Conservation laws and integration of the equilibrium equation of liquid crystals

R. S. Akopyan and B. Ya. Zel'dovich

Institute of Mechanics Problems, USSR Academy of Sciences
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The derivation of the equilibrium equations of the deformed state of a liquid crystal (LC) from a variational principle is considered. The invariance of the form of the free energy to translation in accord to the Noether theorem leads to conservation of the momentum flux, and invariance with respect to the rotation group leads to conservation of the angular-momentum flux. The "spin" and "orbital" momenta are separately conserved in the single-constant approximation. The conservation laws obtained can be used to derive analytic solution of a number of problems concerning the equilibrium of LC with substantially nonplanar distribution of the director: a cholestric liquid crystal (CLC) with longitudinal magnetic field and homeotropic orientation on the wall, a planar-homeotropic CLC cell, and a cell with a twisted nematic.

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1. INTRODUCTION. THE NOETHER THEOREM

The equations that determine the equilibrium configuration of the deformed state of a liquid crystal (LC) are usually obtained from a variational principle i.e., from the requirement that the free energy be a minimum. These nonlinear equations are as a rule quite complicated, and their integration is therefore extremely difficult. A search for any kind of integral for these equation is therefore most desirable. In the absence of external fields, the expression for the free-energy density is invariant to translations and rotations of three-dimensional space; in some particular cases there are also other more subtle symmetry properties.

By virtue of the known Noether theorem, a conserved quantity corresponds to each single-parameter symmetry group of the density of a Lagrangian function (its role is played in our case by the free-energy density F). We recall the corresponding expressions, see Refs. 1 and 2. Assume that we must minimize the functional

$$J=\int_G F(x_k, y_a, y_{a,k})\,dx, \tag{1}$$

possibly subject to the additional condition

$$\Phi(y_a)=0, \tag{2}$$

imposed on the independent variables $y_a=y_a(x)$. Here $x_k (k=1,...,n)$ are the coordinates, $y_{a,k}\equiv\partial y_a/\partial x_k$ and $dx=dx_1dx_2...dx_n$. The Euler-Lagrange equations take then the form

$$\frac{\partial F}{\partial y_a}+\lambda(x)\frac{\partial\Phi}{\partial y_a}-\frac{\partial}{\partial x_k}\left(\frac{\partial F}{\partial y_{a,k}}\right)=0. \tag{3}$$

Here $\lambda(x)$ is an indeterminate Lagrange multiplier obtained when account is taken of condition (2). The symbol $\partial/\partial x_k$ in (3) means

$$\frac{\partial H(x,y,y_{,m})}{\partial x_k}=\left.\frac{\partial H}{\partial x_k}\right|_{y,y_{,m}}+\frac{\partial H}{\partial y}\frac{\partial y}{\partial x_k}+\frac{\partial H}{\partial y_{,m}}\frac{\partial^2 y}{\partial x_k\partial x_m}. \tag{4}$$

Assume that under an infinitesimal transformation (as $\varepsilon\to 0$) of the form

$$x_k^*=x_k+\varepsilon\Lambda_k(x,y),\quad y_a^*(x^*)=y_a(x)+\varepsilon\Omega_a(x,y) \tag{5}$$

both densities $F(x,y,y_k)$ and Φ (7) remain unchanged in first order in ε. We get then a relation of a conservation-law type

$$\frac{\partial N_k}{\partial x_k}=0,\quad N_k=-\Lambda_kF+\sum_{j,a}\Lambda_j\frac{\partial F}{\partial y_{a,k}}y_{a,j}-\sum_a\Omega_a\frac{\partial F}{\partial y_{a,k}}. \tag{6}$$

We shall use below the general formulas for specific types of symmetry. We note specially that the function $\Phi(y_a)$ whose vanishing determines the supplementary condition is contained in Eq. (3) rather than in (6). These results can also be obtained by considering the minimum of the functional $\tilde F=F+A\Phi^2$ with a parameter A independent of the coordinates. The in the limit as $A\to\infty$ we obtain $\Phi\to\infty$, $A\Phi^2\to 0$, and $2A\Phi\to\lambda(x)$, and since $\partial\Phi/\partial y_{ak}=0$, expression (6) turns out to be independent of Φ.

