Nonlinear relaxation of a beam of relativistic electrons in a plasma: nonlinear damping of sound

V. M. Malkin

Institute of Nuclear Physics, Siberian Division of the Academy of Sciences of the USSR
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We consider the interaction between a plasma and a beam of relativistic electrons in that range of parameters where the instability of the system against the excitation of Langmuir waves is eliminated by their scattering by the nonlinearly damped sound. We find the distribution function of the beam electrons and the Langmuir wave spectrum. The range of beam densities covered by the quantitative relaxation theory is hereby broadened considerably—now this range stretches up to the limit of applicability of weak turbulence theory.

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1. INTRODUCTION

In a previous paper[1] we studied the elimination of beam instability by the scattering of Langmuir waves by induced fluctuations in the plasma density. The reciprocal scattering time Γ was assumed to be small compared to the damping rate ν_s of the density fluctuations; this enabled us to consider this process to be a four-plasmon one. In a plasma with an ion temperature higher than the electron one ($T_e \lesssim T_i$) the inequality $\Gamma < \nu_s$ is the same as the condition for the modulational instability of the Langmuir spectrum excited by the beam. When $T_e \gg T_i$ the region $\Gamma > \nu_s$ intersects the region of applicability of weak turbulence theory. The main part of these regions broadens rapidly when the parameter T_e/T_i increases and becomes very extensive in a strongly nonisothermal plasma. The study of this region is of considerable interest: On the one hand, a plasma with $T_e \gg T_i$ occurs naturally in many experiments on beam heating because the energy exchange between the electron and ion components is weak;[1)] on the other hand, it is precisely at $T_e \gg T_i$ that the upper bound on the beam density (n_b) turns out to be very stringent in the regimes studied earlier of the interaction of the beam with the plasma.

The range of beam densities bounded from below by the inequality $\Gamma > \nu_s$ and from above by the condition of applicability of weak turbulence theory is considered in the present paper. As the formal basis of our calculations we use kinetic equations describing the decay interaction between Langmuir and ion-sound waves. We note that the turbulence spectra arising when there is this interaction have been often studied in the past (see, e.g., Ref. 2); both power-law[3] and some other[4] spectra which are possible in "inertial" frequency range were found, but the spectra excited by real sources were not amenable to an analytical study. Numerical studies have also encountered considerable difficulties and have only been performed in the "planar" variant of the problem of the beam excitation.[5] It will become clear in what follows that an analytical solution of this problem in the three-dimensional case differs strongly from the one obtained in Ref. 5: The Langmuir spectrum turns out to be nearly isotropic and to lie completely in the resonance range of frequencies (as in Ref. 1) while the sound spectrum consists of a nearly isotropic background and jets (in which a large part of the sound energy is concentrated). The reason for this difference is that when we change the three-dimensional problem to a two-dimensional one we lose the very important property that the growth rate of the beam instability averaged over angles is negative, and this happens in a very narrow range of wave numbers near $k=\omega_p/c$. This fact, apparently, also leads to the applicability of the so-called satellite approximation which well explains the results of Ref. 5.

2. BASIC EQUATIONS

We use in what follows the dimensionless variables of Ref. 1; references to that paper are preceded with a 1 inside parentheses [e.g., (1.1)]. In terms of those variables the collision term of the kinetic equation for the Langmuir waves has again the form (1.18). The term $\tilde{\gamma}_{\mathbf{k}} N_{\mathbf{k}}$ which describes the induced scattering of Langmuir waves by ions can be omitted as it is small in all cases considered in what follows (in some of them the calculation of $\tilde{\gamma}_{\mathbf{k}} N_{\mathbf{k}}$ would be even an exaggeration of the accuracy). The function $F_{\mathbf{q}}$ in Eq. (1.19) must now be determined from the kinetic equation for the sound waves. Using the fact that the ion sound frequency $\Omega_q = gq$, where g is given by Eq. (1.20), is small as compared to the width of the Langmuir spectrum we can go over to the differential approximation in that equation. The result can conveniently be expressed in terms of the following correlation functions of the spatial Fourier components of the sound perturbations:[2)]

$$\langle n_{\mathbf{q}} n_{-\mathbf{q}'}\rangle = F_{\mathbf{q}}\delta(\mathbf{q}-\mathbf{q}'),$$
$$\left\langle \frac{\partial n_{\mathbf{q}}}{\partial t} n_{-\mathbf{q}'} - n_{\mathbf{q}}\frac{\partial n_{-\mathbf{q}'}}{\partial t}\right\rangle = 2i\Omega_{\mathbf{q}} A_{\mathbf{q}}\delta(\mathbf{q}-\mathbf{q}'). \quad (1)$$

The functions $F_{\mathbf{q}}$ and $A_{\mathbf{q}}$ can be written in terms of the even and odd parts of the spectral density $W^s_{\mathbf{q}}$ of the sound energy:

$$F_{\mathbf{q}} = {}^1/_2(W_{-\mathbf{q}}{}^s + W_{\mathbf{q}}{}^s), \quad A_{\mathbf{q}} = {}^1/_2(W_{-\mathbf{q}}{}^s - W_{\mathbf{q}}{}^s).$$

The collision terms of the equations for $F_{\mathbf{q}}$ and $A_{\mathbf{q}}$ are given within the necessary accuracy by the formulas

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$$\mathrm{St}_{\mathbf{q}}^{(F)}=f_{\mathbf{q}}+\lambda_{\mathbf{q}}A_{\mathbf{q}}-(\nu_q+\tilde{\nu}_{\mathbf{q}})F_{\mathbf{q}},$$

$$\mathrm{St}_{\mathbf{q}}^{(A)}=\lambda_{\mathbf{q}}F_{\mathbf{q}}-(\nu_q+\tilde{\nu}_{\mathbf{q}})A_{\mathbf{q}}; \tag{2}$$

$$\begin{pmatrix} f_{\mathbf{q}} \\ \lambda_{\mathbf{q}} \\ \tilde{\nu}_{\mathbf{q}} \end{pmatrix} = \pi \int d^3k\, d^3k_1 \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{q}) \left(\frac{\mathbf{k}\mathbf{k}_1}{kk_1}\right)^2 \times \begin{pmatrix} \Omega_q{}^2 N_{\mathbf{k}} N_{\mathbf{k}_1} \delta(k^2-k_1{}^2) \\ \Omega_q (N_{\mathbf{k}_1}-N_{\mathbf{k}}) \delta(k^2-k_1{}^2) \\ \Omega_q{}^2 (N_{\mathbf{k}}-N_{\mathbf{k}_1}) \delta'(k^2-k_1{}^2) \end{pmatrix}.$$

