Theory of the passage of particles and waves through randomly inhomogeneous media

I. M. Lifshitz, S. A. Gredeskul, and L. A. Pastur

S. I. Vavilov Institute of Physics Problems, Academy of Sciences of the USSR, Moscow, and Physicotechnical Institute of Low Temperatures, Academy of Sciences of the Ukrainian SSR, Kharkov
(Submitted 28 July 1982)
Zh. Eksp. Teor. Fiz. **83**, 2362–2376 (December 1982)

A study is made of the statistical properties of the transparency of a disordered layer. The asymptotic behavior of the transparency of a thick layer is investigated in quasi-one-dimensional and three-dimensional cases: in particular, a formula is obtained for the relationship between the self-averaging decrement of the transparency and the asymptotic value of the transparency probability density of separate filaments forming a quasi-one-dimensional layer. Expansions of the decrement are obtained in powers of the density of scatterers for different positions of the incident particle energy relative to the spectrum of an infinite disordered system.

PACS numbers: 05.60. + w, 61.80. − x

The problem of the tunnel penetration of particles across a homogeneous potential barrier with randomly distributed scattering centers (impurities) was considered in Ref. 1. In the present paper, which is related to the treatment in Ref. 1, we shall first analyze the statistical properties of some of the fundamental quantities characterizing the tunnel passage (and the transparency throughout the spectrum in the quasi-one-dimensional case) and, second, develop a regular method for expanding the decrements of various characteristics of the passage in powers of the scatterer density.[1]

1. INTRODUCTION

We shall consider the incidence (normal, for the sake of simplicity) of a flux of particles of unit intensity and of energy E on a layer of a disordered medium of thickness L and of cross-sectional area S. This process is described by the Schrödinger equation with a potential $U(\mathbf{r})$ which differs from zero only in the region $0\leqslant x\leqslant L$ occupied by the layer. A homogeneous layer corresponds to the potential $U_0 = \text{const}$ and in the simplest case the disorder can be regarded as generated by a random function of the type

$$v(\mathbf{r})=\sum_{0\leqslant x_j\leqslant L} u(\mathbf{r}-\mathbf{r}_j), \tag{1.1}$$

where the points $\mathbf{r}_j$ are distributed randomly with a density n in the layer and $u(\mathbf{r})$ is the potential of a single impurity (although many of the facts given below are valid also in the case of potentials of more general form). Therefore, the total potential modeling such a layer is

$$U(\mathbf{r})=U_0+v(\mathbf{r}). \tag{1.2}$$

Our task will be to investigate the asymptotic (corresponding to large L) properties of the transparency σ_L of a layer, defined as the ratio of the particles transmitted by the layer to the number of the incident particles:

$$\sigma_L=\frac{1}{S}\int j\,dS, \tag{1.3}$$

where j denotes the x component of the flux density emerging from the layer. We shall consider two cases: three-dimensional and quasione-dimensional when the layer consists of $M = Sb^{-2}$ separate filaments whose transverse size is b and the mean free path representing jumps from one filament to another is large compared with L. This situation is possible only because of the strong anisotropy of the effective mass. In the quasi-one-dimensional case the layer transparency σ_L [Eq. (1.3)] is

$$\sigma_L=\frac{1}{M}\sum_{i=1}^{M} D_{Li}, \tag{1.4}$$

where D_{Li} is the transmission coefficient of a single filament corresponding to the potential

$$U(x)=U_0+\sum_{0\leqslant x_j\leqslant L} u(x-x_j), \quad 0\leqslant x\leqslant L, \tag{1.5}$$

and x_j are the coordinates of the impurities incident on a filament.

In the theory of disordered systems the characteristics that have been investigated much more thoroughly are those which are additive functions of the dimensions.[2] The main distinguishing feature of such characteristics is that after division by the relevant dimensions of a system, they become valid also in the macroscopic limit, i.e., these characteristics are self-averaging. The integral occurring in the definition of the transparency (1.3) is clearly an additive function of the transverse dimensions of the investigated layer and, therefore, for a fixed layer thickness L and for an area S tending to infinity, the transparency becomes self-averaging.

The most usual is a situation corresponding to the passage through a three-dimensional layer of particles of energy $E > E_c$, where E_c is the mobility edge (threshold). In this case the process is described by a transport equation which gives reliable characteristics of the passage when a natural condition $S \gg L^2$, is obeyed and $\langle\sigma_L\rangle$ is inversely proportional to the layer thickness.[3]

For $E < E_c$ in the three-dimensional case and also for all energies in the quasi-one-dimensional case, all the states in a disordered system are localized. For this reason the transparency decreases exponentially on increase in the thickness L and its average value $\langle\sigma_L\rangle$ is obtained, as shown below, only for values of S which are exponentially large compared with L.

In the quasi-one-dimensional situation the effective decrement of the transparency (1.4)

$$\gamma_{\rm eff}=-L^{-1}\ln\sigma_L \tag{1.6}$$

depends on the relationsip between L and S. In the limiting cases this decrement can be expressed in terms of the characteristics of a single filament i.e., in terms of the decrement of the transmission coefficient D_L

$$\bar{\gamma}=-\lim L^{-1}\ln D_L \tag{1.7}$$

or in terms of the decrement of the average transmission coefficient[2]

$$\gamma_D=-\lim L^{-1}\ln\langle D_L\rangle. \tag{1.8}$$

The decrement $\bar{\gamma}$ has the property of self-averaging[4] and is related by the expression

$$\bar{\gamma}=2\gamma_\psi \tag{1.9}$$

to another self-averaging quantity, very thoroughly investigated in Refs. 5 and 2, which is the growth increment γ_ψ of a wave function with a fixed (at some point) logarithmic derivative:

$$\gamma_\psi=\lim_{x\to\infty}(2x)^{-1}\ln[\psi^2(x)+|E|^{-1}\psi'^2(x)]. \tag{1.10}$$

This quantity is an important characteristic of one-dimensional disordered systems because it is identical with the reciprocal of the localization radius of a wave function.

We shall show below [see Eq. (2.15)] that in the case of large values of L and S the effective transparency decrement of a quasi-one-dimensional layer $\gamma_{\rm eff}$ given by Eq. (1.6) is also a self-averaging quantity. The limiting value γ_q of this quantity is given by the following relationships only when the parameter q is either sufficiently small or sufficiently large[3]:

$$q=L^{-1}\ln(S/b^2),\quad S=b^2e^{qL}, \tag{1.11}$$

which are identical with $\bar{\gamma}$ and γ_d, respectively. In the general case of an arbitrary q, the decrement γ_q satisfies the inequality

$$\gamma_D\leqslant\gamma_q\leqslant\bar{\gamma} \tag{1.12}$$

and can be found with the aid of Eq. (2.16) which expresses this quantity in terms of the probability density of the transmission coefficient (2.12). In particular, it follows from Eq. (2.16) that γ_q is a monotonically decreasing function of q.