We note that the conservation of the angular momentum and of the momentum in LC were used to obtain solutions in Refs. 3 and 4.

2. MOMENTUM FLUX TENSOR IN DEFORMED STATE OF A LIQUID CRYSTAL

The state of a cholesteric (CLC) or nematic (NLC) liquid crystal will be described by the three components of the director $\mathbf{n}(\mathbf{r})=(n_x,n_y,n_z)$ which are subject to the supplementary condition $\Phi(\mathbf{n})=n_x^2+n_y^2+n_z^2-1=0$; furthermore, $\mathbf{n}$ and $-\mathbf{n}$ are assumed to be indistinguishable. The free energy density (in erg/cm^3) will be written in the form[5,6]

$$F=\tfrac{1}{2}K_1(\operatorname{div}\mathbf{n})^2+\tfrac{1}{2}K_2(\mathbf{n}\operatorname{rot}\mathbf{n}+q)^2+\tfrac{1}{2}K_3[\mathbf{n}\operatorname{rot}\mathbf{n}]^2+T_{ik}n_in_k. \tag{7}$$

Here $K_{1,2,3}$ are Frank constants of dimensionality dyn, T_{ik} (in erg/cm^3) is a certain tensor that characterizes the anisotropic action of the external fields (e.g., electric or magnetic). If T_{ik} does not depend on the coordinates, the function F in (7) is invariant to translations in a Cartesian coordinate system. We can then make in the general expression (5) the substitutions

$$\varepsilon\Lambda_k\to\varepsilon_\alpha\delta_{k\alpha},\quad \varepsilon\Omega_a\to 0,$$

where $\alpha = x, y, z$ and ε_α is the vector of infinitely small translation. As a result, the "momentum flux" conservation law takes the form

$$\frac{\partial P_{k\alpha}}{\partial x_k}=0,\quad P_{k\alpha}=-F\delta_{k\alpha}+\frac{\partial n_a}{\partial x_\alpha}\frac{\partial F}{\partial(\partial n_a/\partial x_k)}. \tag{8}$$

This expression for the stress tensor turns out in the general case to be symmetric in the indices k and α. This symmetry is of no significance for the integration of the equilibrium equations. We shall therefore not bother to "symmetrize" it by adding a term of the form $\partial\psi_{\alpha k\gamma}/\partial x_\gamma$, where $\psi_{\alpha k\gamma}$ is antisymmetric in the indices k and γ (see the analogous operations in Ref. 7). An expression similar to (8) was first obtained by Ericksen[8] (see also the review[9]).

3. FLUX TENSOR OF "ORBITAL" AND "SPIN" MOMENTA

In the absence of external anisotropic fields (at $T_{ik} = 0$) the free energy (7) is invariant to the rotation group of three-dimensional space. If we put in the general formula (5)

$$\varepsilon\Lambda_k=e_{km\alpha}x_m\varphi_\alpha,\quad \varepsilon\Omega_a=e_{ab\alpha}n_b\varphi_\alpha,$$

where φ_α is an infinitesimal angle of rotation around the axis, we obtain from Eq. (6)

$$\frac{\partial M_{k\alpha}}{\partial x_k}=0,\quad M_{k\alpha}=L_{k\alpha}+S_{k\alpha}, \tag{9}$$

$$L_{k\alpha}=-Fe_{km\alpha}x_m+\frac{\partial F}{\partial(\partial n_a/\partial x_k)}\frac{\partial n_a}{\partial x_j}e_{jm\alpha}x_m, \tag{10}$$

$$S_{k\alpha}=-\frac{\partial F}{\partial(\partial n_a/\partial x_k)}e_{ab\alpha}n_b. \tag{11}$$

It is natural to call $L_{k\alpha}$ in (10) the orbital-momentum flux tensor, and S_k in (11) the spin-momentum flux tensor. In the general case when the Frank constants K_1, K_2, and K_3 are different, only the total angular momentum $M + S$ is conserved.

