Boundary phenomena in a helium II + solid system

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A quasiclassical approximation is constructed of the quantum analog of the dispersion equation for a roton. With the concept of a two-dimensional roton (an elementary excitation localized in helium at the interface with a solid substrate) as the basis, the contribution of rotons to the surface normal component ν is considered. It is shown that at $T > 1$ K the increment to ν, due to the quasiclassical branches of two-dimensional rotons, is comparable with the contribution of the main branch. Allowance for this increment makes it possible to reconcile the results of neutron experiments with measurements of the density of the superfluid component in He II films. In addition, a dispersion law is obtained for a surface wave of the phonon type propagating along the interface between the He II and the substrate. The possibility of decay of a surface phonon is investigated and the phonon damping coefficient at $T=0$ is found. It is proposed that the branches of the two-dimensional rotons are direct continuations of the phonon section of the surface-excitation spectrum, a hypothesis whose experimental investigation would be of interest.

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INTRODUCTION

The properties of few-dimensional systems (electrons over the surface of liquid helium, inversion layers in semiconductors, and others) are under intense investigation at present. Particular interest attaches to surface phenomena in quantum liquids. As far back as in 1966, Andreev[1] has shown that for an incontrovertible interpretation of the experiments of Ref. 2 on the surface tension of weak superfluid solutions of He^3 in He^4 it is necessary to assume the existence of a discrete surface level of an inpurity quasiparticle. Later Andreev and Kompaneets[3] introduced the concept of the surface normal component. Iguchi[4] estimated the contribution of "one-dimensional" rotons to the processes that determine the mobility of charged particles captured by vortex filaments.

Advances in experimental helium physics made it possibel to attempt to investigate the singularities of the elementary excitation spectrum of a liquid in contact with a solid. Recent experiments[5] on inelastic scattering of slow neutrons in He II that fills a finely porous medium have revealed a scattering peak at an energy $\Delta_0 \approx 0.54$ meV, much lower than the energy $\Delta \approx 0.76$ meV of a bulk roton. It was found that the low-energy maximum is connected with the presence of higher helium density near the interface with a solid substrate, and that the corresponding excitation can be identified with a two-dimensional roton. Similar conclusion were drawn by Chester *et al.*[6] from measurements of the density of the superfluid component in He II films, but to reconcile satisfactorily the two-dimensional-roton model with the thermodynamic experimental data the gap of the two-dimensional quasiparticle had to be chosen equal to 0.4 meV, much less than the value of Δ_0 given by the direct method.[5]

I have sown in an earlier paper[7] that at an interface with a solid helium has a set of discrete surface levels correspoding to two-dimensinoal rotons. The cause of such levels is that the condensed (by the Van der Waals interaction with the substrate) He II layer acts as a potential well for the roton. The form of the well is determined by the dependence of the roton parameters, mainly the gap $\Delta(\rho)$, on the local helium density $\rho = \rho_0 + \delta\rho$ (Ref. 8):

$$\Delta(z) = \Delta + (\partial\Delta/\partial\rho)_0 \delta\rho(z) = \Delta - \gamma_3/z^3, \quad \Delta \equiv \Delta(\infty). \tag{1}$$

Only the term linear in the deviation of the density from the bulk value ρ_0 is retained in (1), the explicit form is used for the correction to the density[9,10] when the distances from the plane boundary exceed a, and the notation

$$\gamma_3 \equiv \frac{\Delta}{\rho_0} \left| \frac{\partial \ln \Delta}{\partial \ln \rho} \right|_0 \frac{\hbar\overline{\omega}}{8\pi^2 c^2} \tag{2}$$

is used, where c is the speed of sound in helium; it is also taken into account that $(\partial \ln\Delta / \partial \ln \rho)_0 < 0$ (see Refs. 11 and 12). The frequency $\overline{\omega}$ can be estimated from the formula[13]

$$\overline{\omega} = \frac{\pi}{2^{3/2}} \frac{(n_{sol}^2-1)(n_{He}^2-1)}{[(n_{sol}^2+1)(n_{He}^2+1)]^{1/2}} \frac{\overline{\omega}_{sol}\overline{\omega}_{He}}{(n_{sol}^2+1)^{1/2}\overline{\omega}_{sol}+(n_{He}^2+1)^{1/2}\overline{\omega}_{He}}$$

which contains the refractive indices n_{He} and n_{sol} of the helium and solid substrate, and the characteristic frequencies $\overline{\omega}_{He}$ and $\overline{\omega}_{sol}$ in the corresponding absorption spectra. Assuming $n_{He}^2 - 1 = 0.06$, $\overline{\omega}_{He} = 3.8 \cdot 10^{16}$ sec^{-1} (Ref. 13) and $n_{sol}^2 \approx 5$, $\overline{\omega}_{sol} \approx 1.5 \cdot 10^{16}$ sec^{-1}, and substituting the remianing numerical values,[12] we obtain $\gamma_3 \approx 1.1 \Delta a^3$. The term $U_3 = -\gamma_3/z^3$ in (1) determines the energy of the roton attraction to the substrate.[1)]

In Sec. 1 we obtain in the quasiclassical approximation the wave functions and the level energies for the special case when the two-dimensional roton momentum coincides with the characteristic momentum p_0 of the roton in the bulk. The next section is devoted to calculation of the increment to the surface normal density from the quasiclassical branches. At not too low temperatures this increment turns out to be comparable with the contribution of the main branch, so that it becomes possible to reconcile the results of the experiments of Refs. 6 and 5 (Sec. 4). Finally, we obtain the surface-sound dispersion law and obtain an expression for its damping at $T=0$.

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1. QUASICLASSICAL APPROXIMATION

It is convenient to start from the roton Hamiltonian[14,4]

$$\hat{H}=\Delta+(\hat{\mathbf{p}}^2-p_0^2)^2/8\mu p_0^2+U, \tag{4}$$

which follows from the alternate representation of the spectrum of the Landau elementary excitations near the minimum; here U is the potential energy of the roton, μ is the effective mass, and $\bar{\mathbf{p}} = -i\hbar\nabla$. It can be easily seen that the momentum part of the classical analog of (3) corresponds in the region $p\approx p0$ to the traditional Landau form (see, e.g., Ref. 11), but the latter, in contrast to (3), leads to the appearance of an operator in the form $\sqrt{-\nabla^2}$, which is nonanalytic in the coordinate representation.