These properties of the process of passage are discussed in detail in § 2 (subsections 1 and 2), whereas in § 3 (subsections 1 and 2) we shall obtain expansions of γ_D and $\bar{\gamma}$ in terms of the density of scatters for various positions of the energy E of the incident particles relative to the spectrum of an infinite disordered system with the potential (1.5). In the case when $E_c > E > E_g$, where E_g is the lower boundary of the spectrum of the corresponding infinite disordered system, we can expect the so-called resonance tunneling.[1] Intermediate asymptotes for $\langle\sigma_L\rangle$ in the vicinity of local one-center level are obtained in Ref. 1. In § 2.3 we shall give similar asymptotes for the values of E close to E_g. However, if $E<E_g$, it follows from the classification of Ref. 1 that nonresonance tunneling takes place. In this case, because of the strong subbarrier damping, the three-dimensional problem is largely of one-dimensional nature and, therefore, it is meaningful to calculate the decrement of the average transparency of a three-dimensional layer:

$$\gamma_\sigma=-\lim_{L\to\infty}L^{-1}\ln\langle\sigma_L\rangle. \tag{1.13}$$

Such a calculation is carried out in § in 3.3.

To conclude this section, we shall mention that since the problem of the passage of particles across a layer of a randomly inhomogeneous meidum is, from the point of view of the formalism employed, very close to the corresponding wave problem, it follows that although we shall use only the quantum-mechanical terminology, all the main ideas and conclusions will apply also in the wave case.

2. STATISTICAL PROPERTIES OF THE TRANSPARENCY

1. We shall begin with the following simple example, already considered in Ref. 2. We shall assume that in a filament of length L there are N_L randomly distributed point scatterers [for these scatterers the potential $u(x)$ in Eq. (1.5) is $k_0\delta(x)$]. The transmission coefficient of one impurity is

$$D=(1+k_0^2/4k^2)^{-1},\quad k^2=E. \tag{2.1}$$

Therefore, in the simplest possible approximation when multiple scattering is ignored, the transparency σ_L (which, in the one-dimensional case, is identical with the transmission coefficient D_L) is

$$D_L=D^{N_L}. \tag{2.2}$$

Since the probability that $N_L=N$ in the one-dimensional case should be in the form

$$e^{-n_1L}(n_1L)^N/N!, \tag{2.3}$$

where n_1 is the one-dimensional density of the scatterers, it follows that as $L\to\infty$ we have $N_L\to\infty$ for most of the configurations of the impurity points. However, this means that for almost all the configurations of impurities the transmission coefficient D_L approaches zero exponentially as $L\to\infty$ and the corresponding logarithmic decrement is a nonrandom quantity

$$\bar{\gamma}=-\lim_{L\to\infty}L^{-1}\ln D_L=-n_1\ln D. \tag{2.4}$$

Next, since in this case we have $\langle D_L\rangle=\exp[-n_1L(1-D)]$, it follows that

$$\gamma_D=-\lim_{L\to\infty}L^{-1}\ln\langle D_L\rangle=n_1(1-D). \tag{2.5}$$

The difference between the right-hand side of these two asymptotic formulas illustrates the nonself-averaging nature of the transparency in the one-dimensional case.

The properties of the transparency of a single filament demonstrated in the simplest approximation of independent scatterers, as described by Eqs. (2.1)–(2.3), are retained also in the general case. For example, for any one-dimensional random potential with a finite radius of the statistical correlations in each realization the transparency of a filament decreases exponentially on increase in the filament length and the decrement $\bar{\gamma}$ of Eq. (1.7) is an asymptotically valid (i.e., a self-averaging) quantity.

The filament transparency $\langle D_L \rangle$ averaged over the realizations of a random potential with a finite correlation radius also decreases exponentially on increase in the thickness of the layer for any energy of the incident particle.[3] However, the corresponding logarithmic decrement of Eq. (1.8), which is always smaller than γ_D,

$$\gamma_D<\bar{\gamma}, \tag{2.6}$$

can be calculated only in a few cases.[4] For example, in the range of energies

$$k^{-3}\int B(x)\,dx\ll 1,\quad |k^2-\langle U(x)\rangle|\ll\langle U(x)\rangle, \tag{2.7}$$

where $B(x)=\langle U(x)U(0)\rangle-\langle U(x)\rangle^2$ is the correlation function of a random potential, it is given by[6,7]

$$\gamma_D=\frac{1}{16k^2}\int B(x)\,dx. \tag{2.8}$$

Since in this region, which is "quasiclassical" for a potential in the form of Gaussian white noise,[5] it follows from Ref. 2 that

$$\gamma_\Phi=\frac{1}{8k^2}\int B(x)\,dx,$$

then the relationship (2.8) can be rewritten in the form

$$\gamma_D={}^1/_2\gamma_\Phi.$$

Comparing this relationship with the general formula (1.9), we can see that in the range (2.7) the logarithmic decrement of the average transparency of a filament γ_D is four times less than the transparency decrement of a filament for typical realizations $\bar{\gamma}$.

Another case in which it is possible to indicate a method for approximate calcuation of γ_D is considered in Ref. 1 and in §3. It applies to the potential of Eq. (1.5) in the subbarrier region $k^2<U_0$ and the calculation is carried out utilizing the low density of the scattering centers.

2. We shall now consider a quasionedimensional layer of area $S=\mathrm{Mb}^2$. Here the *rms* deviation δ from the transparency of (1.4) considered in the approximation of single scattering is–according to Eqs. (2.2) and (2.3):

$$\delta\approx M^{-1/2}\exp\left[{}^1/_2 n_1 L(1-D)^2\right],\quad M=e^{qL}. \tag{2.9}$$

The small value of this parameter, i.e., the inequality $q>n_1(1-D)^2$, is the condition for the formation of the average transparency of a single filament and, consequently, the condition that for most realizations we have $\gamma_{\rm eff}\approx\gamma_D$ in the example under discussion. It follows from Eq. (2.9) that these conclusions are valid if the number of filaments in a layer (and, therefore, it cross-section area) are exponentially large compared with its thickness. A "wave field" at the exit from such a quasi-one-dimensional layer should be basically in the form of a dark background with very rare and bright flashes at those points where filaments with representative but untypical[2] numbers of scatterers are located [in the example described Eqs. (2.2)–(2.3), which is under consideration here, these numbers are $n_1L(1-D)/|\ln D|$ and n_1L].

