\varepsilon$$
$$+ \frac{i\varkappa}{\lambda}\psi_1{}^+\varphi^{(1)}\psi_2.$$

We define the matrix $G(x,\lambda)$ as a solution of (15) satisfying the boundary condition $G(x = -L) = I$, where I is the unit matrix. The transition matrix on a finite interval is then defined as the value of $G(x,\lambda)$ at the point $x = L$, namely $T_L(\lambda) = G(x = L,\lambda)$. Using the differential equation method proposed by Sklyanin[9] for the expanded solution $G(x,\lambda) \otimes G(x,\mu)$ we can show that the elements of the transition matrix $T_L(\lambda)$ satisfy the commutation relations (see the Appendix)

$$R_L(\lambda, \mu)T_L(\lambda) \otimes T_L(\mu) = T_L(\mu) \otimes T_L(\lambda)R_L(\lambda, \mu), \tag{19}$$

in which, as found in Refs. 4 and 5, the matrix $R_L(\lambda,\mu)$ coincides fully with the matrix R_L of the nonlinear Schrödinger equation (the NS model).[6-9] The algebra of the transition matrix remains the same in our model and in the NS model when a transition is made to the problem on an infinite interval $-\infty < x < \infty$. For this transition, the commutation relations for the transition matrix on an infinite interval

$$T(\lambda) = \begin{pmatrix} \overline{A}(\lambda) & B(\lambda) \\ -\overline{B}(\lambda) & A(\lambda) \end{pmatrix}$$

retain the form (19) with the matrix $R(\lambda,\mu)$ for the infinite interval:

$$R(\lambda, \mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\alpha = \frac{\lambda - \mu}{\lambda - \mu - i\varkappa}, \quad \gamma = \frac{\lambda - \mu + i\varkappa}{\lambda - \mu + i\delta}, \quad \delta \to +0. \tag{20}$$

We write down here the commutation relations for the operators $A(\lambda)$ and $B(\lambda)$, which we shall need later on:

$$[A(\lambda), A(\mu)] = [B(\lambda), B(\mu)] = 0,$$
$$A(\lambda)B(\mu) = \frac{\lambda - \mu - i\varkappa}{\lambda - \mu}B(\mu)A(\lambda). \tag{21}$$

Using (18), the operators $A(\lambda)$ and $B(\lambda)$ can be written in the form

$$A(\lambda) = 1 + \frac{\varkappa}{i\lambda} \int_{-\infty}^{\infty} dx\, \psi_2^+(x)\, g(x,\lambda)\, \psi_2(x)$$

$$+ i\sqrt{\varkappa} \int_{-\infty}^{\infty} dx\, \varphi(x,\lambda)\, \varepsilon(x) - \frac{\varkappa}{i\lambda} \int_{-\infty}^{\infty} dx\, \psi_1^+(x)\, \varphi(x,\lambda)\, \psi_2(x), \tag{22a}$$

$$B(\lambda) = -\frac{\varkappa}{i\lambda} \int_{-\infty}^{\infty} dx\, e^{ikx} \psi_2^+(x)\, \varphi(x,\lambda)\, \psi_2(x)$$

$$+ i\sqrt{\varkappa} \int_{-\infty}^{\infty} dx\, e^{ikx} \varepsilon^+(x)\, g(x,\lambda)$$

$$- \frac{\varkappa}{i\lambda} \int_{-\infty}^{\infty} dx\, e^{ikx} \psi_2^+(x)\, g(x,\lambda)\, \psi_1(x), \tag{22b}$$

where we have introduced the symbol $k = \varkappa/\lambda - \lambda$, and the functions $g(x,\lambda)$ and $\varphi(x,\lambda)$ are connected with $\varphi^1(x,\lambda)$ and $\varphi^2(x,\lambda)$ by the relations

$$\varphi = \varphi^{(1)} e^{-ikx/2}, \qquad g = \varphi^{(2)} e^{-ikx/2}. \tag{23}$$