We consider a semi-infinite volume of He II bounded by a solid substrate (which we choose to be the xy plane). Inasmuch as in this case U depends only on the distance to the substrate, the motion of the roton along the surface (free motion with momentum $p_\parallel$) is separated from the transverse motion in the stationary Schrödinger equation for the roton wave function belonging to the eigenvalue E. Representing the wave function $\psi(z)$ in the quasiclassical form[15]

$$\psi(z)=\exp[i\sigma(z)/\hbar],\quad \sigma(z)=\sigma_0(z)-i\hbar\sigma_1(z)+\ldots,$$

we obtain (the primes denote derivatives with respect to z)

$$\begin{gathered}[(\sigma_0')^2-p_0^2+p_\parallel^2]^2=8\mu p_0^2[E-\Delta-U(z)],\\ \sigma_1=-{}^1/_2\ln\{\sigma_0'[(\sigma_0')^2-p_0^2+p_\parallel^2]\},\ldots\end{gathered} \tag{4}$$

The determination of the quasiclassical wave function of the roton and of the quantization rule is made complicated by the fact that the longitudinal momentum $p_\parallel$ enters in (4) in a nontrivial manner. The situation is simplified in the special (but realistic) case $p_\parallel = p_0$, when the calculations in the quasiclassical approximation can be carried through to conclusion. Putting $p_\parallel = p_0$ and assuming that $U(z)<0$ and $U(\infty)=0$, we can see that finite motion of the roton corresponds to $E<\Delta$, and the classical turning points are obtained from the equation

$$E-\Delta=U(z_{1,2}).$$

The left-hand turning point z_1 should be determined by the behavior of the potential at distances of the order of a from the boundary, where the macroscopic approximation ceases to hold, so that the deep levels of the two-dimensional rotons cannot be investigated within the framework of the approach described. As for weakly bound states, the corresponding classically allowed regions will substantially exceed the interatomic distance, and the exact form of the conditions imposed on the roton wave function at the interface with the substrate is in fact unimportant for the determination of the energy spectrum. For the sake of argument, we simulate the boundary $z=z_1\sim a$ by a rigid wall on which[16]

$$\psi(z_1)=0,\quad \psi'(z_1)=0. \tag{5}$$

At $p_\parallel = p_0$ the roton wave function in the classically accessible region ($z<z_2$), obtained from (4), is of the form

$$\begin{gathered}\psi(p_0;z)=[E-\Delta-U(z)]^{-3/8}\\ \times[C_1e^{g_1(z)}+C_2e^{-g_1(z)}+C_3e^{ig_1(z)}+C_4e^{-ig_1(z)}],\end{gathered} \tag{6}$$

$$g_1(z)=\frac{1}{\hbar}\int_{z_2}^{z}\{8\mu p_0^2[E-\Delta-U(z)]\}^{1/4}dz. \tag{7}$$

Analogously, in the classically inaccessible region ($z>z_2$), where the solution should attenuate in the interior of the helium ($E<\Delta$),

$$\psi(p_0;z)=[U(z)-E+\Delta]^{-3/8}\left\{A_1\exp\left[-\frac{1+i}{\sqrt{2}}g_2(z)\right]+A_2\exp\left[-\frac{1-i}{\sqrt{2}}g_2(z)\right]\right\}, \tag{8}$$

$$g_2(z)=\frac{1}{\hbar}\int_{z_2}^{z}\{8\mu p_0^2[U(z)-E+\Delta]\}^{1/4}dz. \tag{9}$$

Since the quasiclassical approximation is violated near z_2, a direct transition from (8) to (6) is impossible. The solutions can be matched together by finding, as usual,[15] the exact solution of the Schrödinger equation in the vicinity of z_2 and comparing next its asymptotic forms with (6) and (8). Assuming the potential U to vary slowly enough with the coordinate, we shall effect the matching in that vicinity of z_2 where $U(z)$ can be regarded as linear. It can then be easily shown that in the region

$$[\hbar/Ql]^{4/5}l\ll|z-z_2|\ll l \tag{10}$$

the quasiclassical approach is already applicable, and at the same time the potential still deviates little from linearity, the inequality (10), where $Q\equiv[8\mu p_0^2|U(z_2)|]^{1/4}$, and l is the distance over which $U(z)$ changes noticeably (with $Ql\gg\hbar$), is written with account taken of the quasiclassical-behavior requirement $|d(\hbar/\sigma_0')/dz|\ll1$. For the potential U_3 the condition (10) means

$$[\hbar^4/8\mu p_0^2\gamma_3]^{1/5}z_2^{4/5}\ll|z-z_2|\ll z_2. \tag{11}$$

In the vicinity of z_2, defined by inequalities (11), we have for ψ in fact the equation

$$z_0^5\psi^{IV}+(z-z_2)\psi=0,\quad z_0\equiv[\hbar^4z_2^4/24\mu p_0^2\gamma_3]^{1/5}, \tag{12}$$

which can be solved exactly[15] (see Appendix I). Matching the solution of Eq. (12) with (6) and (8) we get

$$C_1=0,\quad C_3=A_1e^{\pi i/8},\quad C_4=A_1e^{-3\pi i/8},\quad A_2=A_1e^{-\pi i/4}.$$

The remaining unknown constants A_1 and C_2 are determined by the boundary conditions (5) at the point z_1, which lead to the homogeneous system

$$\begin{gathered}C_2\exp(|g_1(z_1)|)+A_1\{\exp[i(-|g_1(z_1)|+\pi/8)]\\ +\exp[i(|g_1(z_1)|-3\pi/8)]\}=0,\\ -C_2\exp(|g_1(z_1)|)+iA_1\{\exp[i(-|g_1(z_1)|+\pi/8)]\\ -\exp[i(|g_1(z_1)|-3\pi/8)]\}=0.\end{gathered} \tag{13}$$

From the condition that the system (13) have nontrivial solutions

$$\sin(|g_1(z_1)|)=0$$

follow the quantization rules[7]

$$\int_{z_1}^{z_2} \{8\mu p_0^2[E-\Delta-U(z)]\}^{1/4}dz=\pi\hbar n,\ n \text{ is an integer.} \tag{14}$$

Substituting in (14) $U(z) = U_3(z)$, we derive for $n \ll 1$

$$E_n(p_0)=\Delta-\frac{(8\mu p_0^2)^3\gamma_3^4 I_0^{12}}{(\pi\hbar n)^{12}},$$

$$I_0\equiv\int_0^1 d\xi(\xi^{-3}-1)^{1/4}=\frac{\Gamma(5/4)\Gamma(1/12)}{3\Gamma(4/3)}\approx 3.9\,, \tag{15}$$

where the argument E_n points clearly to the case $p_{\parallel} = p_0$. Thus, at $p_{\parallel} = p_0$ the number of near-boundary discrete levels of the roton is infinite; the levels condense to the value $E = \Delta$.