Interest attaches also to the special case when we can assume that $K_1 = K_2 = K_3$. In this single-constant case the free energy for nematics, i.e., at $q = 0$, can be reduced to the form (see Ref. 5)

$$F=\frac{1}{2}K\frac{\partial n_a}{\partial x_k}\frac{\partial n_a}{\partial x_k}+\frac{\partial}{\partial x_m}[A_m(\mathbf{n},n_{j,k})]. \tag{12}$$

The term in the form of an exact divergence in F can be discarded, since it does not influence the equilibrium equations, and we have then two separate symmetry groups: rotations of the vector $\mathbf{n}$ with the coordinates unchanged, and rotations of the coordinates with the $\mathbf{n}$ direction unchanged. The first corresponds to conservation of the spin momentum S separately, and the second to conservation of the orbital momentum L separately. In other words, in the single-constant approximation we have

$$K_1=K_2=K_3\to\partial S_{k\alpha}/\partial x_k=0,\quad \partial L_{k\alpha}/\partial x_k=0, \tag{13}$$

where the expressions for $S_{k\alpha}$ and $L_{k\alpha}$ coincide with (11) and (10). In particular,

$$S_{k\alpha}=-Kn_be_{ab\alpha}\partial n_a/\partial x_k. \tag{14}$$

Interest attaches also to the case when there is an external field that specifies some symmetry axis b. We then have only conservation of the projection of the angular momentum on this axis:

$$b_\alpha\partial M_{k\alpha}/\partial x_k=0. \tag{15}$$

We emphasize once more that in the general non-single-constant case the nonlinear equilibrium equations are very cumbersome. Therefore a direct check on the obtained relations (9)–(15) would be exceedingly difficult.

4. CONSERVATION LAWS IN A PLANE PARALLEL CELL WITH LC

We shall apply the results to the problem of equilibrium of an LC in a cell bounded by the planes $z = 0$ and $z = L$. We confine ourselves here to the case when the deformed state of the LC is homogeneous with respect to translations in the (x,y) plane, i.e., $\mathbf{n} = \mathbf{n}(z)$. Then $P_{xy} = P_{yx} = P_{zx} = P_{zy} = 0$, and the quantities $P_{xx}, P_{xz}, P_{yy}, P_{yz}$ do not depend on x or y. Therefore only one out of Eqs. (8) carries nontrivial information, namely $dP_{zz}/dz = 0$, whence

$$P_{zz}=\text{const}=p={}^1/_2\{(K_2+K_{32}n_z^2)(dn_a/dz)^2+K_{12}(dn_z/dz)^2-K_2q^2+\chi_aH^2n_z^2\}. \tag{16}$$

The expression $(dn_a/dz)^2$ implies summation over all three components $a = x,y,z$. We have assumed here than an external magnetic field was applied to the medium (directed, for simplicity, along the z axis, $H = e_zH$). This field introduces into the free energy an additional term $F_H = -\frac{1}{2}\chi_a(\mathbf{n}\cdot\mathbf{H})$, where χ_a is the anisotropy of the magnetic polarizability per unit volume of the liquid crystal. We note that since χ_a is small ($\chi_a \lesssim 10^{-5}$) the magnetic field $\mathbf{H}$ can be regarded as uniform even after the deformation of the LC. To abbreviate the notation we have introduced also the symbols

$$K_{32}=K_3-K_2,\quad K_{12}=K_1-K_2,\quad K_{31}=K_3-K_1, \tag{17}$$

with inequalities $K_{32} > 0$ and $K_{12} > 0$ valid for all the known LC.