Expression (22a) and the system of differential equations for the functions $\varphi(x,\lambda)$ and $g(x,\lambda)$, which can be easily obtained from (18) and (23), make it possible to find the expansion of the operator $\ln A(\lambda)$ in the reciprocal powers of the spectral parameter $i\lambda$:

$$\ln A(\lambda) = \sum_{n=1}^{\infty} a_n (i\lambda)^{-n}. \tag{24}$$

The coefficients a_n in the expansion (24) are the commuting integrals of motion of our model; in particular, the coefficient a_1 coincides with the operator N of the number of quasiparticles of the system, while a_2 coincides with the Hamiltonian (14) of the model:

$$a_1 = \varkappa N, \qquad a_2 = -i\varkappa H. \tag{25}$$

In the derivation of (25) we used a shift, which we shall find useful later on, of the spectral parameter λ in the operator $A(\lambda)$, namely $\lambda \to \lambda + i\varkappa/2$. We note also that the formal justification for the expansion (24) are the results obtained in Ref. 13 for the NS model.

We have thus changed over, with the aid of the auxiliary spectral problem (15), from a description of the model in terms of the local fields $\varepsilon(x)$ and $\psi_\nu(x)$ to a description in terms of variables $A(\lambda)$ and $B(\lambda)$ of the action-angle type. The expansion (24) of the operator $\ln A$ generates an infinite set of commuting integrals of motion of the model, while the operator $B(\lambda)$ will be shown later to be the system quasiparticle creation operator.

§4. EIGENSTATES AND EIGENVALUES OF THE INTEGRALS OF MOTION

We proceed now to describe the eigenstates of the Dicke model. We already know one eigenstate of the system, namely the state in which there are no photons and all the two-level atoms are in the ground state. We shall call this state "vacuum" and designate it by $|0\rangle$; its properties are

$$\varepsilon(x)|0\rangle = 0, \quad \psi_2(x)|0\rangle = 0, \quad \psi_1(x)|0\rangle \ne 0. \tag{26}$$

From this definition and from expression (22a) for the operator $A(\lambda)$ it is clear that $|0\rangle$ is an eigenstate of the operator $A(\lambda)$:

$$A(\lambda)|0\rangle = |0\rangle. \tag{27}$$

We act on the vacuum with the operator $B(\mu)$:

$$|\Psi(\mu)\rangle = B(\mu)|0\rangle. \tag{28}$$

Using the commutation relations for the operators $A(\lambda)$ and $B(\mu)$ [Eq. (21)], we can readily see that (28) is also an eigenstate of the operator $A(\lambda)$:

$$A(\lambda)|\Psi(\mu)\rangle = \frac{\lambda - \mu - i\varkappa/2}{\lambda - \mu + i\varkappa/2} |\Psi(\mu)\rangle, \tag{29}$$

with the eigenvalues of the particle-number operator N and of the Hamiltonian H determined from the expression

$$N|\Psi(\mu)\rangle = |\Psi(\mu)\rangle, \quad H|\Psi(\mu)\rangle = -\mu|\Psi(\mu)\rangle. \tag{30}$$

The first expression in (30) means that (28) is a single-particle state of the model, and the second means that the spectral parameter $-\mu$ is its energy. Retaining the operator $B(\mu)$ [Eq. (22b)] only the terms that yield a nonzero value when acting on a vacuum, we obtain the single-particle state of the system in terms of the local fields:

$$|\Psi(\mu)\rangle = i\sqrt{\varkappa} \int_{-\infty}^{\infty} dx\, e^{ikx} \left[\varepsilon^+(x) + \frac{\sqrt{\varkappa}}{\mu} \psi_2^+(x)\, \psi_1(x) \right] |0\rangle, \tag{31}$$

where $k = \varkappa/\mu - \mu$ plays the role of the quasiparticle momentum. The spectrum of the single-particle state $\omega = \omega(k)(\omega = -\mu)$ consists, as expected, of two polariton branches (Fig. 1):

$$\omega_1(k) = \frac{k}{2} - \left[\left(\frac{k}{2}\right)^2 + \varkappa \right]^{1/2}, \quad \omega_2(k) = \frac{k}{2} + \left[\left(\frac{k}{2}\right)^2 + \varkappa \right]^{1/2}. \tag{32}$$