The term $f_{\mathbf{q}}$ describes the sound generation by the low-frequency beats of the electrical field; $\lambda_{\mathbf{q}}$ and $\tilde{\nu}_{\mathbf{q}}$ are the nonlinear corrections to the damping of the sound caused by elastic and inelastic scattering of the Langmuir waves by them; $\nu_q = \zeta^{-1}\Omega_q$ is the damping rate of the sound due to the ions and electrons in the plasma, and ζ is a coefficient which depends on the ion and electron temperatures.

Equating the collision terms (2) to zero we easily find a relation between the sound and the Langmuir spectra in the stationary regime:

$$F_{\mathbf{q}}=\frac{(\nu_q+\tilde{\nu}_{\mathbf{q}})f_{\mathbf{q}}}{(\nu_q+\tilde{\nu}_{\mathbf{q}})^2-\lambda_{\mathbf{q}}{}^2}, \quad A_{\mathbf{q}}=\frac{\lambda_{\mathbf{q}}f_{\mathbf{q}}}{(\nu_q+\tilde{\nu}_{\mathbf{q}})^2-\lambda_{\mathbf{q}}{}^2}. \tag{3}$$

When the condition $\Gamma < \nu_s$ is satisfied both nonlinear corrections to the damping of the sound are small and the expression for $F_{\mathbf{q}}$ is the same as the one used in Ref. 1. Increasing the beam density is accompanied by an increase in the corrections $\lambda_{\mathbf{q}}$ and $\tilde{\nu}_{\mathbf{q}}$. The first of these becomes important.

3. "ELASTIC" CORRECTION TO THE DAMPING OF THE SOUND

In the formula for the quantity $\lambda_{\mathbf{q}}$ the integration varaible $\mathbf{k}$ ranges over a plane perpendicular to the vector $\mathbf{q}$ and passing through its mid-point. The variable $\mathbf{k}_1$, connected with $\mathbf{k}$ through the relation $\mathbf{k}_1 = \mathbf{k} - \mathbf{q}$, ranges then over a plane which is the mirror image of the first one relative to the origin. Only the anisotropic part of the Langmuir spectrum ($N'_{\mathbf{k}}$) which, according to (1.40), is nonvanishing in the vicinity $\Delta\theta$ of the plane $k_z = 1$ contributes to $\lambda_{\mathbf{q}}$. Taking the foregoing into account we can easily show that the quantity $|\lambda_{\mathbf{q}}|$ has a maximum in the neighborhood of the point $\mathbf{q} = (0,0, \pm 2)$ where

$$|\lambda| \sim gN' \sim gN\Delta\theta \sim \left(\frac{\gamma_0 g}{\zeta\Delta\theta}\right)^{1/2} \sim \left(\frac{\gamma_0 \nu_s}{\Delta\theta}\right)^{1/2}. \tag{4}$$

[In this chain of estimates we used Eqs. (1.32), (1.33), and the definition of ζ.] The "elastic" correction to the damping of the sound is small compared to ν_s when $\gamma_0 < \nu_s \Delta\theta$.

We consider in more detail the case $\gamma_0 > \nu_s$, when the condition that $|\lambda|$ be small is violated for any beam with an angular spread.[3] For the present we shall assume that the second correction $\tilde{\nu}_{\mathbf{q}}$ to the damping of the sound is unimportant.

If the difference $\nu_q - |\lambda_{\mathbf{q}}|$ were to vanish in some point $\mathbf{q}_0$ when the beam density is increased the quantity $F_{\mathbf{q}_0}$ would become infinite and the elastic scattering of plasmons would instantaneously make their spectrum identical on planes passing through the points $\mathbf{k} = \pm \mathbf{q}_0/2$ and perpendicular to $\mathbf{q}_0$. However, for such a spectrum $\lambda_{\mathbf{q}_0} = 0$. This contradiction shows that the denominator in (3) does not vanish; it only becomes rather small after which the scattering of the plasmons symmetrizes the anisotropic part of the spectrum and arrests the growth of $|\lambda|$ at the level ν_s:

$$|\lambda| \sim \nu_s. \tag{5}$$

The antisymmetric part $\delta N'_{\mathbf{k}}$ of the spectrum then turns out to be small compared to the anisotropic part $N'_{\mathbf{k}}$ and instead of (4) we must use the estimate

$$|\lambda| \sim g\delta N'. \tag{6}$$

From (5) and (6) it follows that

$$\delta N' \sim \nu_s/g. \tag{7}$$

Such a small value of $\delta N'$ is guaranteed by the fast scattering of the plasmons from the region where they are excited into the symmetric region. The reciprocal time for this process is larger by a factor $N/\delta N'$ than the beam instability growth rate and is hence $N'/\delta N'$ times larger than Γ:

$$\Gamma^c \sim \gamma \frac{N}{\delta N'} \sim \Gamma \frac{N'}{\delta N'}. \tag{8}$$

The anomalously fast symmetrization of the Langmuir spectrum relative to the plane $k_z = 0$ is connected with the longitudinal sound jets which occur in the vicinity of the points $\mathbf{q} = (0,0, \pm 2)$ and which have a length of the order $\Delta\theta$. The profile of the jet is clearly Lorentzian:

$$F_{\mathbf{q}} \sim \frac{f_{\mathbf{q}}\Delta\theta^2}{\nu_s(\Delta^2+q_\perp{}^2)} \sim \frac{\Gamma\Delta\theta^2}{\Delta^2+q_\perp{}^2}. \tag{9}$$

Substituting this expression into Eq. (1.19) for $\Gamma_{\mathbf{k}}$ we can relate the width Δ of the jet with the reciprocal of the time for symmetrizing the spectrum:

$$\Gamma^c \sim \Gamma\Delta\theta^2 \ln(\Delta\theta/\Delta). \tag{10}$$

The quantity $\Delta/\Delta\theta$ turns out to be exponentially small in terms of the parameter $\Gamma^c/\Gamma\Delta\theta^2$.