We proceed now to the angular momentum. We have $L_{zz}\equiv 0$ for a strain homogeneous in x and y. Out of the three equations (9) $\partial M_{k\alpha}/\partial x_k = 0$, two equations (with $\alpha = x,y$) lead to results that coincide with (16), and in the third we have the identity

$$\partial L_{kz}/\partial x_k\equiv 0.$$

Ultimately, therefore, the zz component of the spin momentum flux is conserved (independently of the ratio of the constants K_1, K_2, and K_3)

$$S_{zz}=\text{const}=m=-(K_2+K_{32}n_z^2)\,\mathbf{n}\,\text{rot}\,\mathbf{n}-K_2q(1-n_z^2). \tag{18}$$

We shall need in what follows also the representation of the vector with the aid of the polar angles:

$$\mathbf{n}=\mathbf{e}_x\cos\varphi\sin\theta+\mathbf{e}_y\sin\varphi\sin\theta+\mathbf{e}_z\cos\theta. \tag{19}$$

The Euler-Lagrange variational equations constitute a system of two nonlinear equations containing $d^2\theta/dz^2$, $d\theta/dz$,

$d^2\varphi/dz^2$, $d\varphi/dz$ and the variables $\theta(z)$ and $\varphi(z)$ themselves. The presence of two conservations laws (16) and (18) enables us to find the solution of the problem in quadratures without writing out the equations themselves.

We shall find it convenient to use an identity that follows from the conditions $|\mathbf{n}| = 1$ and $\mathbf{n} = \mathbf{n}(z)$

$$\left(\frac{d\mathbf{n}}{dz}\frac{d\mathbf{n}}{dz}\right)=\frac{1}{1-n_z^2}\left[(\mathbf{n}\operatorname{rot}\mathbf{n})^2+\left(\frac{dn_z}{dz}\right)^2\right]. \qquad (20)$$

Because of this identity and the relations (16) and (18) we can eliminate the transverse components of the vector $\mathbf{n}$ (i.e., the azimuthal angle φ) and obtain

$$2p+K_2q^2-\chi_aH^2n_z^2$$
$$=(1-n_z^2)^{-1}\left\{\frac{[m+K_2q(1-n_z^2)]^2}{K_2+K_{32}n_z^2}+(K_1+K_{31}n_z^2)\left(\frac{dn_z}{dz}\right)^2\right\}, \qquad (21)$$

whence

$$z=\int\limits_{n_z(z=0)}^{n_z(z)} G(x)\,dx, \qquad (22a)$$

$$G(x)=[(K_1+K_{31}x^2)(K_2+K_{32}x^2)]^{1/2}$$
$$\times[(2p+K_2q^2-\chi_aH^2x^2)(1-x^2)(K_2+K_{32}x^2)-[m+K_2q(1-x^2)]^2]^{-1/2}. \qquad (22b)$$

It is more convenient to deal with the equation for the transverse components of the director by using the azimuthal angle $\varphi(z)$, from which it follows, when account is taken off the relation $\mathbf{n}\operatorname{curl}\mathbf{n} = -(1-n_z^2)d\varphi/dz$ that

$$\partial\varphi/\partial z=[m+K_2(1-n_z^2)q]/(1-n_z^2)(K_2+K_{32}n_z^2), \qquad (23)$$

and therefore

$$\varphi(z)=\varphi(0)+\int\limits_{n_z(z=0)}^{n_z(z)}\frac{m+K_2q(1-x^2)}{(1-x^2)(K_2+K_{32}x^2)}G(x)\,dx. \qquad (24)$$

The parameters m and p in (22) and (24) must be determined from the boundary conditions at $z = L$ (the conditions on the boundary $z = 0$ are already satisfied).

5. CELL WITH TWISTED NEMATIC

We consider, in the absence of a magnetic field, an NLC whose director is pinned to the boundaries at certain angles

$$\theta_0=\theta(z=0),\quad \varphi_0=\varphi(0)\ \text{and}\ \theta_L=\theta(z=L),\quad \varphi_L=\varphi(z=L).$$

If $\theta_0 = \theta_L$, the solution takes the form

$$\theta(z)=\theta_0,\quad \varphi(z)=\varphi_0+(\varphi_L-\varphi_0)z/L. \qquad (25)$$

Of course, this solution, which is valid at arbitrary K_1, K_2 and K_3, could not be obtained without the use of the conservation laws.