The multiparticle state of the system is constructed in analogy with the state (28), namely

$$|\Psi(\mu_1, \ldots, \mu_m)\rangle = B(\mu_1) \ldots B(\mu_m)|0\rangle; \tag{33}$$

in which case

$$N|\Psi(\mu_1, \ldots, \mu_m)\rangle = m|\Psi(\mu_1, \ldots, \mu_m)\rangle,$$

$$H|\Psi(\mu_1, \ldots, \mu_m)\rangle = -\left(\sum_{l=1}^{m} \mu_l\right)|\Psi(\mu_1, \ldots, \mu_m)\rangle. \tag{34}$$
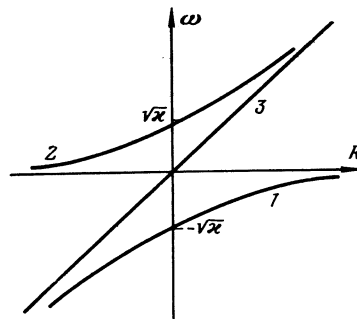

FIG. 1. Curves 1 and 2—polariton branches of "continuous" spectrum, 3—spectrum $\omega = k$ of quantum soliton.
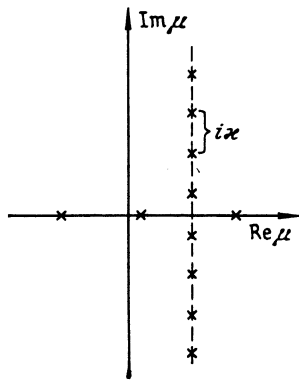
FIG. 2.

If all the μ_l in (33) and (34) are real, the state (33) describes m free quasiparticles of the model, and the connection between the quasiparticle energy $\omega l = -\mu_l$ and its momentum k_l is determined by expressions (32), while the total energy per particle is given by

$$\omega = -\frac{1}{m}\sum_{l=1}^{m}\mu_l.$$

Just as in the NS model (see Refs. 6–8), m quasiparticles can form a bound m-particle state $|\Psi_m(\mu)\rangle$, in which μ_l are complex (Fig. 2):

$$\mu_l = \frac{\mu}{m} + i\varkappa\left(\frac{m+1}{2}-l\right), \quad \mu = \sum_{l=1}^{m}\mu_l. \tag{35}$$

The eigenvalue of the operator $A(\lambda)$ on this state is given by

$$A(\lambda)|\Psi_m(\mu)\rangle = \left(\lambda - \frac{\mu}{m} - i\frac{\varkappa}{2}m\right)$$
$$\times \left(\lambda - \frac{\mu}{m} + i\frac{\varkappa}{2}m\right)^{-1}|\Psi_m(\mu)\rangle, \tag{36}$$

and the energy ω is connected with the momentum per particle $k = (1/m)\Sigma k_l$ by the expression

$$k = \omega\left\{1 - \frac{\varkappa}{m}\sum_{l=1}^{m}\left[\omega^2 + \varkappa^2\left(\frac{m+1}{2}-l\right)^2\right]^{-1}\right\}. \tag{37}$$

The bound state of the quasiparticles (33) and (35) can be called, in analogy with the classical theory,[10] a quantum soliton. The sum in (37) converges rapidly, so that for sufficiently large $m \gg 1$ the soliton spectrum is linear, $\omega = k$, with a small correction proportional to m^{-1}.

It can be easily seen that the wave function of a bound state is limited in the coordinates of all the constituent particles, and the spatial dimension r_0 of the state is inversely proportional to number m of the particles it contains:

$$r_0 \propto (\varkappa m)^{-1}. \tag{38}$$

§5. INHOMOGENEOUSLY BROADENED SYSTEM

We have investigated so far the quasi-one-dimensional quantum model of the Dicke SR under the assumption that the frequency ω_{12} is the same for all the atoms of the gas, and have thus neglected throughout the inhomogeneous reso-

nant-transition line broadening always present in real systems. As already mentioned at the beginning of the article, this simplification is of no importance in principle and the inhomogeneous broadening can be accounted for within the framework of the formalism developed in the present paper.