Strictly speaking, at large distances (in the sense $z > \lambda_0$, see the Introduction) the attraction of the roton to the substrate is described by the potential $U_4(z) = -\gamma_4/z^4$. Despite the utterly negligible contribution of the roton states due to the interaction U_4 to any thermodynamic quantity, it is useful to note that at $p_{\parallel} = p_0$ the discrete spectrum of the two-dimensional rotons in the well $U_4(z)$ contains an infinite number of levels that condense to the value $E = \Delta$ in accord with the law

$$E_n\approx\Delta-\gamma_4\lambda_0^{-4}\exp\{-2\pi\hbar(2/\mu p_0^2\gamma_4)^{1/4}n\}.$$

We have considered so far the special case $p_{\parallel} = p_0$. In the general case the Schrödinger equation for transverse roton motion is of the form

$$\hbar^4\psi^{IV}+(p_0^2-p_{\parallel}^2)[2\hbar^2\psi''+(p_0^2-p_{\parallel}^2)\psi]=8\mu p_0^2[E-\Delta-U(z)]\psi. \tag{16}$$

At values of $p_{\parallel}$ close to p_0 it is possible to regard in (16) the terms containing $(p_0^2 - p_{\parallel}^2)$ as a perturbation. Proceeding in the usual manner[15] we obtain, accurate to $(p_0^2 - p_{\parallel}^2)$, the energy of the n-th level

$$E_n(p_{\parallel})=E_n(p_0)-\frac{p_0^2-p_{\parallel}^2}{4\mu}\frac{\hbar^2}{p_0^2}\int\left|\frac{d\psi_n(p_0;z)}{dz}\right|^2 dz, \tag{17}$$

where $\psi_n(p_0; z)$ is an eigenfunction of Eq. (16) at $p_{\parallel} = p_0$ and belongs to the value $E_n(p_0)$. It follows from (17) that $(dE_n/dp_{\parallel})|_{p_0} > 0$; the branches with different n (terms of the like symmetry) do not intersect.[15] We shall analyze the case of arbitrary $n \gg 1$. Since the spectrum $E_n(p_{\parallel})$ cannot terminate at $E > \Delta$, it is clear that at a certain $p_{\parallel} = p_{n_0} < p_0$ it should pass through a minimum. A simple investigation shows that p_{n0} is given by[7]

$$p_{n0}=p_0\left[1-\hbar^2p_0^{-2}\int|d\psi_n(p_{n0};z)/dz|^2dz\right]^{1/2}, \tag{18}$$

where $\psi_n(p_{n0};z)$ is the normalized wave function of the n-th level of the two-dimensional roton, and is a solution of Eq. (16) at $p_{\parallel} = p_{n0}$. The result (18) can be formulated in the form of the following statement: lowering the number of degrees of freedom of the roton leads to a decrease of the characteristic momentum of the roton compared with the bulk value p_0. We note that no information whatever on the features of the field acting on the roton, other than its one-dimensionality, were used in the derivation of (18).

Assume that Eq. (16) is suitble for the description of deep levels, and let p_{n0} be the momentum corresponding to the minimum of the main branch (in the sense of (18)). Assuming the eigenfunctions $\psi_n(p_{n0};z)$ and the eigenvalues $E_n(p_{00})$ of Eq. (16) to be known in the case $p_{\parallel} = p_{00}$, we obtain in second-order perturbation theory in the operator

$$\{(p_{\parallel}^2-p_{00}^2)^2-2(p_{\parallel}^2-p_{00}^2)[(p_0^2-p_{00}^2)+\hbar^2d^2/dz^2]\}/8\mu p_0^2$$

the energy $E_0(p_{\parallel})$ of the ground branch of the two-dimensional roton in the vicinity of p_{00}:

$$E_0(p_{\parallel})=\Delta_0+(p_{\parallel}^2-p_{00}^2)^2[1-I_1(p_{00})]/8\mu p_0^2,$$

where we put $\Delta_0 = E_0(p_{00})$ and

$$I_1(p_{00})=\frac{\hbar^4}{2\mu p_0^2}\sum_{n\neq 0}\frac{1}{E_n(p_{00})-\Delta_0}\times\left|\int\frac{d\psi_0^*(p_{00};z)}{dz}\frac{d\psi_n(p_{00};z)}{dz}dz\right|^2>0.$$

The effective mass of the ground branch of the two-dimensional roton

$$\mu_0=[\partial^2E_0(p_{\parallel})/\partial p_{\parallel}^2]_{p_{00}}^{-1}=\mu p_0^2\{p_{00}^2[1-I_1(p_{00})]\}^{-1}>\mu \tag{19}$$

turns out to be larger than the corresponding bulk value. We note, however, that whereas the conclusion (18) is perfectly rigorous (at least for weakly bound states), the conclusion (19) is based in fact on the assumption that Eq. (16) is valid for the description of the ground branch.

We note also the following. From Eq. (4), written in the form

$$\sigma_0'=\pm\{p_0^2-p_{\parallel}^2\pm(8\mu p_0^2[E-\Delta-U(z)])^{1/2}\}^{1/2}, \tag{20}$$

it follows that at $p_{\parallel}^2 < p_0^2$ we can obtain a solution $\psi(z)$ that decreases at large z only if $E < \Delta$. Thus, in the region $p_{\parallel} < p_0$ the spectrum of the two-dimensional rotons should lie below the line $E = \Delta$. In the case $p_{\parallel}^2 > p_0^2$, as can be easily seen from (20), the solutions that decrease as $z\to\infty$ and correspond to rotons localized at the substrate boundary are possible if

$$p_{\parallel}^2-p_0^2-[8\mu p_0^2(E-\Delta)]^{1/2}>0,$$

and the line that bounds the spectrum of the two-dimensional rotons at $p_{\parallel} > p_0$ is given by the equation

$$E=\Delta+(p_{\parallel}^2-p_0^2)^2/8\mu p_0^2.$$

2. ROTON CONTRIBUTION TO THE SURFACE NORMAL DENSITY

The quasiclassical method used above is justified for the investigation of weakly bound states of two-dimensional rotons, and owing to the rapid condensation of the levels with increasing n the sums over the discrete set in the thermodynamic functions can be replaced by appropriate integrals. As for the deep branches, they do not enter in the series (15), and their contribution must be calculated separately, choosing the required parameters by comparison with experiment.

An important quantity that describes the boundary of a superfluid liquid is the surface normal density ν introduced by Andreev and Kompaneets.[3] Generalizing their result to the case of the contribution of quasiclassical discrete levels of two-dimensional rotons, we find the corresponding increment ν_d to the normal density

$$\nu_d = -\frac{1}{2(2\pi\hbar)^3}\int dz \int dp_z \int d^2p_\parallel p_\parallel^2 \frac{\partial n_r(E)}{\partial E}, \qquad (21)$$

where n_r is the Boltzmann distribution function that depends on the classical roton energy

$$E(p, z) = \Delta + (p_\parallel^2 + p_z^2 - p_0^2)^2/8\mu p_0^2 - |U_3(z)|. \qquad (22)$$