Let now $\theta_L \neq \theta_0$ and, for the sake of argument, $\theta_L > \theta_0$; in the opposite case it is necessary to interchange $z = 0$ and $z = L$. We consider for this problem first the single-constant approximation. The integrals (22) and (24) can then be explicitly evaluated:

$$\begin{aligned}z(2p/K)^{1/2}&=\arcsin[a\cos\theta_0]\\&\quad-\arcsin[a\cos\theta(z)],\\ \varphi(z)-\varphi(0)&=\arcsin[(a^2-1)^{1/2}\operatorname{ctg}\theta_0]\\&\quad-\arcsin[(a^2-1)^{1/2}\operatorname{ctg}\theta(z)],\\ a^2&=2Kp/(2Kp-m^2).\end{aligned} \qquad (26)$$

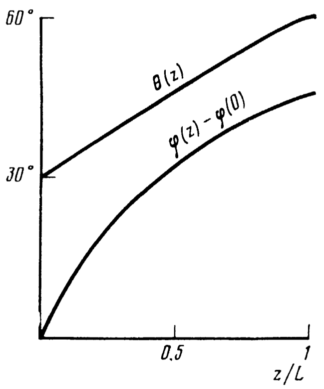

FIG. 1. Equilibrium dependence of the polar $\theta(z)$ and azimuthal $\varphi(z)-\varphi(0)$ angles in a cell with a twisted nematic: $\theta(z=0)=30°$, $\theta(z=L)=60°$, $\varphi(L)-\varphi(0)=45°$ in the single-constant approximation.

The values of p and m must be determined from the boundary conditions at $z = L$.

Figure 1 shows by way of example plots of $\theta(z)$ and $\varphi(z)$ at $\theta_0 = 30°$, $\theta_L = 60°$, $\varphi_L - \varphi_0 = 45°$. The quantities p and m as functions of the cell thickness L vary respectively like L^{-2} and L^{-1}, and for this example, at $K = 6\times10^{-7}$ dyn, their values are $p = 2\times10^{-4}$ dyn/cm^2 and $m = 7\times10^{-6}$ dyn/cm at $L = 5\times10^{-3}$ cm. We note that in the region with smaller θ the azimuthal angle φ varies more rapidly. This is quite natural, since

$$\frac{\partial\mathbf{n}}{\partial x_i}=(\mathbf{e}_y\cos\varphi-\mathbf{e}_x\sin\varphi)\frac{\partial\varphi}{\partial x_i}\sin\theta+O\left(\frac{\partial\theta}{\partial x_i}\right), \qquad (27)$$

i.e., at small θ a finite change of the azimuth φ leads to small changes of the vector $\mathbf{n}$.

Another particular case corresponds to a twisted nematic, $\varphi_0 = \varphi_L$. It can be assumed that the director vector lies in the (xz) plane. The parameter m is here zero, and the solution reduces to an incomplete elliptic integral:

$$z(2p/K_3)^{1/2}=-E(\theta_0;\ \alpha)+E(\theta(z);\ \alpha), \qquad (28)$$

$$E(\theta;\alpha)=\int\limits_0^{\theta}(1-\sin^2\alpha\sin^2\psi)^{1/2}\,d\psi,\quad \sin^2\alpha=K_{31}/K_3.$$

In the single-constant approximation Eq. (28) goes over into $\theta(z) = \theta_0 + (\theta_L - \theta_0)z/L$. A solution of the type (28) was discussed in the literature (see, e.g., Ref. 10). As shown in Ref. 10, the presence of more than one constant in this problem is slight, since usually $|K_1 - K_3|/K_3 \lesssim 0.2$. In essentially nonplanar problems with twisted LC, on the contrary, the differences $K_1 - K_2$ and $K_3 - K_2$ come into play, and are already of the order of the constant K_2 itself; in such problems the single-constant approximation should be much less effective.