The generalization of the classical Bloch-Maxwell equations to include the case of an inhomogeneously broadened system is sufficiently well known and is widely used in the literature. In the quantum case, besides the corresponding generalization of the evolution equations (10), it is necessary also to examine the commutation relations for the local-field operations. We therefore derive here anew the basic equations for the case of inhomogeneously broadened systems.

The Hamiltonian of the system is

$$H = \sum_i \omega_i[n_i+1] - d\sum_i [E^+(\mathbf{r}_i)p_i + \text{H.c.}] + H_0, \tag{39}$$

where H_0 is the Hamiltonian of the free electromagnetic field. In addition to the polarization operator $P(\mathbf{r})$ and the population operator $n(\mathbf{r})$ for a unit volume of the medium (3), we shall find it convenient to use the spectral density of these operators:

$$P(\mathbf{r}) = \int_{-\infty}^{\infty} d\omega\, \chi(\omega) P(\mathbf{r}, \omega), \quad n(\mathbf{r}) = \int_{-\infty}^{\infty} d\omega\, \chi(\omega) n(\mathbf{r}, \omega), \tag{40}$$

where $\chi(\omega)$ is the atom distribution function in the transition frequency, with

$$\int_{-\infty}^{\infty} d\omega\, \chi(\omega) = 1.$$

The commutation relations for the operators $P(\mathbf{r},\omega)$ and $n(\mathbf{r},\omega)$ should be obtained from the condition that the commutation relations (2) hold for the operators $P(\mathbf{r})$ and $n(\mathbf{r})$. These relations are obviously of the form

$$[P^+(\mathbf{r}, \omega), P(\mathbf{r}', \omega')] = \frac{n(\mathbf{r}, \omega)}{\chi(\omega)}\delta(\mathbf{r}-\mathbf{r}')\delta(\omega-\omega'),$$

$$[P(\mathbf{r}, \omega), n(\mathbf{r}', \omega')] = \frac{2P(\mathbf{r}, \omega)}{\chi(\omega)}\delta(\mathbf{r}-\mathbf{r}')\delta(\omega-\omega'), \tag{41}$$

$$[n(\mathbf{r}, \omega), P^+(\mathbf{r}', \omega')] = \frac{2P^+(\mathbf{r}, \omega)}{\chi(\omega)}\delta(\mathbf{r}-\mathbf{r}')\delta(\omega-\omega').$$

After separating in the field and polarization operators the rapidly oscillating parts having the average transition frequency

$$\omega_{12} = \int_{-\infty}^{\infty} \omega\chi(\omega)\,d\omega, \tag{42}$$

transforming to the one-dimensional problem, and correspondingly making the variables dimensionless, the quasiparticle-number operator N and the Hamiltonian become

$$N = \int_{-\infty}^{\infty} dx\{\varepsilon^+(x)\varepsilon(x) + \tfrac{1}{2}[n(x)+1]\}, \tag{43}$$

$$H = \int_{-\infty}^{\infty} \Delta\chi(\Delta)\,d\Delta \int_{-\infty}^{\infty} dx\, n(x,\Delta) - i\int_{-\infty}^{\infty} dx\, \varepsilon^+(x)\partial_x\varepsilon(x)$$

$$-\sqrt{\varkappa}\int_{-\infty}^{\infty}\chi(\Delta)\,d\Delta\int_{-\infty}^{\infty}dx[\varepsilon^{+}(x)p(x,\Delta)+p^{+}(x,\Delta)\varepsilon(x)], \qquad (44)$$

where we have introduced the symbol $\Delta=\omega-\omega_{12}$.