To calculate the integral (21) it is necessary to count the number of bound states of the roton in the quasiclassical approximation. Introducing the dimensionless variables

$$q_\parallel \equiv p_\parallel/p_0,\quad q_z \equiv p_z/p_0,\quad V^2 \equiv 8\mu|U_3(z)|/p_0^2, \qquad (23)$$

and having $V^2 \ll 1$ from simple estimates with account taken of $|U_3| < \Delta$, we get from (22):

$$(q_\parallel^2 + q_z^2 - 1)^2 - V^2 = 8\mu(E-\Delta)/p_0^2. \qquad (24)$$

We consider first the case $E \geqslant \Delta$. From Eq. (4), represented in the form

$$(\sigma_0'/p_0)^2 = 1 - q_\parallel^2 \pm [8\mu(E-\Delta)/p_0^2 + V^2]^{1/2}, \qquad (25)$$

follows the condition of localization of the roton at the boundary

$$q_\parallel^2 - 1 \geqslant [8\mu(E-\Delta)/p_0^2]^{1/2}. \qquad (26)$$

Here all the σ_0' are imaginary, and we can choose a solution $\psi(z)$ that decreases at infinity. From (24) and from the requirement $E > \Delta$ we obtain

$$q_\parallel^2 + q_z^2 \geqslant 1 + V \qquad (27)$$

(the second of the obtained inequalities must be discarded, since it is incompatible with (26)). Eliminating E from (24) and (26) we arrive at the inequality

$$q_z^4 + 2(q_\parallel^2 - 1)q_z^2 - V^2 \leqslant 0,$$

whose solution is

$$0 \leqslant q_z^2 \leqslant [(1-q_\parallel^2)^2 + V^2]^{1/2} + 1 - q_\parallel^2. \qquad (28)$$

Finally from (26)–(28) we obtain the regions

$$\begin{gathered} 0 \leqslant q_z^2 \leqslant [(1-q_\parallel^2)^2 + V^2]^{1/2} + 1 - q_\parallel^2, \quad q_\parallel^2 \geqslant 1 + V, \\ 1 - q_\parallel^2 + V \leqslant q_z^2 \leqslant [(1-q_\parallel^2)^2 + V^2]^{1/2} + 1 - q_\parallel^2, \quad 1 \leqslant q_\parallel^2 \leqslant 1 + V, \end{gathered} \qquad (29)$$

corresponding to the quasiclassical energy levels of the two-dimensional rotons—to finite motion along the z axis with $E \geqslant \Delta$.

We turn now to the case $E \leqslant \Delta$. A solution $\psi(z)$ that decreases as $z \to \infty$ can now be chosen for any sign of the square root in Eq. (25). Expanding the inequality that follows from (24) at $E \lesssim \Delta$, we get

$$1 - q_\parallel^2 - V \leqslant q_z^2 \leqslant 1 - q_\parallel^2 + V,$$

from which we obtain the regions corresponding to quasiclassical localized state of the roton in the field $U_3(z)$ with $E \lesssim \Delta$:

$$\begin{gathered} 1 - q_\parallel^2 - V \leqslant q_z^2 \leqslant 1 - q_\parallel^2 + V, \quad 0 \leqslant q_\parallel^2 \leqslant 1 - V, \\ 0 \leqslant q_z^2 \leqslant 1 - q_\parallel^2 + V, \quad 1 - V \leqslant q_\parallel^2 \leqslant 1 + V. \end{gathered} \qquad (30)$$

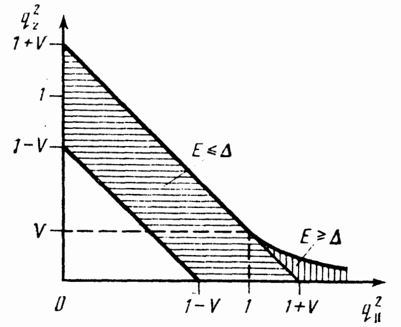

FIG. 1. Region of integration over the momenta in (32) (schematic).

Figure 1 shows the region of integration with respect to the momenta in (21). Transforming (21), we obtain

$$\nu_d = \frac{p_0^5}{8\pi^2\hbar^3 T} e^{-\Delta/T} \int_{z_1}^{\infty} dz e^{W^2} G(z), \qquad (31)$$

where

$$W^2 \equiv |U_3(z)|/T,$$

$$G(z) = \iint_{(d)} dq_z dq_\parallel q_\parallel^3 \exp[-p_0^2(q_\parallel^2 + q_z^2 - 1)^2/8\mu T] \qquad (32)$$

and the symbol (d) denotes that the integration is over the region shaded in Fig. 1 and corresponding to discrete states.

Taking the foregoing into account and separating the contribution to ν_0 from the deep levels to the surface normal density separately, we obtain

$$\nu_0 = -\frac{1}{2(2\pi\hbar)^2} \sum_i \int p_\parallel^2 \frac{\partial n_{ri}(E)}{\partial E} d^2 p_\parallel = \sum_i \frac{p_{0i}^3}{2\hbar^2}\left(\frac{\mu_i}{2\pi T}\right)^{1/2} \exp\left(-\frac{\Delta_i}{T}\right), \qquad (33)$$

where Δ_i, p_{0i}, and μ_i are the parameters of the deep-lying branches of the two-dimensional rotons.

Finally, the complete expression for ν should contain a term corresponding to the excess (on account of the field $u_3(z)$) number of rotons in the continuous spectrum. This term can be represented in the form

$$\nu_c = \frac{p_0^5}{8\pi^2\hbar^3 T} e^{-\Delta/T} \int_{z_1}^{\infty} dz (e^{W^2} - 1) H(z), \qquad (34)$$

where

$$H(z) = \iint_{(c)} dq_z dq_\parallel q_\parallel^3 \exp[-p_0^2(q_\parallel^2 + q_z^2 - 1)^2/8\mu T], \qquad (35)$$

and the integration in (35) should be over the unshaded region of Fig. 1, which corresponds to the continuous (c) spectrum.

Combining now (31), (33), and (34) and carrying out the calculation, we obtain at high temperatures, when $\gamma_3/z_1^3 < T$ (see Appendix II),

$$v=v_0+v_d+v_c=\sum_i \frac{p_{0i}^3}{2\hbar^2}\left(\frac{\mu_i}{2\pi T}\right)^{1/2}$$
$$\exp\left(-\frac{\Delta_i}{T}\right)+\frac{2p_0^4}{3\pi^2\hbar^3}\left(\frac{2\mu\gamma_3}{z_1}\right)^{1/2}$$
$$\times\left\{1+\frac{3}{8}\Gamma\left(\frac{1}{4}\right)\left(\frac{2\mu T}{p_0^2}\right)^{1/4}-\frac{2}{5}\left(\frac{8\mu\gamma_3}{p_0^2 z_1^3}\right)^{1/4}\right.$$
$$\left.+\frac{1}{8}\left(\frac{\pi\gamma_3}{T z_1^3}\right)^{1/2}\right\}\frac{1}{T}\times\exp\left(-\frac{\Delta}{T}\right). \qquad (36)$$

3. SURFACE WAVES

It is known[17] that oscillations having an acoustic spectrum can propagate over the interface between a liquid and a crystal. These oscillations have an acoustic spectrum and are damped in the interior of both the liquid and the solid. If the parameters of the liquid (the sound velocity c and the density) are much smaller than the corresponding parameters of the crystal, the surface-wave velocity is close to c and the energy of such a wave is concentrated mainly in the liquid. The distinguishing feature of helium is that such oscillations (surface sound) exist also as $T\rightarrow 0$ and constitute elementary excitations that contribute to the thermodynamic quantities. Allowance for the capillary phenomena[18] make it possible to find also the dispersion of the surface sound.