On the other hand, if the number of filaments in a layer is not exponentially large, then with a probability tending to unity we can expect $L^{-1}\ln\sigma_L$ to be identical with the logarithm of each term in the sum (1.4) divided by L, i.e., in this case we can expect $\gamma_{\rm eff}\approx\bar{\gamma}$ with the same probability as before.

We shall now analyze $\gamma_{\rm eff}$ of Eq. (1.6) for high values of L and arbitrary q in the relationship (1.11) between L and S. We shall assume that

$$\ln D_{Li}=-L\gamma_i,\quad \gamma_i=\bar{\gamma}-\xi_i,\quad \langle\xi_i\rangle=0,\quad \xi\geqslant\bar{\gamma} \tag{2.10}$$

and rewrite Eq. (1.4) in the form

$$\sigma_L=e^{-L\bar{\gamma}}\frac{1}{M}\sum_{i=1}^{M}e^{L\xi_i}. \tag{2.11}$$

The random quantities ξ_i are statistically independent (because the configurations of impurities on different filaments are independent) and their total probability density $p_L(\xi)$ can be represented in the form

$$p_L(\xi)=Ae^{-L\varphi(\xi)},\quad \varphi(0)=0,\quad A=[L\varphi''(0)/2\pi]^{1/2}, \tag{2.12}$$

where A is the normalization constant. This representation is valid for any random potential with a finite correlation radius when $\xi<\bar{\gamma}$, which can be proved (for example) by the method of Ref. 4. It should be noted that in discussing the resonance effects (see Ref. 1 and subsection 3 below) we find that the main role is played by the configurations characterized by $\sigma\approx 1$, and $\gamma\sim L^{-1}$. Therefore, although a representation of the (2.12) type does apply, this is due to a completely different reason than in the $\xi<\bar{\gamma}$ case. The function $\varphi(\xi)$ is an analog of entropy in statistical physics and, like entropy it is a convex function of ξ, i.e., we have $\varphi''\geqslant 0$.

For the model of Eqs. (2.1)–(2.3) we have

$$\varphi(\xi)=n_1\tilde{\varphi}(\xi/\bar{\gamma}),\quad \tilde{\varphi}(t)=t+(1-t)\ln(1-t),\quad t<1. \tag{2.13}$$

In the range of energies defined by the inequalities in Eq. (2.7) we find that for any random potential with a finite radius of the statistical correlations the function $\varphi(\xi)$ has the universal form:

$$\varphi(\xi)=\xi^2/4\bar{\gamma},\quad \xi<\bar{\gamma}, \tag{2.14}$$

which can be confirmed using the results of Refs. 6 and 2.

We shall show that the quantity $\gamma_{\rm eff}$ becomes reliable for any fixed value of q and large values of L, i.e.,

$$\gamma_{\rm eff}\,|_{L\to\infty}=\langle\gamma_{\rm eff}\rangle|_{L\to\infty}\equiv\gamma_q, \tag{2.15}$$

and its limiting value γ_q has the form

$$\gamma_q=\bar{\gamma}+\min_{\varphi(\xi)\leqslant q}\{\varphi(\xi)-\xi\}. \tag{2.16}$$

Using the properties of the function $\varphi(\xi)$, we can rewrite the above formula as follows:

$$\gamma_q=\bar{\gamma}+q-\xi_0(q),\quad q\leqslant q_1, \tag{2.17a}$$

$$\gamma_q=\gamma_D,\quad q\geqslant q_1, \tag{2.17b}$$

where $\xi_0(q)>0$ is a solution of the equation $\varphi(\xi_0)=q$, $q_1=\varphi(\xi_1)$, and ξ_1 is a point where the function $\varphi(\xi)-\xi$ has a more minimum, i.e., where $\varphi'(\xi_1)=1$.

It is clear from Eq. (2.17) that if q is the smallest parameter with the dimensions of the reciprocal length, then $\gamma_q\approx\bar{\gamma}$. Conversely, if $q>q_1$, then a minimum in Eq. (2.16) coincides with a maximum over the whole range $\xi\leqslant\bar{\gamma}$, i.e., with γ_D.

In the approximation of single scattering described by Eqs. (2.1)–(2.3) the decrements $\bar{\gamma}$ and γ_D have the form of Eqs. (2.4) and (2.5); moreover, according to Eq. (2.13), we have $q_1=n_1(1-D+D\ln D)$ and for $q\ll n_1$, the quantity $n_1\ln D$ is characterized by $\gamma_q\approx\gamma_D$.

In the case by described by Eq. (2.14) a minimum in Eq. (2.16) is always reached at the edge of the interval, i.e., at the point $\xi_0(q)$, and it is of the form

$$\gamma_q=\bar{\gamma}(1-\sqrt{q/\bar{\gamma}})^2,\quad q\leqslant\bar{\gamma}/4,$$
$$\gamma_q=\bar{\gamma}/4,\quad q>\bar{\gamma}/4.$$

Here γ_q approaches a constant value $\gamma_D=\bar{\gamma}/4$ when $q_1=\varphi(\bar{\gamma})=\bar{\gamma}/4$, whereas for $q\ll\bar{\gamma}$, we have $\gamma_q\approx\bar{\gamma}$.

The validity of Eqs. (2.15)–(2.17) follows essentially from the following simple considerations. Using Eqs. (2.10) and (2.11), we shall write down $\gamma_{\rm eff}$ in the form

$$\gamma_{\rm eff}=\bar{\gamma}-\frac{1}{L}\ln\int\limits_{-\infty}^{\bar{\gamma}}e^{L\xi}\frac{1}{M}n(\xi)\,d\xi, \tag{2.18}$$

where

$$n(\xi)=\sum_{i=1}^{M}\delta(\xi-\xi_i). \tag{2.19}$$

It is clear that at high values of L the function $\eta(\xi)$ can be replaced with $n_a(\xi)$ smoothed out over any fixed interval a. Then,

$$\langle n_a(\xi)\rangle=M\int\delta_a(\xi-\xi')p_L(\xi')\,d\xi'\approx e^{L(q-\varphi(\xi))},$$

$$(\langle n_a{}^2\rangle-\langle n_a\rangle^2)/\langle n_a\rangle^2\approx\langle n_a\rangle^{-1}\approx e^{-L(q-\varphi(\xi))}.$$