Just as in §3, we express the operators $n(x,\Delta)$ and $p(x,\Delta)$ in terms of the operators $\psi_{\nu}(x,\Delta)$ with $\nu=1$ and 2:

$$n(x,\Delta)=\psi_{2}^{+}(x,\Delta)\psi_{2}(x,\Delta)-\psi_{1}^{+}(x,\Delta)\psi_{1}(x,\Delta), \qquad (45)$$

$$p(x,\Delta)=\psi_{1}^{+}(x,\Delta)\psi_{2}(x,\Delta),$$

where the operators $\psi_{\nu}(x,\Delta)$ satisfy the anticommutation relations

$$\{\psi_{\nu}(x,\Delta),\ \psi_{\nu'}^{+}(y,\Delta')\}=\frac{\delta_{\nu\nu'}}{\chi(\Delta)}\delta(x-y)\delta(\Delta-\Delta') \qquad (46)$$

and the additional completeness relation

$$\psi_{1}^{+}(x,\Delta)\psi_{1}(x,\Delta)+\psi_{2}^{+}(x,\Delta)\psi_{2}(x,\Delta)=1. \qquad (47)$$

The Hamiltonian (44) and the commutation relations for the spectral density of the polarization and population operators determine the system of Bloch-Maxwell evolution equations for the case of an inhomogeneously broadened system:

$$i(\varepsilon_{t}+\varepsilon_{x})=-\sqrt{\varkappa}\int_{-\infty}^{\infty}d\Delta\chi(\Delta)p(x,\Delta)=-\varkappa^{1/2}p(x),$$

$$ip_{t}(x,\Delta)=\Delta p(x,\Delta)+\varkappa^{1/2}\varepsilon(x)n(x,\Delta), \qquad (48)$$

$$in_{t}(x,\Delta)=2\sqrt{\varkappa}\,[\varepsilon^{+}(x)p(x,\Delta)-p^{+}(x,\Delta)\varepsilon(x)].$$

The feasibility of solving exactly the Cauchy problem for the classical analog of the system (48) by the inverse-problem method was indicated in Ref. 14 (see also Ref. 15), and we shall use here, just as in §3, these results to construct the auxiliary quantum spectral problem.

The spectral problem for an inhomogeneously broadened system assumes as before the form (15), but now the matrix $Q(x,\lambda)$ should be written in the form

$$Q(x,\lambda)=i\begin{pmatrix} -k/2 & \varkappa^{1/2}\varepsilon^{+}(x) \\ \varkappa^{1/2}\varepsilon(x) & k/2 \end{pmatrix}$$
$$+i\varkappa\int_{-\infty}^{\infty}\frac{\chi(\Delta)\,d\Delta}{\lambda+\Delta}\begin{pmatrix} \psi_{2}^{+}(x,\Delta)\psi_{2}(x,\Delta) & \psi_{2}^{+}(x,\Delta)\psi_{1}(x,\Delta) \\ \psi_{1}^{+}(x,\Delta)\psi_{2}(x,\Delta) & -\psi_{2}^{+}(x,\Delta)\psi_{2}(x,\Delta) \end{pmatrix},$$
$$\qquad (49)$$

where

$$k=\varkappa\int_{-\infty}^{\infty}\frac{\chi(\Delta)\,d\Delta}{\lambda+\Delta}-\lambda; \qquad (50)$$

if the spectral parameter λ is real, the integration contour circles around the singularity from below. This definition of the integration in (49) and (50) is necessary to preserve the possibility of analytically continuing the operator $A(\lambda)$ into the upper complex λ half-plane.