With in the volume of the solid (assumed for simplicity to be isotropic), the oscillations that led to the surface sound are described by the usual equations of elasticity theory[19]

$$\rho_{\rm sol}\,\ddot{u}_i=\partial\sigma_{ik}/\partial x_k, \qquad (37)$$

where $\rho_{\rm sol}$ is the density of the solid, $\mathbf{u}$ is the displacement vector of the medium,

$$\sigma_{ik}=\rho_{\rm sol}(c_l^2-2c_t^2)u_{jj}\delta_{ik}+2\rho_{\rm sol}c_t^2 u_{ik}$$

is the stress tensor expressed in terms of the strain tensor u_{ik}, and the velocities c_l and c_t of the longitudinal and transverse waves, connected in known manner with the hydrostatic compression and shear moduli, have been introduced. The oscillations of the helium in the volume satisfy the acoustic equation[1]

$$c^2\nabla^2P=\ddot{P}, \qquad (38)$$

and the pressure P is connected with the liquid velocity $\mathbf{v}=\Delta\varphi$ expressed in terms of the potential φ, and with the density ρ_0, by the relation

$$\nabla P=-\rho_0\dot{\mathbf{v}}.$$

Definite boundary conditions must be satisfied in the helium–solid interface. The general case of the interface of two solids was considered recently by Andreev and Kosevich.[18] On the boundary between a liquid and a solid we should have, in the linear approximation

$$\sigma_{iz}+P\delta_{iz}=\partial g_{i\beta}/\partial x_\beta. \qquad (39)$$

Here $g_{i\beta}$ is the surface-stress tensor (see Ref. 18) and the Greek subscripts denote coordinates in the plane of the boundary; the origin, as before, is chosen on the unperturbed interface and the z axis is directed in the interior of the helium. Conditions (39) were written for the case when there is no surface mass (it can be made equal to zero by suitable choice of the interface).

When account is taken of the surface stresses, as shown in Ref. 18, the z-component of the displacement vector is no longer continuous on the interface[2)] and the difference $\dot{\Delta}_z=\mathrm{v}_z-\dot{u}_z$ can be represented in the form

$$\dot{\Delta}_z=a_{z\beta\gamma}\dot{u}_{\beta\gamma}+b_{zz}\dot{\sigma}_{zz},$$

where the matrix $a_{z\beta\gamma}$, which is symmetric with respect to the last two indices, is defined by the derivatives[18]

$$a_{z\beta\gamma}=-(\partial^2\tilde{\alpha}/\partial\sigma_{zz}\partial u_{\beta\gamma})_0$$

of the thermodynamic potential $\tilde{\alpha}=\alpha-\sigma_{zz}\Delta_z$, which is expressed in terms of the surface energy α, and the essentially positive constant

$$b_{zz}=(\partial^2\alpha/\partial\sigma_{zz}\partial\sigma_{zz})_0$$

is the only (ideal liquid) nonzero component of a matrix whose general form is given in Ref. 18; the subscript zero in the last expressions means that the derivatives are taken at $\sigma_{zz}=u_{\beta\gamma}=0$.

Recognizing 18 that $g_{z\beta}=0$ and $g_{\beta\gamma}=g_{\gamma\beta}$, expressing $g_{\beta\gamma}$ in terms of σ_{zz} and $u_{\beta\gamma}$, and choosing the x axis in the wave propagation direction, we obtain ultimately the conditions on the boundary $z=0$ in the form

$$(c_l^2-2c_t^2)u_{xx}+c_l^2u_{zz}-\rho\dot{\varphi}/\rho_{\rm sol}=0,$$
$$2c_t^2u_{xz}=[h_{xxxx}/\rho_{\rm sol}-(c_l^2-2c_t^2)a_{zxx}]\,\partial u_{xx}/\partial x-c_l^2a_{zxx}\partial u_{zz}/\partial x, \qquad (40)$$
$$\partial\varphi/\partial z-\dot{u}_z=[a_{zxx}+\rho_{\rm sol}(c_l^2-2c_t^2)b_{zz}]\,\dot{u}_{xx}+\rho_{\rm sol}c_l^2b_{zz}\dot{u}_{zz},$$

where the constant is

$$h_{xxxx}=(\partial^2\tilde{\alpha}/\partial u_{xx}\partial u_{xx})_0.$$

Representing the solution as usual,[19] in the form $\mathbf{u}=\mathbf{u}_l+\mathbf{u}_t$ where

$$\operatorname{div}\mathbf{u}_t=0,\quad \operatorname{rot}\mathbf{u}_l=0,$$

and substituting (w_l and w_t are the amplitudes)

$$u_{lx}=w_lk\exp(ikx+\varkappa_{\mathbf{k}}^{(l)}z-i\omega t),$$
$$u_{lz}=-iw_l\varkappa_{\mathbf{k}}^{(l)}\exp(ikx+\varkappa_{\mathbf{k}}^{(l)}z-i\omega t),$$
$$u_{tx}=w_t\varkappa_{\mathbf{k}}^{(t)}\exp(ikx+\varkappa_{\mathbf{k}}^{(t)}z-i\omega t),$$
$$u_{tz}=-iw_tk\exp(ikx+\varkappa_{\mathbf{k}}^{(t)}z-i\omega t)$$

in the acoustic equations (37), we obtain

$$\varkappa_{\mathbf{k}}^{(l,t)}=(k^2-\omega^2/c_{l,t}^2)^{1/2}.$$

Choosing in turn

$$\varphi=\varphi_0\exp(ikx-\varkappa_{\mathbf{k}}z-i\omega t),$$

we have from (38)

$$\varkappa_{\mathbf{k}}=(k^2-\omega^2/c^2)^{1/2}.$$

The right-hand sides of (40) describe the dispersion of the surface wave, and we can put in them $\varkappa_{\mathbf{k}}^{(l,t)}=k$ and $\omega=ck$ since, as will be shown below, the surface sound velocity is $c_s\approx c\ll c_{l,t}$. As $k\rightarrow 0$ one can leave out the terms that describe the dispersion. Putting $\omega_s(k)=c_sk$ and solving the secular equation obtained from (40) by substituting the explicit forms of **u** and φ, we obtain, accurate to the principal terms,[17]

$$c_s=c(1-\varepsilon^2/2)<c, \qquad \varepsilon=\rho c^2/2\rho_{sol}c_t^2(1-c_t^2/c_l^2). \tag{41}$$