It means that if $\varphi(\xi)\leqslant 9$, then the average value of the function $n_a(\xi)$ is exponentially large and its relative fluctuations are correspondingly small. In the additional interval where $\varphi(\xi)>q$, the average $\langle m_a(\xi)\rangle$ is exponentially small. Therefore, the integral in Eqs. (2.18) has the asymptotic value

$$\int\limits_{\varphi(\xi)\leqslant q}\exp\{L(\xi-\varphi(\xi))\}\,d\xi,$$

and this is clearly equivalent to Eqs (2.15) and (2.16). We shall now estimate the order of magnitude of fluctuations of $\gamma_{\rm eff}$. Let us assume that $q<q_1$; we shall postulate that

$$N(t)=\int\limits_{\xi_0+t}^{\bar{\gamma}}n(\xi)\,d\xi,$$

e the quantity ξ_0 is described by the condition $\langle N(0)\rangle=1$. It follows from Eqs. (2.12) and (2.19) that this definition yields $\varphi(\xi_0)=q+0(\ln L/L)$, i.e., in the asymptotic limit it is identical with the above definition $\varphi(\xi_0)=q$ making it more precise for the subsequent calculations. We shall use $-\eta/L$ to denote the difference between $\gamma_{\rm eff}-\bar{\gamma}$ and the quantity

$$-L^{-1}\ln\left(\frac{1}{M}\int\limits_{-\infty}^{\xi_0}e^{L\xi}\langle n(\xi)\rangle d\xi\right)=q-\xi_0-\frac{1}{L}\ln\frac{\lambda}{1-\lambda},$$

$$\lambda=\varphi'(\xi_0)<1,$$

which is asymptotically identical with Eq. (2.17a). Then, using as the integration variable the function $\langle N(t)\rangle$, we shall represent η in the following form:

$$\eta=\ln\left[1+\frac{1-\lambda}{\lambda}\int\limits_0^1 x^{-1/\lambda}\frac{dN}{dx}dx+\frac{1-\lambda}{\lambda}\int\limits_1^\infty x^{-1/\lambda}\left(\frac{dN}{dx}-1\right)dx\right], \tag{2.20}$$

$$\langle N(t)\rangle\equiv x,\quad\langle dN/dx\rangle-1=0.$$

We shall denote the integrals in the above expressions by J_1 and J_2, respectively. We can readily show that if

$$\langle N(t_1)\rangle=x_1,\quad\langle N(t_2)\rangle=x_2>x_1,$$

then the probability that in the interval $(\xi_0+t_1,\ \xi_0+t_2)$ there are N values of ξ_i, i.e., that the function $N(x_1,x_2)=N(t_2)-N(t_1)$ assumes the value N, is

$$(x_2-x_1)^N e^{-(x_2-x_1)}/N!.$$

The integrals J_1 and J_2 correspond to the different ranges of ξ and are therefore independent. In the calculation of $\langle\eta\rangle,\langle\eta\rangle^2,\ldots$ we can average over J_1 for a fixed value of J_2 and in the result so obtained we can allow for fluctuations of J_2. It is readily shown that the dispersion is $\sigma J_2\sim 1$ and the other of $\delta\eta$ is determined by the averaging over J_1. The above averaging procedure rapidly converges in respect of N and, therefore, we can estimate $\delta\eta$ simply by considering the case $N(0,1)=1$. A random value of J_1 in terms of the variable ξ is $J_1=e^{L\xi}$, where $\xi_0<\xi\leqslant\bar{\gamma}$. Going back to the variable x, we find that

$$\langle\eta\rangle=\int\limits_0^1\ln\left(1+J_2+\frac{1-\lambda}{\lambda}x^{-1/\lambda}\right)dx\sim\lambda^{-1},$$

and similarly that $\delta\eta\sim\lambda^{-1}$. Therefore, $\langle\eta^2\rangle\approx 2\lambda^{-2}$ and

$$\delta\gamma_{\rm eff}=\delta\eta/L\sim[L\varphi'(\xi_0)]^{-1}. \tag{2.21}$$

This estimates changes in two cases. If $q>q_1$, fluctuations of $\delta\gamma_{\rm eff}$ are exponentially small and in the simplest case when $q_1=\varphi(\bar{\gamma})$, they are of the order of $q\lesssim L^{-1}$. If $q\lesssim L^{-1}$, we then find that $\delta g_{\rm eff}\sim(L\varphi''(\xi_0))^{-1/2}$, i.e., there is a continuous transition to the case of a single filament ($qL\ll 1$, $M=1$, $\gamma_{\rm eff}=\bar{\gamma}-\xi$), when an estimate of this kind is a direct consequence of Eq. (2.12) for the probability density $p_L(\xi)$.

3. As shown in Ref. 1, in an analysis of the tunneling it is necessary to distinguish two cases: nonresonance and resonance. The resonance case is encountered when the energy of incident particle lies within the spectrum of the corresponding infinite system, but below the mobility edge ($E_c>E>E_g$)

it is then found that at every energy there are low-probability configurations for which the transparency is close to unity. However, it is these configurations that dominate the contribution to the average transparency because in the case of typical configurations the transparency is extremely low and, therefore, the probabilities of resonance configurations and their energy widths determine $\langle\sigma_L\rangle$.

The vicinity of a one-impurity local level where, on the one hand, the resonance effects appear most strongly and, on the other, the wave functions corresponding to the resonance configurations have a clear structure, belongs to this range and it is considered in Ref. 1. The configurations in question correspond to fairly symmetric distributions of impurities so that the amplitudes of the wave function of a particle are identical at the exit and entry to the layer. However, the considerations used here can also be applied to other parts of a discrete spectrum.

We have to distinguish here two cases. In the first the density of the states $\rho(E)$ is a smooth function in the vicinity of the energy under consideration. In this case an expansion of the optimal parameter with a resonance level E of width $(\delta E)_L \lesssim e^{-\alpha L}$, where $\alpha^2 = U_0 - E$, near the center of the layer again gives $\sigma_2 \sim 1$. The probability of this even is

$$W(E)\sim C(E)\,e^{-C(E)L}, \tag{2.22}$$

where $C(E) = \rho(E)(\delta E)_L \sim \rho(E)e^{-\alpha L}$ is the concentration of the resonant clusters.