The remaining calculations are perfectly analogous to those of the preceding sections (see also the Appendix). It turns out here that the R matrix retains its form (20), and the coefficients a_{1} and a_{2} of the expansion of the operator $\ln A(\lambda)$ in reciprocal powers of the spectral parameter $i\lambda$ are expressed as before in terms of the quasiparticle-number operator N (43) and the Hamiltonian of the system (44):

$$a_{1}=\varkappa N,\qquad a_{2}=-i\varkappa H. \qquad (51)$$

We now continue the calculations for the particular case of a Lorentz contour of the transition line, i.e., we specify the atom distribution function in the frequency $\chi(\Delta)$ in the form

$$\chi(\Delta)=\frac{\eta}{\pi}\frac{1}{\Delta^{2}+\eta^{2}}. \qquad (52)$$

In this case the spectrum of the single-particle state of the system (28) is determined from Eqs. (50) and (52):

$$\mathrm{Re}\,k=\omega\left(1-\frac{\varkappa}{\omega^{2}+\eta^{2}}\right), \qquad (53)$$

where $\omega=-\mu$ and $\mathrm{Im}\,\mu=0$. The presence of an imaginary part of the momentum of the single-particle state

$$\mathrm{Im}\,k=\varkappa\eta/(\omega^{2}+\eta^{2}) \qquad (54)$$

does not make it possible to normalize the wave function of this state over the infinite interval $-\infty<x<\infty$. The single-particle state, however, has a real physical meaning; for example, it can be excited by a photon incident on the interface of the system with the vacuum. We note also that the results obtained in the present paper for the single-particle state (28), but in the case of an unbroadened and in the case of an inhomogeneously broadened system, are not connected with the restrictions used in the construction of the model. These polariton states can be exactly obtained within the framework of linear electrodynamics. The damping of the polariton state in an inhomogeneously broadened system is due to dephasing of the electric-field intensity and to polarization of the medium (see, e.g., Ref. 16). It is interesting that the contributions of these processes to the momentum of the motion, as a whole, of the bound complex of quasiparticles $|\Psi_{m}(\mu)\rangle$ (33), (35) cancel each other. In the general case of an m-particle bound state, when account is taken of (50) and (52), the momentum of the l-th particle is given by

$$k_{l}=\begin{cases} \varkappa/(\mu_{l}+i\eta)-\mu_{l}, & \mathrm{Im}\,\mu_{l}>0 \\ \varkappa/(\mu_{l}-i\eta)-\mu_{l}, & \mathrm{Im}\,\mu_{l}<0 \end{cases} \qquad (55)$$

and consequently the momentum of the bound complex

$$k=\frac{1}{m}\sum_{l=1}^{m}k_{l}$$

is a real quantity.

Concluding thereby the discussion of the Dicke effect in inhomogeneously broadened systems, we note that the results of the present section make it also possible to investigate the behavior of the system in interesting cases other than that of a Lorentz line (52). For example, choosing $\chi(\Delta)$ to be a two-hump function with maxima at the points $\Delta=\Delta_{1}$ and $\Delta=\Delta_{2}$, we can investigate the SR effect in a mixture of two gases with different transition frequencies and study the dependence of the Dicke pulse parameters on the relative concentration of the gases. The last remark pertains to the theory of self-induced transparency, in which the classical analog of the system (48) is used.

6. CONCLUSION

As shown in the present paper, the formalism of the method of the quantum inverse scattering problem provides perfectly adequate means for a quantum description of collective spontaneous emission in extended systems. Transformation to action-angle variables has made it possible to determine the eigenfrequency and the spectrum of the quasiparticle-number operator and of the Hamiltonian of the system.

The eigenstates of the system include both continuous-spectrum states corresponding to ordinary spontaneous emission,and bound m-particle complexes that can be set in correspondence to Dicke SR pulses.

It is clear that the initial excited state of the system, usually taken to mean a state having only N excited (inverted) atoms and which can be readily seen not to be an eigenstate of the system Hamiltonian, decays in the course of time into a set of eigenstate of the Hamiltonian with a total number of quasiparticles N and with a total energy $\omega_{12} N$. In the limit as $t \to \infty$ we should therefore obtain asymptotically, with a definite probability, a set of bound complexes having, generally speaking, different number of quasiparticles, as well as a certain number of continuous-spectrum quasiparticles.