6. CHOLESTERIC IN A MAGNETIC FIELD AT A HOMEOTROPIC ORIENTATION ON THE WALLS

The problem of a cholesteric with homeotropic pinning of the director at the walls, i.e., with boundary conditions

$$\mathbf{n}(z=0)=\mathbf{n}(z=L)=\mathbf{e}_z, \qquad (29)$$

was considered in Refs. 4 and 11. Here we discuss those changes which arise when account is taken of an external magnetic field **H** directed along the same z axis. As before, we confine ourselves to solutions in which **n** depends only on the coordinate z. The boundary condition (29), satisfied on at least one of the walls, predetermines the vanishing of the angular-momentum transfer: $M_{zz} = m = 0$.

Before we proceed to discuss the general solution (22), (24) we note that under the boundary conditions (29) there always exists a solution in the form $\mathbf{n}(z) \equiv \mathbf{e}_z$. In the absence of a magnetic field, however, and at a cell thickness $L > \pi q^{-1} K_3/K_2$, this solution is unstable. The manifestation of such an instability was named in Ref. 11 a zero-field Fréedericksz transition. The threshold of this transition changes in the presence of a magnetic field $\mathbf{H} = H\mathbf{e}_z$.

Examination of the linearized equation for small perturbations of $\theta(z,t)$ analogous to that carried out in Refs. 4 and 11, yields a solution in the form

$$\theta(z,t) = \sum_{m=1}^{\infty} \theta_m \sin\frac{\pi m z}{L} \exp(\Gamma_m t),$$
$$\Gamma_m = \frac{K_3}{\eta}\left[\left(\frac{K_2}{K_3}q\right)^2 - \frac{\chi_a H^2}{K_3} - \left(\frac{m\pi}{L}\right)^2\right]. \tag{30}$$

We have introduced in (30) an orientation-relaxation constant η with the dimension of poise. Thus, the magnetic field stabilizes the homeotropic structure, at $\chi_a > 0$ and destabilizes it at $\chi_a < 0$. The instability manifests itself first in the mode with $m = 1$, and the threshold of the Fréedericksz transition is determined by the relation $\Gamma_1 = 0$.

In the single-constant approximation it follows from (24), with allowance for $m = 0$, that

$$\varphi(z) - \varphi(z=0) = qz$$

independently of the behavior of $\theta(z)$ and of the magnetic field. For $\theta(z)$, Eq. (22) yields an explicit relation in the form of an elliptic integral:

$$\int_0^{\theta(z)} \frac{d\psi}{(\sin^2\theta_m - \sin^2\psi)^{1/2}} = \left(q^2 - \frac{\chi_a H^2}{K}\right)^{1/2} z. \tag{31}$$

To determine the above-threshold stationary value $\theta_{max} = \theta(z = L/2)$ we must solve the equation

$$\int_0^{\theta_m} \frac{d\psi}{(\sin^2\theta_m - \sin^2\psi)^{1/2}} = \frac{L}{2}\left(q^2 - \frac{\chi_a H^2}{K}\right)^{1/2}. \tag{32}$$

In particular, at a slight excess above threshold we have

$$\sin^2\theta_m = \frac{4}{\pi}\left\{\left[(qL)^2 - \frac{\chi_a H^2}{K}L^2\right]^{1/2} - \pi\right\}. \tag{33}$$

The general solution in the non-single-constant approximation is given by the integrals (2) and (24). It makes it possible in principle to calculate the stationary above-threshold distribution of the director. The corresponding analysis is very sensitive to the numerical values of the ratios K_{32}/K_3, K_{31}/K_3, and will therefore not be presented here. We note only that here, just as in the absence of a magnetic field, hysteresis of the Fréedericksz phenomenon is possible (cf. Ref. 4).