Such a situation occurs in the fluctuation region of the spectrum. Here, $\rho(E)$ obtained in Refs. 8 and 2 has the form

$$\rho(E)\approx\exp\left[-\lambda n_d(E-E_g)^{-d/2}\right], \tag{2.23}$$

where d is the dimensionality of space and the factor λ is expressed in terms of the effective mass of a particle moving in an infinite medium where all the scatterers are located at the minimum distance a. We find from Eqs. (2.16)–(2.18) that the contribution made to $\langle\sigma_L\rangle$ by the resonance configurations of the fluctuation type is

$$\sigma_I\approx\exp\left[-\lambda n_d(E-E_g)^{-d/2}-\alpha L\right]. \tag{2.24}$$

The contribution of the other (nonresonance) configurations is already a smooth function and, therefore, in the case of a low density it can be obtained in the form of a cluster expansion of its powers (see Ref. 2 and §3). Denoting this contribution of σ_{II}, we find that

$$\sigma_{II}=\exp(-\gamma_\sigma^{(d)}L), \tag{2.25}$$

where

$$\gamma_\sigma^{(d)}=\gamma_\sigma \quad \text{for} \quad d=3,$$

$$\gamma_\sigma^{(d)}=\gamma_D \quad \text{for} \quad d=1.$$

We then find in the (E, L) plane a line which is defined by

$$\lambda n_d(E-E_g)^{-d/2}+\alpha L=\gamma_\sigma^{(d)}L \tag{2.26}$$

and which divides in this plane two regions, in one of which the average transparency for sufficiently high values of L has the form given by Eq. (2.24), whereas in the other it is given by Eq. (2.25) in accordance with the nature of the configurations (the resonance or nonresonance) that determine $\langle\sigma_L\rangle$. It is clear from Eqs. (2.26), (3.6), and (3.12) that at low concentrations of impurities we find that Eq. (2.19) is valid in the case of sufficient thicknesses of the layer if the energy $E > E_g$ is fixed, and at energies not too close to E_g, it is valid if the layer thickness L varies within specified limits.

The contribution of multiple resonances is found to be negligible. In fact, in the case of a chain of N clusters the resonance width is, as in Ref. 1, $(\delta E)_{L,N} \sim e^{-\alpha L/N}$ and, instead of Eq. (2.22), we now have

$$\frac{C^N}{N!}\exp(-CL)=\frac{\rho^N}{N!}\exp(-CL-\alpha L).$$

The factor $\rho^N/N!$ which is then obtained ensures the smallness of the corresponding contribution compared with that already under discussion.

The second possible case of resonance tunneling occurs at those energies E_0 for which the density of states has a sufficiently sharp peak of integrated amplitude $C_0 = C(E_0)$ and of width δE_0. As in the former case, C_0 can be regarded as the density of the corresponding resonance levels, but the value of δE_0 is now not determined by the length of a filament and the number of centers in a resonance chain but by the structure of the density of states in the vicinity of E_0. [For example, in the case of a peak due to two impurities located at the minimum distance a, we have $C_0 \sim n^2a^{-3d}$ and according to Ref. 2 δE_0 is of the order of $\delta E_0 \propto \exp(-2k_0 n^{-1/d}$, where n is the impurity concentration and $k_0^2 = |E_0|$.] The condition for the sharpness of a peak is the inequality

$$\delta E_0\ll e^{-\alpha L/N}, \quad N\sim\left[\frac{\alpha L}{|\ln(n/a^d)|}\right]^{1/2},$$

where on the right we have–according to Ref. 1–the energy width of a resonance at which N is the number of impurities in an optimal chain. In the example of a two-center level mentioned above, this inequality becomes

$$2k_0n^{-1/d}>\left[\alpha L\ln(n/a^d)\right]^{1/2}.$$

As in the case of Eq. (2.25), variation of L at low values of N causes "switching" from one intermediate asymptote to another.

The upper limit to the resonance energies in the three-dimensional case is located in the vicinity of the mobility edge E_c, which separates the range of high energies where the wave functions extend over the whole crystal, from the region below E_c where they are localized at impurity atom clusters or, more commonly, at some more or less local fluctuations of the random potential. The localization radius R_c of the states (correlation radius of the wave function) becomes infinite at $E = E_c$. In accordance with the percolation scaling concepts near the mobility edge we have $R_c \propto (E_c - E) - \nu$, where $\nu > 0$. On the other hand, it is quite natural to assume that the damping length of the incident wave in the range $E < E_c$ should also be R_c. Therefore, near E_c but in the range $E < E_c$, we have the relationship

$$-L^{-1}\ln\langle\sigma_L\rangle\propto(E_c-E)^{\nu},\quad L\gg R_c. \tag{2.27}$$

It follows from similar considerations, at $E<E_c$ and for sufficiently large thicknesses when the diffusion coefficient can be used, that

$$\langle\sigma_L\rangle\propto L^{-1}(E-E_c)^{\Delta}. \tag{2.28}$$

When the impurity concentration is increased, a region of delocalized states may appear near the center of the impurity energy band formed from the individual impurity levels. The boundary of this region in the (n, E) plane is given by the curve $n=n_k(E)$ (see Ref. 2), which also corresponds to becoming infinite:

$$R_c\propto[n_k(E)-n]^{-\nu_1},\quad \nu_1>0.$$

Therefore, formulas analogous to Eqs. (2.27) and (2.28) should apply also in this situation, but it is then more natural to express them in the form

$$-L^{-1}\ln\langle\sigma_L\rangle\propto[n_k(E)-n]^{\nu_1},$$
$$\langle\sigma_L\rangle\propto L^{-1}[n-n_k(E)]^{\Delta_1}.$$

3. EXPANSION OF TRANSPARENCY DECREMENTS IN POWERS OF THE DENSITY OF SCATTERERS

1. We shall consider first a quasione-dimensional system. The corresponding three-dimensional formulas are obtained analogously and will be given in subsection 3. As in §2, we shal assume that all the scatterers are point-like, i.e., we shall take $u(x)$ in Eq. (1.5) in the form $u(x)=k_0\delta(x)$.

Since the quasione-dimensional case corresponds to a layer in the form of a bundle of thin filaments with an independent dynamics of particles on each of them, the wave function $\psi(\mathbf{r})$ at $\mathbf{r}=\mathbf{r}_L=\mathbf{r}|_{x=L}$ is governed by the coordinates $\mathbf{r}_1...\mathbf{r}_N$ of only those particles which are on a filament terminating at the point $\mathbf{r}_L$. We shall assume that the impurities are distributed continuously and homogeneously throughout the layer (nonresonance passage is then possible if the impurities are repulsive with $k_0>0$ and $E<U_0$) and we shall write down the transparency in the form

$$\langle\sigma_L\rangle=\sum_{m=0}^{N}C_N^m\frac{(V-V_1)^{N-m}}{V^N}\int\limits_{V_1}\sigma(\mathbf{r}_1,\ldots,\mathbf{r}_m)\,d\mathbf{r}_1\ldots d\mathbf{r}_m,$$

$$V_1=Lb^2. \tag{3.1}$$

Hence, it follow that if $M=Sb^{-2}\gg1$,

$$\langle\sigma_L\rangle=\langle D_L\rangle=\sum_{m=0}^{\infty}\frac{n_1^m}{m!}e^{-n_1L}\int\limits_0^L D_L(x_1,\ldots,x_m)\,dx_1\ldots dx_m, \tag{3.2}$$

where $n_1=nb^2$ is the average number of scatterers per unit length of the filament; $D_L(x_1,\ldots,x_m)$ is the transmission coefficient of a one-dimensional barrier with m impurities at the points $x_1,\ldots x_m$. Expanding Eq. (3.2) in powers of n_1, taking logarithms of the resultant series, and reexpanding again in powers of n_1, we obtain

$$\gamma_D=-L^{-1}\ln D_L^{(0)}-n_1L^{-1}\int\limits_0^L\left(\frac{D_L(x_1)}{D_L^{(0)}}-1\right)dx$$
$$-\frac{n_1^2}{2}L^{-1}\iint\limits_0^L\left(\frac{D_L(x_1,x_2)}{D_L^{(0)}}-\frac{D_L(x_1)D_L(x_2)}{D_L^{(0)2}}\right)dx_1\,dx_2+\ldots. \tag{3.3}$$