The foregoing reasoning is to some degree analogous to the known results on the exact solution of the Cauchy problem in the classical inverse-problem method,[10] where the initial perturbation also breaks up in the course of time into states of a "continuous" spectrum and into soliton states of the system. In the case of a large ("quasiclassical") initial perturbation, the contribution of the "continuous" spectrum in the final state is exponentially small compared with the soliton contribution. If we use this analogy for our quantum case, we can state that an almost completely inverted initial state of the system (as is the case in the reported experiments[2,3]) should have an overwhelming probability of decaying into a set of quantum solitons with a small admixture of states of the "continuous" spectrum. Thus, the SR pulses should contain an appreciable fraction of the energy intially stored in the atomic subsystem.

We note also that the simultaneous existence of several complexes of quasiparticles that make up an eigenstate of the system explains fully the experimentally observed so-called oscillatory SR regime, and no additional arguments are necessary in this case (see Ref. 3).

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APPENDIX

We obtain the commutation relations (19) for the matrix element of the transition in the most general spectral problem (15), (49). Using the integral form of the spectral problem for the matrix $G(x, \lambda)$, we can establish the following commutation relations:

$$[\psi_1{}^+(x, \Delta), \ G(x+0, \lambda)] = \frac{-i\varkappa}{\lambda + \Delta} \psi_2{}^+(x, \Delta)(\sigma^+ G),$$

$$[\psi_2{}^+(x, \Delta), G(x+0, \lambda)] = \frac{-i\varkappa}{\lambda + \Lambda}$$

$$\times \{(\sigma^z G) \psi_2(x, \Delta) + (\sigma^+ G) \psi_1(x, \Delta)\},$$

$$[\varepsilon^+(x), \ G(x+0, \lambda)] = -i\varkappa (\sigma^- G), \tag{A1}$$

where $\sigma^\pm = \frac{1}{2}(\sigma^x \pm i\sigma^y)$; σ^x, σ^y, and σ^z are Pauli matrices. The commutators

$$[\psi_\nu(x, \Delta), \ G(x-0, \lambda)] = [\varepsilon(x), \ G(x-0, \lambda)] = 0. \tag{A2}$$

We introduce, by definition

$$H_{12}(x) = G(x+0, \lambda_1) \otimes G(x-0, \lambda_2) \equiv G_1(x+0) \otimes G_2(x-0).$$

Using (15) we have

$$dH_{12}(x)/dx = :Q_1(x+0) G_1(x+0): \otimes G_2(x-0)$$
$$+ G_1(x+0) \otimes :Q_2(x-0) G_2(x-0). \tag{A3}$$

After reducing the right-hand side of (A3) to a normally ordered form, we obtain with allowance form (A1) and (A2)

$$dH_{12}(x)/dx = :\Gamma_{12}(x) H_{12}(x):,$$

where the matrix Γ_{12} is given by

$$\Gamma_{12} = Q_1 \otimes I + I \otimes Q_2 - \varkappa(\sigma^- \otimes \sigma^+)$$

$$- \varkappa^2 \int_{-\infty}^{\infty} \frac{\chi(\Delta) d\Delta}{(\lambda_1 + \Delta)(\lambda_2 + \Delta)} \{\psi_2{}^+\psi_2(\sigma^z \otimes \sigma^z + \sigma^+ \otimes \sigma^-)$$

$$+ \psi_1{}^+\psi_1(\sigma^- \otimes \sigma^+) + \psi_2{}^+\psi_1(\sigma^z \otimes \sigma^+) + \psi_1{}^+\psi_2(\sigma^- \otimes \sigma^z)\}.$$

By a direct check with account taken of the completeness condition (47) we find that the matrix Γ_{12} satisfies the inequality

$$R_L \Gamma_{12} = \Gamma_{21} R_L, \tag{A4}$$

where

$$R_L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta & \alpha & 0 \\ 0 & \alpha & \beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \alpha = \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2 - i\varkappa},$$

$$\beta = \frac{-i\varkappa}{\lambda_1 - \lambda_2 - i\varkappa}$$

and coincides with the R_L matrix of the NS model. The commutation relations (19) are a direct consequence of (A4). The transition to the problem on a infinite interval is by the standard method.[6-7]

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