Taking into account the dispersion, we obtain

$$\omega_s(\mathbf{k})=c_s k(1+B\varepsilon^2 k), \qquad B=a_{zxx}+2\rho_{sol}c_t^2 b_{zz}. \tag{42}$$

it may turn out that $B<0$, and in this case the branch $\omega_s(k)$ in the region of small k is absolutely stable at $T=0$. If, however, $B>0$ (we recall that $b_{zz}>0$), the spectrum (42) is damped at small **k**—the surface wave becomes unstable to decay into two quanta already at $T=0$, and the situation is analogous in this case to that in the instability of a volume phonon.[20] To take into account the corresponding three-phonon processes it is necessary to retain in the Hamiltonian the principal anharmonic terms. It can be easily seen that the most substantial contribution to the perturbation Hamiltonian that describes the processes of interest to us, namely the decay of a surface phonon with $\mathbf{k}\to 0$, is made by the terms

$$\int_0^\infty dz\int d^2r_\parallel\left[\frac{1}{2}\nabla\hat{\varphi}\hat{\rho}'\nabla\hat{\varphi}+\frac{1}{6}\left(\frac{d}{d\rho}\frac{c^2}{\rho}\right)_0\hat{\rho}'^3\right], \tag{43}$$

where $r_\parallel$ is the two-dimensional radius vector in the plane of the boundary. In the derivation of (43) it was recognized that practically the entire energy of the surface phonon is concentrated in the liquid, and the nonlinear processes in the solid were neglected. The quantities in (43) have the following meaning:

$$\hat{\varphi}=\frac{1}{S^{1/2}}\sum_{\mathbf{k}}\left(\frac{\hbar c\varkappa_{\mathbf{k}}}{\rho_0 k}\right)^{1/2}\{\hat{c}_{\mathbf{k}}\exp[i[\mathbf{k}\mathbf{r}_\parallel-\omega_s(\mathbf{k})t]]$$
$$+c_{\mathbf{k}}^+\exp[-i[\mathbf{k}\mathbf{r}_\parallel-\omega_s(\mathbf{k})t]]\}\exp(-\varkappa_{\mathbf{k}}z)$$

is the velocity-potential operator and

$$\hat{\rho}'=\frac{i}{S^{1/2}}\sum_{\mathbf{k}}\left(\frac{\rho_0\hbar k\varkappa_{\mathbf{k}}}{c}\right)^{1/2}\{\hat{c}_{\mathbf{k}}\exp[i[\mathbf{k}\mathbf{r}_\parallel-\omega_s(\mathbf{k})t]]$$
$$-\hat{c}_{\mathbf{k}}^+\exp[-i[\mathbf{k}\mathbf{r}_\parallel-\omega_s(\mathbf{k})t]]\}\exp(-\varkappa_{\mathbf{k}}z)$$

is the operator of the helium-density perturbation. The operators are normalized by the condition

$$\int d^2r_\parallel\int_0^\infty dz\left[\frac{1}{2}\rho_0(\nabla\hat{\varphi})^2+\frac{c^2}{2\rho_0}\hat{\rho}'^2\right]$$
$$=\sum_{\mathbf{k}}\hbar\omega_s(\mathbf{k})\left(\hat{c}_{\mathbf{k}}^+\hat{c}_{\mathbf{k}}+\frac{1}{2}\right).$$

here S is the area of the interface; $\hat{c}_{\mathbf{k}}^+$ and $\hat{c}_{\mathbf{k}}$ are the operators for the creation and annihilation of surface-sound quanta, and wherever it cannot lead to misunderstanding we put $c_s=c$ and $\varkappa_{\mathbf{k}}=0$.

The differential probability of the decay of a surface phonon with wave vector **k** into two phonons with wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ (Ref. 20) is

$$dw=\frac{2\pi}{\hbar}|V_{fi}|^2\delta(E_f-E_i)\frac{S^2d^2k_1d^2k_2}{(2\pi)^4}. \tag{44}$$

It is expressed in terms of the transition matrix element

$$|V_{fi}|=S^{-3/2}\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)(3!/2)(2\pi)^2\hbar^{3/2}(ckk_1k_2\varkappa_{\mathbf{k}}\varkappa_{\mathbf{k}_1}\varkappa_{\mathbf{k}_2}/\rho_0)^{1/2}$$
$$\times(\varkappa_{\mathbf{k}}+\varkappa_{\mathbf{k}_1}+\varkappa_{\mathbf{k}_2})^{-1}\{1+(\rho_0^2/3c^2)[d(c^2/\rho)/d\rho]_0\},$$

where

$$E_i=\hbar\omega_s(\mathbf{k}), \qquad E_f=\hbar\omega_s(\mathbf{k}_1)+\hbar\omega_s(\mathbf{k}_2)$$

are the energies of the initial (E_i) and final (E_f) states with dispersion law (42). Upon integration of (4) the final answer is obtained precisely when (42) is used, whereas calculation with a linear dispersion law leads to divergence in the two-dimensional case (it is known that no such divergence occurs in the three-dimensional case[20]). As a result we obtain for the surface-phonon damping coefficient at $T=0$ in the case of small k

$$\Gamma=\frac{\hbar w}{2}=\frac{3\hbar^2c^2}{256\pi\rho_{sol}c_t^2(1-c_t^2/c_l^2)}\left[1+\frac{\rho_0^2}{3c^2}\left(\frac{d}{d\rho}\frac{c^2}{\rho}\right)_0\right]B^{-1/2}k^{9/2}. \tag{45}$$

4. DISCUSSION OF EXPERIMENTAL RESULTS

In the experimental investigation of the temperature dependence of the surface normal density[6] one introduces a length $L(T)$ in accordance with the rule

$$L(T)=L_0+\nu(T)/\rho_s(T), \tag{46}$$

where $\rho_s(T)$ is the bulk density of the superfluid component and L_0 is a constant, so that $L(T)-L_0$ is the so-called "healing length."[6] In the roton-temperature range, $\nu(T)$ is given by (36). Since the resultant expression $L(T)$ contains the unknown phenomenological constants p_{0i}, Δ_i, and μ_i, a complete quantitative comparison with experiment[6] is impossible.

According to neutron experiments[5] the depth, i.e., the binding energy, of the discrete level of a two-dimensional roton is ≈ 2.5 K. In Ref. 6 the experimental $L(T)-L_0$ dependence was fitted to a formula corresponding to a single term of the sum (33). This made it necessary to choose for the binding energy of the two-dimensional roton the value ≈ 4.2 K, which differs greatly from that in Ref. 5.

Expression (36) obtained above for $\nu(T)$ makes it possible in principle to reconcile the experimental results of Refs. 6 and 5. By way of estimates we confine ourselves to the assumptions that all the two-dimensional roton levels, with the sole exception of the ground one, are shallow. Although the quantities p_{00} and μ_0 that characterize the ground branch of the two-dimensional roton are unknown, some information on the relation between them can be obtained by starting from the fact that in Ref. 5 are given the energies of the two-dimensional roton for two value of the wave vector. By suitable reducing the experimental data of Ref. 5 and putting $p_{00}=p_0$, we obtain $\mu_0\approx 1.2\,\mu$ (the latter does not contradict the result (19)).