The scale of the changes of the function $\lambda_{\mathbf{q}}$ with respect to $q_\perp$ in the vicinity of the jet is of the order $\Delta\theta$, so that Eq. (9) is applicable in the region $q_\perp \lesssim \Delta\theta$. For large values of $q_\perp$ the correction $\lambda_{\mathbf{q}}$ is small and the sound spectrum is nearly isotropic. Anisotropic sound occurring in the neighborhood of the points $\mathbf{q} = (0,0, \pm 2)$ leads, due to its finite angular width, not only to symmetrization but also to some smoothing of the anisotropic part of the Langmuir spectrum. It is easy to show, however, that it is automatically weaker than the smoothing caused by the isotropic sound. Indeed, the additional smoothing is diffusive and the diffusion coefficient contains an integral of the type $\int F_{\mathbf{q}} q_\perp^2 d\mathbf{q}_\perp$ over the region $q_\perp \lesssim \Delta\theta$; one can estimate this integral to be approximately $\Gamma\Delta\theta^4$ and hence the correction to the reciprocal Γ of the smoothing time is of order $\Gamma\Delta\theta^2$. It is noteworthy that it is indepedent of the width Δ of the jet. This is explained by the fact that the expression $F_{\mathbf{q}} q_\perp^2$ is integrable as $\Delta \to 0$.

The role of the jet is thus reduced to the symmetrization of the Langmuir spectrum relative to the plane $k_z = 0$. The beam instability is again eliminated by the scattering by the isotropic sound so that the estimates of the energy W of the

Langmuir waves, the beam relaxation length l, and the anisotropic addition to the Langmuir spectrum remain the same as in Ref. 1. The analytical solution of the problem of a beam with a small angular spread also reduces to the one obtained in Ref. 1. We show how this is done.

We split the sound spectrum into the jet (9), an isotropic background, and an anisotropic extra term:

$$F_{\mathbf{q}}=F^c(q_z)\,\delta(\mathbf{q}_\perp)+F_q+F_{\mathbf{q}}' \tag{11}$$

(we can assume the jet profile to be δ-shaped since its finite width does not affect the further calculations). Substituting (11) into Eq. (1.19) and retaining in the equation $\mathrm{St}_{\mathbf{k}}=0$ the largest of the anisotropic terms we get

$$2\gamma_{\mathbf{k}}N_k-\Gamma_kN_{\mathbf{k}}'-\Gamma_{\mathbf{k}}^c(N'_{k_\perp,k_z}-N'_{k_\perp,-k_z})=0, \tag{12}$$

where Γ_k is given by Eqs. (1.39),

$$\Gamma_k^c=2\pi\left(\frac{k_\perp^2-k_z^2}{k^2}\right)^2\frac{F^c(2k_z)}{2|k_z|}. \tag{13}$$

According to (12) the anisotropic part of the Langmuir spectrum is nonvanishing in a neighborhood $\Delta\theta$ of the planes $k_z=\pm1$ and given by the formula

$$N_{\mathbf{k}}^-\equiv N'_{k_\perp,-k_z}\Big|_{k_z\approx1}=\frac{\Gamma_{\mathbf{k}}^c}{2\Gamma_{\mathbf{k}}^c+\Gamma_k}\frac{2\gamma_{\mathbf{k}}}{\Gamma_k}N_k,$$

$$N_{\mathbf{k}}^+\equiv N'_{k_\perp,k_z}\Big|_{k_z\approx1}=\frac{\Gamma_{\mathbf{k}}^c+\Gamma_k}{2\Gamma_{\mathbf{k}}^c+\Gamma_k}\frac{2\gamma_{\mathbf{k}}}{\Gamma_k}N_k. \tag{14}$$

In this neighborhood, apart from a small part that adjoins the cone $k_\perp^2=k_z^2$ (on which $\Gamma_{\mathbf{k}}^c=0$), we have $\Gamma_{\mathbf{k}}^c\gg\Gamma_k$ and the Langmuir spectrum is practically even:

$$N_{\mathbf{k}}^+\approx N_{\mathbf{k}}^-\approx\gamma_{\mathbf{k}}N_k/\Gamma_k.$$

Substituting (14) into the equation $\mathrm{St}_{\mathbf{k}}=0$ averaged over the angles [see (1.41)], we again are led to the relation

$$\gamma_k^{\mathrm{eff}}N_k=0. \tag{15}$$

Everywhere, except in a narrow neighborhood of the sphere $k^2=2$,

$$\gamma_k^{\mathrm{eff}}=2\gamma_k-\nu_e+(2\pi\Gamma_k)^{-1}\int do_k\gamma_{\mathbf{k}}^2$$

and Eq. (15) reduces to (1.43) by the substitution $\gamma_0\to2\gamma_0$ (with a fixed ratio ν_e/γ_0).

The function $F^c(q_z)$ which describes the distribution of the sound energy along the jet (9) is determined from the condition $\lambda_q|_{q_\perp=0}\approx\nu_q=\zeta^{-1}\Omega_q$ and is given by the equation

$$F^c(q_z)={}^1/_2\zeta\left|\int d\mathbf{k}_\perp\gamma_{\mathbf{k}}N_k\right|_{k_z=q_z/2}.$$

To complete the picture we briefly give the results of a study of the Langmuir spectrum in the vicinity of the sphere $k^2=2$. It turns out that a hump in the odd part of the Langmuir spectrum, arising near the circle $k_\perp=k_z=1$ and due to the fact that $\Gamma_{\mathbf{k}}^c$ is small in that region, leads to the appearance of a weak sound jet[4] lying on the cone $q_\perp^2=q_z^2$ at distances $\propto\Delta\theta^{4/7}$ from the origin. Scattering by this jet, which is not taken into account in Eqs. (12) and (14), changes them somewhat near the circles $k_\perp=|k_z|=1$; the changes are such that when $k^2\approx2$ a dip occurs in the effective growth rate γ_k^{eff} and a spherical gap of width $\propto\Delta\theta^{4/7}$ occurs in the Langmuir spectrum. This gap does not affect the integral characteristics of the Langmuir turbulence; in particular, the quantitative formulas of Ref. 1 for the energy of the Langmuir waves and for the beam relaxation length remain valid (apart from the substitution mentoined above). We note that the same result is obtained when Langmuir turbulence is excited by two identical counter-beams of half the density.