Using the representation (1.10) from Ref. 1, we can readily show that if $\alpha L\gg1$, then

$$D_L^{(0)}=e^{-2\alpha L},\quad D_L(x)/D_L^{(0)}=(1-\mu_1)^2,\quad D_L(x_1,x_2)/D_L^{(0)}=(1-\mu_1)^4(1-\mu_1^2e^{-2\alpha|x_2-x_1|})^{-2}, \tag{3.4}$$

where $\mu_1=k_0(2\alpha+k_0)^{-1}$ is the "subbarrier" scattering amplitude for one center. Therefore, we finally obtain:[6]

$$\gamma_D=2\alpha+n_1\mu_1(2-\mu_1)-n_1^2S_2+\ldots, \tag{3.5}$$
$$S_2=\mu_1^2(1-\mu_1)^4\int\limits_0^\infty\frac{e^{-2\alpha x}(2-\mu_1^2e^{-2\alpha x})}{(1-\mu_1^2e^{-2\alpha x})^2}dx$$
$$=\frac{(1-\mu_1)^4}{2\alpha}\ln(1-\mu_1)^2-\frac{\mu_1^2(1-\mu_1)^3}{2\alpha(1+\mu_1)}.$$

The method used to obtain the expansion (3.5) is analogous to that employed in the derivation of the virial expansion in statistical physics[9]; it allows us also to find higher terms. This method is equally suitable for the discrete case when impurities can occupy with a probability c_1 the sites in a regular lattice with a period a (and even in a more general case when impurities cannot approach one another to distances shorter than a certain minimum). In this case the nonresonance energies lie below the boundary of the spectrum of the periodic system in which all the sites are occupied by scatterers and the corresponding expansion has the form

$$\gamma_D=2\alpha+\frac{c_1}{\alpha a^2}\mu_1(2-\mu_1)+\frac{c_1^2}{2a}\mu_1^2(1-\mu_1)^2-\frac{c_1^2}{a}S_2+\ldots,$$
$$S_2=\mu_1^2(1-\mu_1)^4\sum_{m=1}^{\infty}\frac{e^{-2\alpha am}(2-\mu_1^2e^{-2\alpha am})}{(1-\mu_1^2e^{-2\alpha am})^2}. \tag{3.6}$$

The additional (compared with the continuous case) term of the order of c_1^2 appears because two impurities cannot occupy the same point in the lattice.

We can similarly obtain an expansion in terms of the concentration and of the value of $\bar{\gamma}$ from Eq. (1.8) which–according to Eq. (2.17)–is a decrement of σ_L for small areas S. We then have to apply the above considerations to the expression $L^{-1}\ln[\psi^2(L)+\alpha^{-2}\psi'^2(L)]$, which on the strength of Eqs. (1.9) and (1.10) is asymptotically identical with $\bar{\gamma}$. As a result we obtain:[7]

$$\bar{\gamma}=2\alpha-n_1\ln(1-\mu_1)^2+n_1\bar{S}_2+\ldots, \tag{3.7}$$
$$\bar{S}_2=2\int\limits_0^\infty\ln(1-\mu_1^2e^{-2\alpha x})\,dx.$$

The quantity $\bar{\gamma}$ can be expanded in powers of the density also in these cases when the energy of the incident particles lies within the spectrum of an infinite disordered system with a potential described by Eq. (1.5). In particular, for a continuous distribution of attractive point scatterers when $E<U_0$

(subbarrier resonance region according to the terminology of Ref. 1), the formula for $\bar{\gamma}$ differs from that given by Eq. (3.7) only because in the expression for the coefficient $\overline{S}_2$ in the integrand there should be a logarithm of the modulus since in the region under discussion $\mu_1 = |k_0| (|k_0| - 2\alpha)^{-1}$ may be greater than unity. This expansion becomes meaningless[8] only for $|\mu_1| \gg 1$, i.e., in the direct vicinity of a local level $(\alpha \approx |k_0|/2)$. The cause of this are resonance effects in the passage process, which are discussed in Ref. 1 and in §2.3.

2. An expansion in powers of the density in the investigated case of statistically independent point scatterers can also be obtained in the suprabarrier region $E > U_0$. For simplicity, we shall consider the repulsion case with $k_0 > 0$. The attraction case corresponds to some modification of the intermediate formulas and to the substitution $k_0 \to |k_0|$ in the final results given by Eqs. (3.8) and (3.9).

It is clear from the meaning of $\bar{\gamma}(E)$ that in the range $E > 0$ this quantity represents the real part of the analytic continuation of $\bar{\gamma}(E)$ from the subbarrier nonresonance region $E < 0$ (we recall that at this stage we are assuming that $U_0 = 0$) and the doubled number of states $2N(E)$ is the imaginary part of the same continuation. In the expansion coefficients of Eq. (3.7) such a continuation is made by replacing α with $-ik$, for which μ_1 becomes $k_0/(k_0 - 2ik)$. After this substitution and separation of the real and imaginary parts in the resultant expressions, we obtain the following formulas:

$$\bar{\gamma}(E) = n_1 \left[\ln \left(1 + \frac{k_0^2}{4k^2}\right) + \frac{n_1}{k} \overline{B}_2 + O\left(\frac{n_1^2}{k^2}\right) \right], \quad E = k^2 > 0, \tag{3.8}$$

$$\overline{B}_2 = 2 \int_0^{n_1/k} \left(\varkappa \operatorname{arctg} \frac{\varkappa + n_1/k}{2} + \ln \left[\left(\varkappa + \frac{n_1}{k} \right)^2 + 4 \right] \frac{d\varkappa}{\varkappa^2 + 4} \right.$$
$$\left. - 2 \operatorname{arctg} \frac{n_1}{2k} \ln \frac{4 + n_1^2/k^2}{2} \right);$$

$$N(E) = \frac{k}{\pi} \left[1 - \frac{n_1}{k} \operatorname{arctg} \frac{k_0}{k} + O\left(\frac{n_1^2}{k^2}\right) \right]. \tag{3.9}$$

In the Appendix we shall describe a different metod for obtaining these expansions and enabling definite conclusions to be made about the rate and radius of their convergence.