Figure 2 shows a plot of $L(T)$ obtained on the basis of Eq. (36), in which the contribution of the deep levels to $\nu(T)$ is approximated by the single term

$$[p_{00}^3\mu_0^{1/2}/2\hbar^2(2\pi T)^{1/2}]\exp(-\Delta_0/T),$$

and we put $L_0=11$ Å, $\gamma_3=1.1\Delta a^3$, $p_{00}^3\mu_0^{1/2}=1.1\,p_0^3\mu^{1/2}$, and the gap of the main branch is taken to be $\Delta_0=6.15$ K,

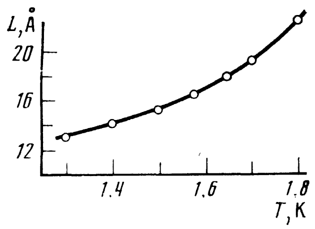

FIG. 2. $L(T)$ plotted in accord with Eqs. (46) and (36) and corresponding to inclusion of one term in ν_0. Points—experiment.[6]

which is close to the experimental value.[5] The quantity z_1 can be interpreted as the thickness of the frozen helium layer on the solid surface; in accord with the estimates,[5] we assume $z_1 \approx 2a$. Estimates show that the contribution of the shallow levels to $\nu(T)$ in the region of the "roton" temperatures is by far not small, and is predominant at $T \gtrsim 1.4$ K.

Although with the aid of the employed approximation of $\nu(T)$ it is possible to obtain satisfactory agreement with experiment at a reasonable choice of the two-dimensional roton parameters, the question of whether the ground level is the only deep one (with a binding energy of the order of the temperature in the interval $1.2 < T < 1.8$ K investigated in Ref. 6) remains open. In the presence of other levels with relatively high binding energy it would be necessary to take into account the additional terms of φ_0, and for reconciliation with an experiment of the type in Ref. 6 this would necessitate a suitable correction of the numerical values of the parameters (in particular, a decrease of $p_{00}^3/\mu_0^{1/2}$ in accord with the result of Ref. 18). It seems unlikely, however, that there are many such levels.

Finally, we call attention to the following. We have dealt so far with surface phonons and rotons as being two types of quasiparticle. One cannot exclude the possiblity that actually the surface roton branch (at least the ground one) is a direct continuation of the phonon part of the spectrum of the two-dimensional excitation (cf. the volume situation). It would be of interest to verify this in experiment.

I am grateful to A. F. Andreev for interest in the work and for valuable remarks, and to I. A. Fomin for numerous helpful discussions.

APPENDIX I

At $E < \Delta$ the equation

$$(\hbar^4/8\mu p_0^2)\psi^{IV} - \gamma_3 z^{-3}\psi - (E-\Delta)\psi = 0$$

take the form (2) in the vicinity of the classing turning point $z_2 = [\gamma_3/(\Delta - E)]^{1/2}$, where the potential can be regarded as linear. Solving (12) by the Laplace method (see Ref. 15) and changing to the dimensionless variable $\zeta = (z - z_2)/z_0$, we obtain

$$\psi(\zeta) = \text{const} \int_C \exp(\vartheta^5/5 + \zeta\vartheta)\, d\vartheta, \qquad (I.1)$$

where the ends of the contour C should go off into infinity in the shaded region of the ϑ plane in Fig. 3, where $\mathrm{Re}(\vartheta^5) < 0$.

The asymptotic form of (I.1) at large $|\zeta|$ can be obtained by the saddle-point method. At $\zeta > 0$ the extremum of the argument of the exponential in the integrand of (I.1) is

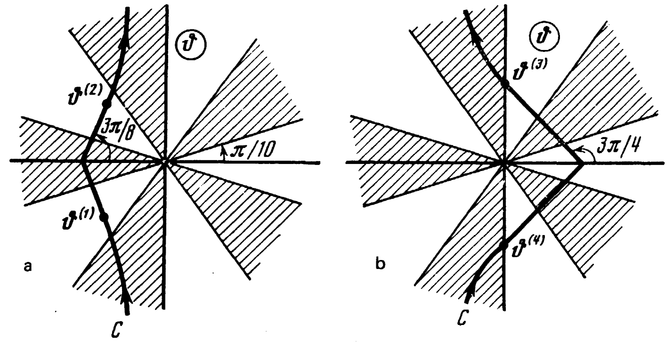

FIG. 3. Integration contour in (I.1) as $|\zeta| \to \infty$; a) $\zeta > 0$, b) $\zeta < 0$.

reached at the points $\vartheta = \pm(1 \pm i)/\zeta^{1/4}/2^{1/2}$, and the contour C should pass through two of them: $\vartheta^{(1,2)} = -(1 \pm i)\zeta^{1/4}2^{1/2}$ in the direction of the steepest descent at the respective angles $\mp 3\pi/8$ relative to the real axis (Fig. 3a). Expanding the exponent near these extremal points we obtain

$$\psi(\zeta) \approx \text{const}\Big\{\exp\Big(\frac{3\pi i}{8}\Big)\int_{-\infty} \exp\Big(-\frac{4}{5}\frac{1-i}{\sqrt{2}}\zeta^{5/4} - 2\zeta^{3/4}R^2\Big)dR$$
$$-\exp\Big(-\frac{3\pi i}{8}\Big)\int_{-\infty}^{\infty}\exp\Big(-\frac{4}{5}\frac{1+i}{\sqrt{2}}\zeta^{5/4} - 2\zeta^{3/4}R^2\Big)dR\Big\},$$

whence

$$\psi(\zeta) = i\cdot\text{const}\frac{(2\pi)^{1/2}}{\zeta^{3/8}}\exp\Big(-\frac{2^{3/2}}{5}\zeta^{5/4}\Big)\sin\Big(\frac{2^{3/2}}{5}\zeta^{5/4} + \frac{3}{8}\pi\Big),$$
$$\zeta \to \infty. \qquad (I.2)$$

(If the two other extremal points were considered, we would obtain an expression that diverges at large ζ.)

At negative ζ with large absolute values the integral (I.1) builds up mainly in the vicinity of the extremal points $\vartheta^{(3,4)} = \pm i|\zeta|^{1/4}$, where the steepest-descent lines pass respectively at angles $\pm 3\pi/4$ to the real axis (Fig. 3b). Expanding again the argument of the exponential in terms of the deviation from the extremal point and integrating, we find

$$\psi(\zeta) = i\cdot\text{const}\frac{(2\pi)^{1/2}}{|\zeta|^{3/8}}\sin\Big(\frac{4}{5}|\zeta|^{5/4} + \frac{\pi}{4}\Big). \qquad (I.3)$$

We note that we have excluded from consideration the extremal point $\vartheta = |\zeta|^{1/4}$, since its inclusion would lead to an increase of the solution as $|\zeta| \to \infty$. On the other hand, the contribution made by the extremal point $\vartheta = -|\zeta|^{1/4}$ turns out to be exponentially small compared with (I.3).