4. NONLINEAR SOUND DAMPING

1. Estimates

After taking into account the "elastic" corrections to the damping of sound the upper bound on the beam density can be connected with the assumption that the "inelastic" corrections are small ($\nu_s>\tilde\nu\propto g^2W$) or with the condition that the kinetic equations which describe the decay interaction between Langmuir and ion-sound waves be applicable. The standard form of these equations is certainly applicable if the ion-sound frequency is considerably larger than the reciprocal of the time for the nonlinear processes ($g>\Gamma$ for an even excitation, $g>\Gamma^c$ in the opposite case) while the renormalization of this frequency is small ($W<1$). The first of these conditions turns out to be not very important. Indeed, the elastic scattering of Langmuir waves by density fluctuations occurs in the zeroth approximation in the parameter g and proceeds exactly as in the case of time-independent fluctuations. It is described by the standard kinetic equation as long as the reciprocal scattering time is less than the width of the Langmuir spectrum. As to the validity of the kinetic equation for the sound, in the case of a Langmuir spectrum which changes with the characteristic scattering time a necessary condition for it would, of course, be $\Gamma<g$ ($\Gamma^c<g$). However, in the stationary case one requires for the validity of this equation only that the renormalization of the sound frequency and the randomness of the phases of the Langmuir waves be small. This can be checked using the generalized kinetic equations obtained in Ref. 6, which differ from the standard ones by the renormalization of the sound frequency and the presence of the higher-order time derivatives of the sound perturbation correlators: In the stationary case the time derivatives are unimportant. One can thus use the weaker restriction:

$$\Gamma<1\quad(\Gamma^c<1).$$

Comparing the conditions listed here one easily establishes that for an even excitation the limit of applicability of the results obtained is determined by the nonlinear sound damping if $\nu_s<g^2$, and by the renormalization of the sound frequency if $\nu_s>g^2$. When the sound is smeared out ($\Delta\theta\sim1$) these conditions can be written in the form

$$\gamma_0<\min\{\nu_s/g^2,\ g^2/\nu_s\}. \tag{16}$$

If the excitation is odd one must use the condition $\Gamma^c<1$, which for smeared-out sound reduces to the inequality

$$\gamma_0<\nu_s^{1/2}. \tag{17}$$

When $g^3<\nu_s<g^{3/2}$ inequality (17) is more stringent than (16) and the region of applicability of the theory is restricted by

the requirement that the phases of the Langmuir waves be random. Let $\nu_s < g^2$ when we are dealing with an even growth rate, and $\nu_s < g^3$ in the opposite case. In that case, in the region

$$\gamma_0 > \nu_s / g^2, \tag{18}$$

adjoining the limit of applicability of the results obtained above, the Langmuir turbulence is still weak and the "inelastic" correction to the damping of the sound is already important for any beam with an angular spread. This correction is sometimes called "nonlinear sound damping." We shall also use this term in what follows. We must merely bear in mind that in spite of the prevailing opinion (see, e.g., Ref. 2) the nonlinear sound damping is not necessarily positive even when the Langmuir spectrum is isotropic.

Since Eqs. (1.32) remain valid under the same assumptions as in Ref. 1, to close the system of estimates we need to establish a connection between the time for the scattering of Langmuir waves and their energy. If we use for the nonlinear sound damping the rough estimate $\tilde{\nu} \sim g^2 W$ we obtain $\Gamma \sim g^2 W^2/\tilde{\nu} \sim W$ and, hence, using (1.32), $W \sim \gamma_0/\Delta\theta^3$. The condition $\tilde{\nu} > \nu_s$ that the nonlinear sound damping be small, which was assumed above, is, indeed, satisfied because of (18). Nonetheless, it is impossible to assume that the estimates obtained here are substantiated, as there are a number of other principally different, but equally probable possibilities. For instance, the quantity $\tilde{\nu}_{\mathbf{q}}$ could be negative, but the sum $\nu_q + \tilde{\nu}_{\mathbf{q}}$ positive and in some region close to zero. In that case the value of Γ given by the estimate (1.32) would be reached for a relatively low energy of Langmuir waves: $W \sim \tilde{\nu}/g^2 \sim \nu_s/g^2$. On the level of estimates it is impossible to choose a particular variant. We therefore now turn to an exact solution of the problem.

2. EQUATION FOR THE SPECTRUM

As in Ref. 1, we obtain an analytical solution of the relaxation problem for a beam with a narrow angular spread ($\Delta\theta \ll 1$). The anisotropic part of the Langmuir spectrum excited by such a beam is small. The isotropic parts of the quantities $\Gamma_{\mathbf{k}}$, $u_{\mathbf{k}}$, $F_{\mathbf{q}}$ are again given by Eqs. (1.39) and we need solely replace in the last of them the linear damping of the sound by the total damping[5]:

$$F_{2k_1} = \frac{\pi^2 g^2 k_1}{\nu_{2k_1} + \tilde{\nu}_{2k_1}} \int_{k_1}^{\infty} dk_2 k_2 \left(1 - 2\frac{k_1^2}{k_2^2}\right)^2 N_{k_2}^2,$$
$$\tilde{\nu}_{2k_1} = \pi^2 g^2 k_1 \left[N_{k_1} + 4 \int_{k_1}^{\infty} \frac{dk_2}{k_2} \frac{k_1^2}{k_2^2} \left(1 - 2\frac{k_1^2}{k_2^2}\right) N_{k_2} \right]. \tag{19}$$