7. CELL IN COMBINED ELECTRIC AND MAGNETIC FIELDS

Let an electric and a magnetic field be applied to the cell. The latter we write in the form $\mathbf{E} = -\operatorname{grad}\psi(\mathbf{r})$, where $\psi(\mathbf{r})$ is the potential, and we regard the quantity $\psi(\mathbf{r})$ as an independent field when deriving the variational equations. The supplementary free-energy term connected with the field **E** is of the form

$$F_E = -(8\pi)^{-1}\varepsilon_{ik}(\mathbf{r})(\partial\psi/\partial x_i)(\partial\psi/\partial x_k),$$
$$\varepsilon_{ik}(\mathbf{r}) = \varepsilon_\perp\delta_{ik} + \varepsilon_a n_i(\mathbf{r}) n_k(\mathbf{r}). \tag{34}$$

Here ε_a is the anisotropy of the dielectric constant, $\varepsilon_a = \varepsilon_\parallel - \varepsilon_\perp$. The boundary conditions for ψ are of the form

$$\psi(z=0, x, y) = \psi_1, \quad \psi(z=L, x, y) = \psi_1 + V,$$

where V is the voltage applied to the cell from an external source. We note first of all that in the absence of free charges the free energy F is invariant t the transformations $u(\mathbf{r}) \to \psi(\mathbf{r}) + \text{const}$. Therefore the Euler-Lagrange equations that follow from (34), when varied with respect to $\psi(\mathbf{r})$

$$\frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial(\partial\psi/\partial x_i)}\right) = 0 \to \operatorname{div}\mathbf{D}(\mathbf{r}) = 0,$$
$$D_i(\mathbf{r}) = -\varepsilon_{ik}(\mathbf{r})\frac{\partial\psi(\mathbf{r})}{\partial x_k}, \tag{35}$$

can be simultaneously be regarded also as conservation equations of the Noether-theorem type. The equation $\operatorname{div} D(\mathbf{r}) = 0$ under the condition $\mathbf{E} = -\nabla\psi$ is the fundamental equation of electrostatics, which we now obtain from the same variational principle.

For the problem homogeneous in the transverse coordinates $\psi(\mathbf{r}) \equiv \psi(z)$ we have $E(\mathbf{r}) = \mathbf{e}_z E(z)$. Owing to possible distortions of the director, however, the induction $\mathbf{D}(z)$ has generally speaking all three Cartesian coordinates.

The equation $\operatorname{div}\mathbf{D} = 0$ yields in this case one condition: $D_z(z) = D_0 = \text{const}$; in terms of the potential $\psi(z)$ it takes the form

$$\varepsilon_{zz}\frac{d\psi}{dz} = \frac{d\psi}{dz}(\varepsilon_\perp + \varepsilon_a n_z^2(z)) = D_0 = \text{const}. \tag{36}$$

The momentum flux density acquires an additional term of the type of the Maxwell stress tensor, so that on the whole

$$P_{zz} = P_{zz}|_{E=0} - \frac{1}{8\pi}[\varepsilon_\perp + \varepsilon_a n_z^2(z)]\left(\frac{\partial\psi}{\partial z}\right)^2$$
$$= P_{zz}|_{E=0} - \frac{1}{8\pi}\frac{D_0^2}{\varepsilon_\perp + \varepsilon_a n_z^2(z)} = p = \text{const}. \tag{37}$$

Expression (18) for the momentum flux M_{zz} does not change when an electric field directed along the z axis is turned on. Using the conservation laws (37) and (38) we easily obtain a solution for $\theta(z)$ and $\varphi(z)$. It takes the form (22), (24), in which p must be replaced by $p + (D_0^2/8\pi)\times(\varepsilon_\perp + \varepsilon_a n_z^2(z))$. We emphasize that the foregoing analysis takes full account of the distortion that the deformation (in general, not planar) of the director produces in the electric field inside the cell.

If initially (i.e., without the electric and magnetic fields) the configuration corresponds to a twist-cell, i.e., $\theta(z) \equiv \pi/2$, $\varphi(L) - \varphi(0) = \Delta\varphi$, application of the fields **E** and **H** can cause a transition of the Fréedericksz type. The threshold of this transition is determined by the conditioned

$$A=\chi_a H^2+\frac{\varepsilon_a}{4\pi}\left(\frac{V}{L}\right)^2-2K_2q\frac{\Delta\varphi}{L}-(K_3-2K_2)\left(\frac{\Delta\varphi}{L}\right)^2-K_1\left(\frac{\pi}{L}\right)^2=0.$$

In the particular cases $V=q=0, H\neq 0$ or $H=q=0$, $V\neq 0$ this expression coincides with the published results (see Refs. 6, 12, and 13).