Comparing the asymptotic form of (I.3) with the solution (6) we conclude that we can put with quasiclassical accuracy

$$C_1 = 0,\quad C_3 = -\text{const}(\pi/2)^{1/2}\exp(-i\pi/4),$$
$$C_4 = \text{const}(\pi/2)^{1/2}\exp(i\pi/4); \qquad (I.4)$$

as for the constant C_2, it is of course different from zero, but its value cannot be determined from the asymptotic expansions and must be determined by the boundary conditions (5). A simple analysis of the system (13) shows that

$$C_2 \sim \text{const} \cdot \exp(-|g_1(z_1)|),$$

so that in the region where the solution (6) is matched to (I.3) the contribution made by the term with C_2 is exponentially small compared with (I.3) to the extent that the inequality $|g_1(z_1)| \gg 4/5|\xi|^{5/4}$ is satisfied. Comparison of (I.2) with (8) yields

$$A_1 = -\text{const}(\pi/2)^{1/2}\exp(-3\pi i/8),$$
$$A_2 = \text{const}(\pi/2)^{1/2}\exp(3\pi i/8), \qquad (I.5)$$

after which we arrive at the system (13) and the quantization rules (14) given in the text.

APPENDIX II

The contribution of the quasiclassical levels to the excess normal density ν can be represented in the form (see (31)–(35))

$$\nu_d + \nu_c = \frac{p_0^5}{8\pi^2\hbar^3 T} \times e^{-\Delta/T}\left\{\int_{z_1}^{\infty} dz\,(e^{W^2}-1)[G(z)+H(z)] + \int_{z_1}^{\infty} dz\,G(z)\right\}. \qquad (II.1)$$

The expression $G(z) + H(z)$ does not depend on z and can be easily determined:

$$G(z)+H(z) = 2\int_0^{\infty} dq_\parallel q_\parallel^3 \int_0^{\infty} dq_z \exp[-p_0^2(q_\parallel^2+q_z^2-1)^2/8\mu T] \approx {}^4/_3(2\pi\mu T/p_0^2)^{1/2} \qquad (II.2)$$

where, as usual, we employed the inequality $p_0^2 \gg 2\mu T$.

Introducing the dimensionless variables

$$\tau \equiv (q_\parallel^2+q_z^2-1)/V, \qquad s \equiv q_z/V^{1/2}$$

and taking the definitions of V and W into account (see (23) and (32)), we have

$$G(z) = V^{1/2}\left\{\int_{-1}^{1} d\tau e^{-W^2\tau^2}\int_0^{s_1(\tau)} ds[1+V(\tau-s^2)] + \int_1^{\infty} d\tau e^{-W^2\tau^2}\int_0^{s_2(\tau)} ds[1+V(\tau-s^2)]\right\}$$
$$= V^{1/2}\left\{\frac{2}{3\sqrt{V}}\int_{-1}^{1} d\tau e^{-W^2\tau^2}(1+V\tau)^{3/2} + \int_1^{\infty} d\tau e^{-W^2\tau^2}[\tau-(\tau^2-1)^{1/2}]^{1/2}\left[1+\frac{2}{3}V\tau+\frac{V}{3}(\tau^2-1)^{1/2}\right]\right\}, \qquad (II.3)$$

where the integration limits are

$$s_1(\tau) = (\tau+1/V)^{1/2}, \qquad s_2(\tau) = [\tau-(\tau^2-1)^{1/2}]^{1/2}.$$

At "high" temperatures $T > \gamma_3/z_1^3 \approx 1$ K we have $W^2 < 1$ for all $z > z_1$ and carrying out in (II.3) the appropriate expansion we obtain, accurate to the principal terms[3)]

$$G(z) = \frac{4}{3}V\left\{1+\frac{3}{8}\Gamma\left(\frac{1}{4}\right)\left(\frac{2\mu T}{p_0^2}\right)^{1/4} - V^{1/2}\right\}. \qquad (II.4)$$

Integrating now in (II.1) using (II.2) and (II.4), we arrive at Eq. (36) of the text.

[1)]Expression (1) is valid for z that are small compared with the wavelength $\lambda_0 \sim 1000$ Å, which is typical of the helium absorption spectrum.[9] If $z > \lambda_0$, the energy of roton attraction to the substrate is determined by $-\gamma_3 z^{-4}$, where $\gamma_4 > 0$ is a constant.[9]

[2)]We are investigating the case of an interface between a solid and an ideal liquid (helium at $T = 0$) so that no conditions need be imposed on the tangential components of the displacement.

[3)]See Ref. 21 for expressions of (II.3) in terms of special functions.

[1]A. F. Andreev, Zh. Eksp. Teor. Fiz. **50**, 1415 (1966) [Sov. Phys. JETP **23**, 939 1966)].
[2]K. R. Atkins and Y. Narahara, Phys. Rev. **138A**, 437 (1965).
[3]A. F. Andreev and D. A. Kompaneets, Pis'ma Zh. Eksp. Teor. Fiz. **13**, 720 (1971) [JETP Lett. **13**, 511 (1971)]. Zh. Eksp. Teor. Fiz. **61**, 2459 (1971)] [Sov. Phys. JETP. **34**, 1316 (1972)].
[4]I. Iguchi, Phys. Rev. **A6**, 1087 (1972).
[5]W. Thomlinson, J. A. Tarvin, and L. Passel, Phys. Rev. Lett. **44**, 266 (1980).
[6]M. Chester, G. D. Webster, E. Webster, and T. Oestereich, Phys. Rev. Lett. **45**, 464 (1980).
[7]V. N. Bondarev, Pis'ma Zh. Eksp. Teor. Fiz. **33**, 511 (1981) [JETP Lett. **33**, 495 (1981)].
[8]V. N. Bondarev, Pis'ma Zh. Eksp. Teor. Fiz. **18**, 693 (1973) [JETP Lett. **18**, 405 (1973)].
[9]I. E. Dzyaloshinskiĭ, E. M. Lifshitz, and L. P. Pitaevskiĭ, Usp. Fiz. Nauk **73**, 381 (1961) [Sov. Phys. **4**, 153 (1961)].
[10]J. Mahanty and B. W. Ninham, J. Chem. Phys. **59**, 6157 (1973).
[11]I. M. Khalatnikov, Introduction to the Theory of Superfluidity, Benjamin, 1965.
[12]R. F. Harris-Lowe and K. A. Smee, Phys. Rv. **A2**, 158 (1970). O. W. Dietrich, E. H. Graf, C. H. Huang, and L. Passell, Phys. Rev. **A5**, 1377 (1972).
[13]E. S. Sabisky and C. H. Anderson, Phys. Rev. **A7**, 790 (1973).
[14]E. M. Lifshiftz and L. P. Pitaevskiĭ, Zh. Eksp. Teor. Fiz. **33**, 535 (1957) [Sov. Phys. JETP **6**, 418 (1958)].
[15]L. D. Landau and E. M. Lifshitz, Quantum Mechanics, Nonrelativistic Theory, Pergamon, 1977.
[16]I. Iguchi, J. Low Temp. Phys. **4**, 637 (1971).
[17]L. M. Berkovskikh, Waves in Layered Media, Academia, 1960, §§22–24.
[20]E. M. Lifshitz and L. P. Pitaevskii, Statistical Physics, Part 2, Pergamon, 1980.
[21]Is. Gradshtein and I. M. Ryzhik, Tables of Integrals, Series, Sums, and Products, Academic, 1965.

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