We note that $\bar{\gamma}$ obtained from Eq. (3.8) is a small quantity. However, the cases when $\bar{\gamma}$ is small are the most interesting. In fact, in such cases the thickness of the layer can be very large and the transparency for typical realizations $\exp(-\bar{\gamma}L)$ need not be very small. The parameter $q = L^{-1} \ln(S/b^2)$ of Eq. (1.11) is small because the cross-sectional area is in reality finite. However, it then follows from the general formula (2.17) that γ_{eff} is essentially identical with $\bar{\gamma}$.

A question may arise whether it is possible to expand the decrement of the average transparency $\gamma_D(E)$ in powers of the density in the suprabarrier region. In attempting to obtain such an expansion by a method which yields Eq. (3.5), we find that integrals of the same type as in the above formula appear in the expression for the coefficient S_2. However, if $E > 0$, these integrals diverge because instead of the damped exponential functions, we now have oscillatory ones. Nor can such an expansion cannot be obtained by the analytic continuation method because $\gamma_D(E)$ is determined by the mean-square value of the modulus of the wave function and, therefore, does not have such good analytic properties as $\bar{\gamma}(E)$. The reason for these formal difficulties is the possibility of the appearance of characteristic resonance situations[1] in the case when the energy of the incident particles lies within the spectrum of an infinite disordered system with a potential given by Eq. (1.5), so that the dependence of $\bar{\gamma}(E)$ on the density of scatterers in the $E > 0$ range is not generally analytic.

3. We shall now consider the three-dimensional case and assume, as before, that impurities are point-like. It is known that in the three-dimensional case this means that they are attractive and the amplitude u is related to the radius r_0 by the relationship $u \propto r_0^{-2}$, where r_0 is the smallest parameter with the dimensions of length in our problem. In view of the attractive nature of impurities, a nonresonance range of energies exists (as in the one-dimensional case) only when impurities from a lattice gas (in the more general case, they cannot approach each other to distances smaller than a certain fixed value). Following essentially the same procedure as in the derivation of Eq. (3.3), we find that an expansion of the damping decrement of the average transparency is

$$\gamma_\sigma = -L^{-1} \ln \sigma_L^{(0)} - cL^{-1} \sum_{0 \leqslant x_j \leqslant L} \left(\frac{\sigma_L(\mathbf{r}_j)}{\sigma_L^{(0)}} - 1 \right)$$
$$- \frac{c^2}{2} L^{-1} \sum_{0 \leqslant x_j \leqslant L} \left(\frac{\sigma_L(\mathbf{r}_j)}{\sigma_L^{(0)}} - 1 \right)^2 \tag{3.10}$$
$$- \frac{c^2}{2} L^{-1} \sum_{0 \leqslant x_i \neq x_j \leqslant L} \left(\frac{\sigma_L(\mathbf{r}_i, \mathbf{r}_j)}{\sigma_L^{(0)}} - \frac{\sigma_L(\mathbf{r}_i)\sigma_L(\mathbf{r}_j)}{\sigma_L^{(0)2}} \right) + \ldots,$$

where c is the impurity concentration.

As shown in Ref. 1, in calculating the quantities $\sigma_L(\mathbf{r}_1,\ldots,\mathbf{r}_m)$ in the case when $\alpha L \gg 1$, we can ignore the waves scattered from the boundaries $x = 0$ and L on the internal part of the barrier, i.e., we may assume that the boundaries of a homogeneous barrier are moved away from the layer to infinite distances on each side. Bearing this point in mind [see Eqs. (4.4) and (4.6) in Ref. 1], we find that the coefficient of c in Eq. (10) is

$$\mu/\alpha a^3, \tag{3.11}$$

where μ is the amplitude of subbarrier scattering by one center in the three-dimensional case, introduced in Ref. 1. This much simpler (than in the one-dimensional case) expression is obtained here because the term of the second power in μ, containing the sum over the layer of $h^2(\mathbf{r})$, where $h(\mathbf{r}) = r^{-1} e^{-\alpha r}$ is the Green function of the three-dimensional Schrödinger equation for $U(\mathbf{r}) = 0$ and $E = -\alpha^2$, is found–because of the more rapid decay of this function in the three-dimensional case, i.e., because we now have $r^{-1} e^{\alpha r}$ instead of $e^{-\alpha|x|}$–to be of the order of $\ln(\alpha L)$ and not αL. For the same reason the first of the terms of the order of c^2 in Eq.

(3.10) due to the lattice nature of the distribution of impurities is also of the order of ln (αL). Therefore, in the three-dimensional case the corrections to the "virial" expansion of γ_σ in powers of c are of the order of $(\alpha L)^{-1}\ln(\alpha L)$, and not of $(\alpha L)^{-1}$ as in the one-dimensional case. A calculation of the coefficients of c^2 requires very lengthy although basically simple operations, and we shall give directly the final expression for ϕ_σ:

$$\gamma_\sigma=2\alpha+c\mu/\alpha a^3-c^2S_2+\dots,$$

$$S_2=\frac{\mu^2}{\alpha^3a^6}\Big[1-4\mu\alpha^2a^3\sum_{\mathbf{r}\neq0}\frac{h^2(\mathbf{r})}{1-\mu^2h^2(\mathbf{r})}-4\mu^2\alpha^2a^3\sum_{\mathbf{r}\neq0}e^{-\alpha x}\frac{h^3(\mathbf{r})}{1-\mu^2h^2(\mathbf{r})}\Big] \tag{3.12}$$

(both sums in the brackets apply to the whole infinite lattice).

It follows from the above conclusion and the resultant formulas that an expansion of the decrement of the average transparency γ_σ in powers of the concentration is applicable in the subbarrier range of energies. Consequently, the convergence radius of the series (3.5), (3.6), and (3.12) on approach of the energy to the actual boundary of the spectrum E_g. However, the expansion obtained can also be used in the case when $E>E_g$, provided only that $E-E_g\ll E_g$. In fact, as shown in §2 (see also Ref. 2), the spectrum in this range of energies consists of fluctuation states which are realized at clusters of impurities of sufficiently large size. Therefore, at these energies we obtain large values of the coefficients of the very high powers of the concentration, which are of the order of the number of impurities in a fluctuation cluster amounting to $[\alpha(E-E_g)^{1/2}]^{-\alpha}$. The appearance of these terms is the result of activation of the mechanism of resonance tunneling at fluctuation levels discussed in Ref. 1 and in §2. Hence, it is clear that in the part of the expansion in terms of the concentrtaion which corresponds to the small numbers of impurities represents the contribution of nonresonance configurations and can therefore be used as an approximate expression for γ_σ on condition that the right-hand side of Eq. (2.21) is less than the left-hand side. Using a more mathematical terminology, this can be formulated as the statement that under these conditions the cluster expansion becomes an asymptotic series for γ_σ.