At a small excess above threshold, i.e., at $0<AL^2/K_1\pi^2\ll 1$, an explicit expression can be obtained for $\alpha_m=\pi/2-\theta(z=L/2)$. In this case we have in a stationary above-threshold structure

$$\varphi(z)=\Delta\varphi\frac{z}{L}+o(\alpha_m^2),\quad \theta(z)=\frac{\pi}{2}-\alpha_m\sin\frac{\pi z}{L}+o(\alpha_m^3)$$

and α_m is given by

$$\alpha_m\approx\pm(AB)^{1/2}, \tag{38a}$$

$$B^{-1}=\frac{1}{2}K_1^{-1}\left\{K_3\chi_a H^2+\frac{\varepsilon_a V^2}{4\pi L^2}\left(K_3+3K_1\frac{\varepsilon_a}{\varepsilon_\perp}\right)-2[K_1K_3-2K_1K_2+K_2K_3]q\frac{\Delta\varphi}{L}-[K_1K_2^{-1}K_3^2-K_1K_3+11K_1K_2+K_3^2-2K_2K_3]\left(\frac{\Delta\varphi}{L}\right)^2\right\}. \tag{38b}$$

In other words, α_m is proportional to the square root of the relative excess above the transition threshold.

8. HOMEOTROPICALLY PLANAR CHOLESTERIC

Let a cholesteric liquid crystal be poured into a cell whose one plane ($z=0$) imposes rigidly a homeotropic orientation, $\mathbf{n}(z=0)=\mathbf{e}_z$, and the other ($z=L$) maintains rigidly a planar orientation, $\mathbf{n}(z=L)=\mathbf{e}_x$. In the absence of an electric and of a magnetic field, $\theta(z)$ is given by the integral (22) with $H=0$ and $m=0$; the latter follows from the homeotropic pinning on the $z=0$ wall. The constant p is determined from the condition $\theta(z=L)=\pi/2$, and the course of $\varphi(z)$ is given by the equation

$$d\varphi/dz=K_2q/(K_2+K_{32}\cos^2\theta(z)). \tag{39}$$

At $qL\gg 1$, i.e., for a "thick cell," we have the asymptotic relations

$$qz\ll 1\to\theta(z)\approx\left(\frac{K_2}{K_3}\right)^{1/2}qz,\quad \varphi(z)\approx\frac{K_2}{K_3}qz,$$
$$qz\gg 1\to\varphi(z)\approx(z-L)q, \tag{40}$$

$$\theta(z)\approx\frac{\pi}{2}-4\sqrt{2}C\exp\left\{-qL\left(\frac{K_3}{K_1}\right)^{1/2}\right\}\operatorname{sh}\left\{q(L-z)\left(\frac{K_3}{K_1}\right)^{1/2}\right\}. \tag{41}$$

The constant C stands here for the definite integral

$$C=\exp\left\{\frac{K_3}{(K_1K_2)^{1/2}}\int_0^1\ln(1-x)\frac{df}{dx}dx\right\},$$

$$f^2(x)=\frac{(K_3-K_{31}x^2)(K_3-K_{32}x^2)}{K_3^2(1+x)^2}. \tag{42}$$

Just as in the case of an orientation homeotropic on both walls (see Ref. 4), the deviation from $\theta=0$ at the middle of the cell is exponentially small, $\propto\exp\{-0.5qL(K_3/K_1)^{1/2}\}$.

The case $qL\ll 1$ is equivalent to the problem of a homeotropically invariant nematic (see above).

Thus, the use of the Noether theorem to obtain the conservation law not only helps to better understand the qualitative picture of the equilibrium deformation of a liquid crystal, but also to obtain in a large number of cases an analytic solution of the problem. The most useful is the application of the Noether theorem to problems in which the director does not lie in some single plane.

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