The derivation of the relations for the Langmuir spectrum is completely analogous to the one given in Ref. 1: To begin with the anisotropic part of the spectrum can be expressed in terms of the isotropic one, and by averaging $\mathrm{St}_{\mathbf{k}}$ over angles we derive for the latter an equation which contains the average growth rate of the beam instability and its square—the first of these can be evaluated at once and the second after finding the angular distribution function for the beam electrons, which turns out to be universal and the same as in Ref. 1; afterwards the equation is written in a form convenient for further study. Using the condition for the external stability of the spectrum we can write the result in the form of Eqs. (1.61) and (1.62). The function $N(\omega)$ in these equations is again introduced in accord with (1.56) and satisfies the normalization condition (1.57); the quantity a is given by the formula

$$W = \gamma_0 A a / 2\pi^2 \Delta\theta^3, \tag{20}$$

where A is the number (1.55), $\tilde{\Gamma}(\omega)$ is given by Eq. (1.63), while the function $\Gamma(\omega)$ is connected with Γ_k through the relation

$$\Gamma_k = \frac{\pi W}{2ak^5} \Gamma(k^2), \tag{21}$$

that is, through Eq. (1.58) in which

$$F(\omega_1) = \frac{\psi(\omega_1)}{\Phi(\omega_1)}, \qquad \psi(\omega_1) = \int_{\omega_1}^{\infty} d\omega_2 (\omega_2 - 2\omega_1)^2 N^2(\omega_2),$$
$$\Phi(\omega_1) = \delta + \omega_1 \left[N(\omega_1) + 4 \int_{\omega_1}^{\infty} \frac{d\omega_2}{\omega_2^2} (\omega_2 - 2\omega_1) N(\omega_2) \right],$$
$$\delta = 16\pi\Delta\theta^3 / A g \zeta \gamma_0 \ll 1. \tag{22}$$

To find the Langmuir spectrum we must thus solve a nonlinear integral equation. It depends on the angular spread of the beam through the parameter δ. It is impossible to neglect the small quantity δ in (22) inasmuch as at $\delta = 0$ the integral $F(\omega_1)$ in (1.58) diverges at the lower limit. Moreover, it is not clear *a priori* whether δ is small compared to the second terms in $\Phi(\omega)$: Ony one of the two possible kinds of solution mentioned at the start of this section satisfies that condition. Moreover, the above-mentioned simplest possibilities by no means exhaust the list of plausible variants. Omitting an enumeration of these variants and a proof of the initial contradiction of most of them, we dwell in detail on the self-consistent solution of the problem which is, apparently, unique.

The spectral density of the Langmuir waves turns out to be nonvanishing in a spherical layer

$$\omega_m < \omega < \omega_M, \quad \omega_m > 1.$$

According to (1.61) and (1.62) inside this layer $\Gamma(\omega) = \tilde{\Gamma}(\omega)$ and outside it $\Gamma(\omega) \geqslant \tilde{\Gamma}(\omega)$. Hence

$$\left.\frac{d\Gamma(\omega)}{d\omega}\right|_{\omega=\omega_M+0} \geqslant \frac{d\tilde{\Gamma}(\omega_M)}{d\omega_M} = \left.\frac{d\Gamma(\omega)}{d\omega}\right|_{\omega=\omega_M-0},$$
$$\left.\frac{d\Gamma(\omega)}{d\omega}\right|_{\omega=\omega_m-0} \leqslant \frac{d\tilde{\Gamma}(\omega_m)}{d\omega_m} = \left.\frac{d\Gamma(\omega)}{d\omega}\right|_{\omega=\omega_m+0}. \tag{23}$$

Using the definition (1.58) of the funciton $\Gamma(\omega)$ we can write the boundary coditions (23) in the form

$$F(\omega_M + 0) \geqslant F(\omega_M - 0), \tag{24}$$
$$F(\omega_m - 0) \leqslant F(\omega_m + 0). \tag{25}$$

The function $F(\omega)$ which has the meaning of the spectral density of the sound energy is everywhere non-negative and vanishes in the region $\omega > \omega_M$. Therefore (24) is satisfied only when

$$F(\omega_M-0)=0. \tag{26}$$

We can, by using (22), replace the condition (25) by the requirement that

$$N(\omega_m-0)\geqslant N(\omega_m+0),$$

from which it follows that

$$N(\omega_m+0)=0. \tag{27}$$

Since (23) includes equalities, the following conditions must also hold:

$$\left.\frac{d^2\Gamma(\omega)}{d\omega^2}\right|_{\omega=\omega_M+0}\geqslant\frac{d^2\tilde{\Gamma}(\omega_M)}{d\omega_M{}^2}=\left.\frac{d^2\Gamma(\omega)}{d\omega^2}\right|_{\omega=\omega_M-0},$$

$$\left.\frac{d^2\Gamma(\omega)}{d\omega^2}\right|_{\omega=\omega_m-0}\geqslant\frac{d^2\tilde{\Gamma}(\omega_m)}{d\omega_m{}^2}=\left.\frac{d^2\Gamma(\omega)}{d\omega^2}\right|_{\omega=\omega_m+0}.$$

However, they do not lead to additional restrictions as they turn out to be equivalent to the requirements

$$\left.\frac{dF(\omega)}{d\omega}\right|_{\omega=\omega_M-0}\leqslant 0,\qquad \left.\frac{dN(\omega)}{d\omega}\right|_{\omega=\omega_m+0}\geqslant 0,$$

which are automatically valid for the functions $N(\omega)$ and $F(\omega)$ which are positive in the range (ω_m,ω_M).