APPENDIX

We shall obtain an expansion of $\bar\gamma$ in powers of the concentration using a formula which follows from Eq. (1.9) and from Refs. 5 and 2:

$$\bar\gamma=2\int zp(z)\,dz. \tag{A.1}$$

Here, z is the logarithmic derivative of the wave function and its probability density $p(z)$ satisfies, according to Ref. 11, the following equation (when $U_0=0$):

$$(z^2+k^2)p(z)=n_1\int_{z-k_0}^{z}p(z')\,dz'+N(E),$$

where $N(E)$ is the number of states of energy not exceeding E per unit length of the system. If in this equation we regard the integral term as a perturbation, we find that

$$\bar\gamma(E)=\frac{B}{A},\quad B=2k\sum_{m=1}^{\infty}B_m\left(\frac{n_1}{k}\right)^m,\quad A=\sum_{m=0}^{\infty}A_m\left(\frac{n_1}{k}\right)^m; \tag{A.2}$$

$$N^{-1}(E)=k^{-1}A,$$

$$A_m=\int_{-\infty}^{\infty}f_m(\zeta)\,d\zeta,\quad B_m=\int_{-\infty}^{\infty}\zeta f_m(\zeta)\,d\zeta. \tag{A.3}$$

The functions $f_m(\zeta)$ then satisfy the following recurrence relationships:

$$f_m(\zeta)=\frac{1}{\zeta^2+1}\int_{\zeta-k_0/k}^{\zeta}f_{m-1}(\zeta')\,d\zeta',\quad f_0(\zeta)=\frac{1}{\zeta^2+1},$$

which can be used to show that for each fixed energy E the series in Eqs. (A.2) and (A.3) converge for all values of n_1/k at a rate no less than that for an exponential function, i.e., these series are complete analytic functions of the parameter n_1/k. It follows that $N^{-1}(E)$ is a function of the same kind. As for $\bar\gamma(E)$ and $N(E)$, the series for these quantities are obtained by reexpression of the fractions in Eqs. (A.2) and (A.3) in powers of n_1/k, and their radius of convergence is equal to the distance to the nearest zero of the denominator, which–because the coefficients A_m are positive–is not located on the positive semiaxis of the complex plane of the parameter n_1/k. For the same reason both $\bar\gamma(E)$ and $N(E)$ can be expanded as series in the vicinity of any point of this semiaxis.

In the limit $E\to0$ the convergence of the series for $\bar\gamma$ deteriorates. This is due to two factors: firstly, the parameter of the expansion is not the density n_1 but the ratio n_1/k and, secondly, the coefficients B_m increase [for example, $B_1\sim\ln(k_0/k)$ as $k\to0$]. Therefore, we can expect the appearance of a singularity in respect of the density for the decrement $\bar\gamma(E)$ as $E\to0$. This can be demonstrated by considering a somewhat simpler case in which the amplitudes of the scatterers are also random independent quantities with the probability densities $k_0^{-1}\exp(-k/k_0)$. In this example, $p(z)$ for $E=0$ is found explicitly[12]:

$$p(z)=Cz^{-2}\theta(z)\exp(-z/k_0-n_1/z),$$

where C is the normalization constant. Now using Eq. (A.1) to calculate $\bar\gamma(0)$, we find that if $n_1\ll k$, then

$$\bar\gamma(0)\approx-n_1\ln(n_1/k_0). \tag{A.4}$$

[1)]We shall consider only a one-particle problem in which the particles experience only elastic collisions with randomly distributed force centers.

[2)]Since we are interested in quantities which decrease exponentially on increase in the layer thickness, the symbol $\lim_{L\to\infty}$ should not be understood literally. In reality, we have to calculate the first terms of the asymptote (which are generally intermediate) with respect to L for the decrements in question.

[3)]The corresponding inequalities will be made clear later.

[4)]The inequality (2.6) follows directly from the inequality between the arithmetic and geometric mean values: $\langle e^{\xi}\rangle\geqslant e^{\langle\xi\rangle}$.

[5)]For a potential of the type (1.5) this expression is equal to the reflection coefficient of one impurity calculated in the Born approximation and multiplied by $n_1/2$.
[6)]In Ref. 1 the coefficient S_2 is given incorrectly with a multiplier 1/2.
[7)]In Ref. 1 the value of the coefficient $\overline{S}_2$ is given incorrectly.
[8)]The same conclusions are obtained by employing the approach of Ref. 10.

[1]I. M. Lifshitz and V.Ya. Kirpichenkov, Zh. Eksp. Teor. Fiz. **77**, 989 (1979) [Sov. Phys. JETP **50**, 499 (1979)].
[2]I. M. Lifshitz, S. A. Gredeskul, and L. A. Pastur, Vvedenie v teoriyu neuporyadochennykh sistem (Introduction to the Theory of Disordered Systems), Nauka, M. 1982.
[3]S. Chandrasekhar Radiative Transfer, Clarendon Press, Oxford 1950 (Russ. Transl., IIL, M., 1953).
[4]L. A. Pastur and É. P. Fel'dman, Zh. Eksp. Teor. Fiz. **67**, 487 (1974) [Sov. Phys. JETP **40**, 241 (1975)].
[5]K. Ishii, Prog. Theor. Phys. Suppl. No. **53**, 77 (1973).
[6]G. C. Papanicolaou, SIAM J. Appl. Math. **21**, 13 (1971).
[7]T. N. Antsygina, L. A. Pastur, and V. A. Slusarev, Fiz. Nizk. Temp. **7**, 5 (1981) [Sov. J. Low. Temp. Phys. **7**, 1 (1981)].
[8]I. M. Lifshitz, Usp. Fiz. Nauk **83**, 617 (1964) [Sov. Phys. Usp. **7**, 549 (1965)].
[9]L. D. Landau and E. M. Lifshitz, Statisticheskaya fizika, Ch. I. Nauka, M., 1976 (Statistical Physics, Vol. 1, 3rd ed., Pergamon Press, Oxford, 1980).
[10]Yu. A. Bychkov and A. M. Dykhne, Zh. Eksp. Teor. Fiz. **51**, 1923 (1966) [Sov. Phys. JETP **24**, 1290 (1967)].
[11]H. L. Frisch and S. P. Lloyd, Phys. Rev. **120**, 1179 (1960).
[12]M. M. Benderskiĭ and L. A. Pastur, Zh. Eksp. Teor. Fiz. **57**, 284 (1969) [Sov. Phys. JETP **30**, 158 (1970)].

Translated by A. Tybulewicz