3. Construction of the spectrum

In the range $\omega<\omega_m$ the functional form of $F(\omega)$ is known and $\Phi(\omega)$ and $\psi(\omega)$ are here second degree polynomials:

$$\Phi(\omega)=\delta+\omega(I_1-\omega I_2),\qquad \psi(\omega)=J_0-J_1\omega+J_2\omega^2.$$

The coefficients are given by the formulas

$$I_1=4\int_{\omega_m}^{\omega_M}\frac{d\omega}{\omega}N(\omega),\qquad I_2=8\int_{\omega_m}^{\omega_M}\frac{d\omega}{\omega^2}N(\omega); \tag{28}$$

$$J_0=\int_{\omega_m}^{\omega_M}d\omega\,\omega^2N^2(\omega),\qquad J_1=4\int_{\omega_m}^{\omega_M}d\omega\,\omega N^2(\omega),$$

$$J_2=4\int_{\omega_M}^{\omega_M}d\omega N^2(\omega). \tag{29}$$

Knowing $F(\omega)$ in the range $\omega<\omega_m$ we can evaluate the functions

$$\Gamma_0(\omega)=\int_0^{\omega_m}d\omega_1(\omega-2\omega_1)^2F(\omega_1),\qquad \Gamma_1(\omega)=\tilde{\Gamma}(\omega)-\Gamma_0(\omega). \tag{30}$$

We can further find $F(\omega)$ in the range $\omega_m<\omega<\omega_M$ from the integral equation

$$\Gamma_1(\omega)=\int_{\omega_M}^{\omega}d\omega_1(\omega-2\omega_1)^2F(\omega_1).$$

This equation is to be solved under the condition

$$\Gamma_1(\omega_m)=0. \tag{31}$$

The solution is

$$F(\omega)=\hat{R}_1\frac{1}{\omega^2}\frac{d\Gamma_1(\omega)}{d\omega}, \tag{32}$$

where $\hat{R}_1$ is the linear operator (1.69).

The ratio of the functions $\psi(\omega)$ and $\Phi(\omega)$ is thus known for all frequencies. If we knew the functions $\psi(\omega)$ and $\Phi(\omega)$ themselves, we could find the Langmuir spectrum using one of the formulas:

$$N^2(\omega)=\frac{1}{\omega^2}\hat{R}_2\frac{d\psi}{d\omega}, \tag{33}$$

$$N(\omega)=\hat{R}_2\frac{\Phi(\omega)-\delta}{\omega}, \tag{34}$$

where $\hat{R}_2$ is the linear operator (1.70). Now these formulas enable us only to rewrite the oringinal integral equation in a different form:

$$\left[\hat{R}_2\frac{\Phi(\omega)-\delta}{\omega}\right]^2+\frac{1}{\omega^2}\hat{R}_2\frac{d}{d\omega}[F(\omega)\Phi(\omega)]=0. \tag{35}$$

As will become clear from the results, the quantity δ is important only for the evaluation of $\Gamma_1(\omega)$ and can be omitted in (34) and (35). This does, however, not greatly simplify Eq. (35). It is possible to simplify it considerably near the upper limit of the spectrum in the region

$$(\omega_M-\omega)/\omega_M\ll 1. \tag{36}$$

There

$$F(\omega)\approx F'(\omega_M-\omega),\qquad F'\equiv-\left.\frac{dF(\omega)}{d\omega}\right|_{\omega=\omega_M-0};$$

$$\hat{R}_2\approx 1,\qquad N(\omega)\approx\frac{\Phi(\omega)}{\omega}; \tag{37}$$

$$N^2(\omega)+F'\frac{d}{d\omega}\frac{\omega_M-\omega}{\omega_M}N(\omega)=0. \tag{38}$$

One can easily solve Eq. (38):

$$N(\omega)=BF'/(\omega_M-\omega+B\omega_M). \tag{39}$$

The integration constant B can, as becomes clear in what follows, be written in the form

$$B=\delta^{\xi},\qquad 0<\xi<1,$$

where ξ is a parameter which depends on ν_e/γ_0. We consider first the case

$$\xi\gg(\ln(1/\delta))^{-1}, \tag{40}$$

where $B\ll 1$. In this case the main contribution to $\psi(\omega)$ comes from a narrow vicinity of the upper limit of the spectrum and this enables us to evaluate $\psi(\omega)$ in the whole range $\omega<\omega_M$ using Eq. (39) for $N(\omega)$:

$$\psi(\omega)=\frac{(2\omega-\omega_M)^2}{\omega_M}BF'^2\left(1-\frac{B\omega_M}{\omega_M-\omega+B\omega_M}\right)[1+O(B)]. \tag{41}$$

Knowing $\psi(\omega)$ one can easily find the whole Langmuir spectrum. Clearly, now Eq. (33) is unsuitable for this purpose but in turn we can evaluate

$$\Phi(\omega)=\psi(\omega)/F(\omega)$$

and use Eq. (34). The function $N(\omega)$ found in this way depends on six parameters whose values are as yet unknown[6)]

$$\omega_m,\ \omega_M,\ B\ (\text{or}\ \ \xi),\ F',\ I_1,\ I_2.$$

They are uniquely determined by the conditions (26), (27), (28), (31), and (37).

All further calculations are appreciably simplified if we neglect terms which are small in the parameter $[\ln(1/B)]^{-1}$. In this approximation only the region (36) contributes to $(\hat{R}_2-1)\Phi(\omega)\omega^{-1}$ and this enables us at once to use Eq. (34) to evaluate $N(\omega)$:

$$N(\omega)=\frac{BF'}{\omega_M-\omega+B\omega_M}+\frac{4BF'}{\omega_M}\left(\frac{\omega_M}{\omega}\right)^{1/2}\ln\left(\frac{\omega_M-\omega}{B\omega_M}+1\right)$$
$$\times\left[\cos\left(\frac{\sqrt{7}}{2}\ln\frac{\omega_M}{\omega}\right)+\frac{1}{\sqrt{7}}\sin\left(\frac{\sqrt{7}}{2}\ln\frac{\omega_M}{\omega}\right)\right]. \qquad (42)$$

The first term in (42) is important in the region

$$(\omega_M-\omega)/\omega_M\lesssim(\ln(1/B))^{-1}.$$

It must be omitted when writing out the condition (27). As a result it follows from (27) that

$$\frac{\omega_M}{\omega_m}=\exp\left\{\frac{2}{\sqrt{7}}(\pi-\operatorname{arctg}\sqrt{7})\right\}\approx 4.31. \qquad (43)$$

Using (42) we can easily express the parameters (28) in terms of the others:

$$I_1=\frac{4\sqrt{2}BF'}{\omega_M}\left(\frac{\omega_M}{\omega_m}\right)^{1/2}\ln\frac{1}{B},$$
$$I_2\equiv\frac{1}{\omega_m}[I_1-(I_1-\omega_m I_2)]=\frac{I_1}{\omega_m}\left[1+O\left(\frac{1}{\ln B}\right)\right].$$

Turning to the calculations connected with the function $\tilde{\Gamma}(\omega)$, we restrict ourselves to the case[7] $\nu_e\ll\gamma_0$. In that case it turns out that

$$1\ll\omega_m^{5/2}\ll\gamma_0/\nu_e, \qquad (44)$$

and $\tilde{\Gamma}(\omega)$ can be simplified in the region of the spectrum:

$$\tilde{\Gamma}(\omega)=\omega^2-\frac{1}{2}\omega-\frac{\nu_e}{2\gamma_0}\omega^{7/2}.$$

The function $\Gamma_0(\omega)$ is simple from the very beginning:

$$\Gamma_0(\omega)=c_2\omega^2-c_1\omega+c_0. \qquad (45)$$

The necessary information about the coeffficients of quadratic trinomial (45) is given by the relations

$$c_2\approx\frac{\omega_M^2F'}{4\sqrt{2}}\left(\frac{\omega_m}{\omega_M}\right)^{1/2}\frac{1-\xi}{\xi},$$
$$c_0\sim c_1\omega_m\sim c_2\omega_m^2\frac{\ln\ln(1/B)}{\ln(B/\delta)}. \qquad (46)$$

Therefore, $\Gamma_1(\omega)$ has the form

$$\Gamma_1(\omega)=(1-c_2)\omega^2+\left(c_1-\frac{1}{2}\right)\omega-c_0-\frac{\nu_e}{2\gamma_0}\omega^{7/2}. \qquad (47)$$

Because of (31)

$$1-c_2=\frac{1}{2\omega_m}-\frac{c_1\omega_m-c_0}{\omega_m^2}+\frac{\nu_e}{2\gamma_0}\omega_m^{7/2}. \qquad (48)$$

The small corrections to c_2 are not always important so that it is useful to bear in mind together with (48) also the cruder condition

$$c_2\approx 1. \qquad (49)$$

Substituting (47) into (32) we can easily evaluate $F(\omega_M)$ and F', and eliminate, by using (48), the difference $1-c_2$ from the result:

$$F(\omega_M)=\sqrt{2}\left(\frac{\omega_m}{\omega_M}\right)^{1/2}\left\{\frac{1}{2\omega_m^2}-\frac{c_1\omega_m-c_0}{\omega_m}-\frac{\nu_e}{2\gamma_0}\frac{3}{11}\left(\frac{35}{2\sqrt{2}}\frac{\omega_M}{\omega_m}+8\right)\omega_m^{1/2}\right\}, \qquad (50)$$

$$F'=\sqrt{2}\left(\frac{\omega_m}{\omega_M}\right)^{1/2}\left\{\frac{1+c_1}{\omega_m^3}+\frac{\nu_e}{2\gamma_0}\frac{3}{22}\left(\frac{35}{2\sqrt{2}}\frac{\omega_M}{\omega_m}+8\right)\omega_m^{-1/2}\right\}. \qquad (51)$$

When

$$\omega_m\frac{\ln\ln(1/B)}{\ln(B/\delta)}\ll 1 \qquad (52)$$

the terms containing c_0 and c_1 are negligibly small and it follows from (26), (50), and (51) that

$$\omega_m=\left[\frac{11}{3(35\omega_M/2\sqrt{2}\omega_m+8)}\frac{\gamma_0}{\nu_e}\right]^{2/5}\approx 0.33\left(\frac{\gamma_0}{\nu_e}\right)^{2/5}, \qquad (53)$$

$$F'=5/(2\omega_m\omega_M)^{3/2}\approx 5.7(\nu_e/\gamma_0)^{6/5}. \qquad (54)$$

Combining condition (49) with Eq. (46) we can find ξ:

$$\xi=5/16\omega_M\approx 0.22(\nu_e/\gamma_0)^{2/5}. \qquad (55)$$

The assumptions (40) and (52) turn out to be equivalent and are satisfied in the region

$$\gamma_0/\nu_e\ll(\ln(1/\delta))^{5/2}. \qquad (56)$$

Knowing the Langmuir spectrum we can evaluate all the characteristics of the relaxation. For the sake of simplicity we restrict ourselves to estimates.[8]

The energy of the Langmuir waves is given by Eqs. (1.57) and (20):

$$W\sim\frac{\nu_e}{\Delta\theta^3}\delta^{\xi}\ln\frac{1}{\delta}. \qquad (57)$$

If the angular spread of the beam is not too small:

$$\ln\frac{1}{\Delta\theta}<\left(\frac{\gamma_0}{\nu_e}\right)^{2/5}, \qquad (58)$$

we can write the estimate (57) in the form

$$W\sim\frac{\nu_e}{\Delta\theta^3}\left(\frac{\nu_s}{g^2\gamma_0}\right)^{0.22(\nu_e/\gamma_0)^{2/5}}\ln\frac{g^2\gamma_0}{\nu_s}. \qquad (59)$$

According to (1.52) the angular spread of the beam in the region (58) changes as

$$\Delta\theta^5=\Delta\theta_0^5+z/l, \qquad (60)$$

where

$$l\sim\frac{l_0}{\nu_e\ln(g^2\gamma_0/\nu_s)}\left(\frac{\gamma_0}{\nu_e}\right)^{1/5}\left(\frac{g^2\gamma_0}{\nu_s}\right)^{0.22(\nu_e/\gamma_0)^{2/5}} \qquad (61)$$

is the angular relaxation length.

The fraction of energy lost by the beam is connected with the angular spread through the relation

$$\varepsilon\sim(\nu_e/\gamma_0)^{4/5}\Delta\theta^2. \qquad (62)$$

At the point $z_c\sim l$ where the instability is cut off (due to the isotropization of the beam) this fraction is

$$\varepsilon\sim(\nu_e/\gamma_0)^{4/5}. \qquad (63)$$

All estimates given here referred to the case (56). If the opposite condition holds,

$$\gamma_0/\nu_e \gg (\ln(1/\delta))^{5/2}, \tag{64}$$

the Langmuir spectrum is independent of ν_e and remains qualitatively the same as on the boundary of the regions (56) and (64). The dependence of all physical quantities on the boundary of the region (56) on δ is known so that when (64) is satisfied it is not difficult to obtain a universal equation (without any parameter) for the spectrum and to solve that equation numerically. Relations analogous to (57) to (61) are obtained from the latter by substituting $\gamma_0[\ln(1/\delta)]^{-5/2}$ for ν_e. In particular, the relation (60) is unchanged and the estimates (59) and (61) take the form

$$W\sim\frac{\gamma_0}{\Delta\theta^3}\left(\ln\frac{g^2\gamma_0}{\nu_s}\right)^{-1/2}, \quad l\sim\frac{l_0}{\gamma_0}\ln^2\frac{g^2\gamma_0}{\nu_s}. \tag{65}$$

The energy lost by the beam is given by the equation

$$\varepsilon\sim\frac{\nu_e}{\gamma_0}\left(\ln\frac{g^2\gamma_0}{\nu_s}\right)^{1/2}\Delta\theta^2.$$

We note that (65) is the same as the rough estimate in Sec. 4.1 apart from the logarithmic factor. The appearance of the latter is caused by the shift of the spectrum into the region

$$k_m\sim\omega_m^{1/2}\sim(\ln(g^2\gamma_0/\nu_s))^{1/4}\gg1$$

and by the logarithmically large contribution from the region $\omega\leqslant\omega_m$, where $\tilde{\nu}\propto\omega$, to $\Gamma(\omega)$. Indeed, knowing these facts enables us to improve the estimates of (1.29), (1.30), and Γ as follows:

$$\frac{\gamma}{k_m^2}N\sim\Gamma N', \quad \frac{\gamma}{k_m^2}N'\Delta\theta\sim\frac{\gamma_0}{k_m^3}N,$$

$$\Gamma\sim W\ln\frac{1}{\delta}\sim Wk_m^2. \tag{66}$$

The estimates (66) lead to correct expressions, i.e., expressions agreeing with the exact solutions of the problem corresponding to the case (64), for the energy, the scattering time, and the anisotropic addition to the Langmuir wave spectrum:

$$W\sim\frac{\gamma_0}{k_m^3\Delta\theta^3}, \quad \Gamma\sim\frac{\gamma_0}{k_m\Delta\theta^3}, \quad \frac{N'}{N}\sim\frac{\Delta\theta}{k_m}. \tag{67}$$

It is remarkable that the spectrum due to the shift to the region $k\gg1$ turns out to be almost isotropic even when $\Delta\theta\sim1$.

5. CONCLUSION

The results obtained above are applicable as long as the renormalization of the ion sound frequency is small and the phases of the Langmuir waves are random. The first condition is satisfied when

$$W<\omega_m, \tag{68}$$

the second is satisfied provided the reciprocal scattering time of the waves is less than the scale of the changes in frequency of the Langmuir spectrum. If the growth rate is even we must have

$$\Gamma<\omega_m B, \tag{69}$$

and in the opposite case

$$\Gamma^c<\omega_m B. \tag{70}$$

In both cases the condition that the phases be random is the more stringent one. Condition (68) can compete with it only for an even excitation in the region of the universal spectrum (64). In that case (68) is equivalent to (69) and for a smeared beam it reduces, apart from a logarithmic factor, to the inequality

$$\gamma_0<1. \tag{71}$$

Comparing the region (71) with the region where the results of Ref. 1 are applicable one establishes easily that the former is broader by a factor ν_s^{-1}. In the case where the sound is damped by the electrons in a deuterium plasma of temperature $T_e=T=20$ keV we have

$$\nu_s\sim(m_e/m_i)^{1/2}g\sim10^{-3}.$$

As for the effect of the magnetic field and of the electromagnetic waves on the relaxation, everything said at the end of Ref. 1 remains valid.

The study of the interaction of beams with a plasma that is even denser than assumed above requires going beyond the framework of weak turbulence theory. At the present time it is not known how the transition takes place to strongly turbulent relaxation regimes which, possibly, are described by a phenomenological theory of the kind considered in Ref. 7. The only thing which is clear is that the occurrence of Langmuir collapse[8] may qualitatively change the whole picture of the relaxation.

[1)] For instance, in a deuterium-tritium plasma with $T_e=10$ keV the time for the equalization of the electron and ion temperatures is close to the Lawson time.

[2)] The weak spatial inhomogeneity of the spectra which is connected with the smooth change of the beam parameters has been neglected here for the sake of simplicity. It is well known that taking it into account leads to the appearance in the kinetic equation of linear terms containing spatial derivatives of the spectral functions. These terms are unimportant for what follows for the same reasons as in Ref. 1.

[3)] The condition $\Gamma<\nu_s$ that the sound "static," which is equivalent to the inequality $\gamma_0<\nu_s\Delta\theta^3$, is then also violated for any beam with an angular spread.

[4)] The total sound energy in this jet is of the order $\Gamma\Delta\theta^{6/7}$.

[5)] Using (19) one can easily construct examples of isotropic spectra for which $\tilde{\nu}_q$ is negative in some interval of q.

[6)] The parameters (29) are already expressed in terms of the rest by virtue of (41).

[7)] If $\nu_e\gg\gamma_0$, the instability is cut off at a small angular spread of the beam and the effective heating of the plasma cannot be large.

[8)] The numerical coefficients in the following formulas can be established only by retaining small terms in the parameter $[\ln(1/B)]^{-1}$. Otherwise ξ is determined up to terms of order $\xi[\ln(1/B)]^{-1}\sim[\ln(1/\delta)]^{-1}$ while $B=\delta^\xi$ up to a factor of order unity.

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