Excitation of a quantum linear oscillator in a thermostat with a nonmonochromatically varied frequency by an external force

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An averaging method is used to obtain an analytic solution for the density matrix of a quantum linear oscillator located in a thermostat and having a variable frequency $\omega(t) = \omega_0 + \mu(t)\cos 2\pi_0 t$; the modulation amplitude $\mu(t)$ is small compared with ω_0, changes little over the period $2\pi/\omega_0$, and is in all other respects an arbitrary function of time. Conditions are investigated under which the sensitivity of the oscillator as a sensor for an external force $f(t)$ increases substantially under the influence of $\mu(t)$ compared with the case $\mu = 0$. For a definite type of external-force spectrum, such an increase is impossible if the change of the frequency $\omega(t)$ is monochromatic ($\mu(t) = \text{const}$), but becomes possible if $\mu(t)$ is a variable function of time. It is shown that quantum nondestructive measurements of the oscillator are possible at any $\mu(t)$; an operator that realizes such measurements is constructed.

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1. INTRODUCTION

Very many physical phenomena lead to the problem of an oscillator with variable frequency (see, e.g., Refs. 1–8). Widespread use of a physically verified model of a linear oscillator with variable frequency is hindered, however, by the fact that even the classical problem with arbitrary dependence of the oscillator frequency $\omega(t)$ on the time t has no analytic solution, whereas the problem of a classical linear oscillator with friction, excited by an arbitrary external force $f(t)$, can be easily solved completely. Some information on the properties of an oscillator with variable frequency could be obtained by using stochastic methods. These methods are effective if $\omega(t)$ is a random function of the time whose partial or total characteristics are delta-correlated in time.[1–5] This assumption cannot always be physcially justified. Without the use of stochastic methods, to our knowledge, a palpable analytic solution was obtained only for the case of a small monochromatic (or harmonic) variation of $\omega(t)$ (see, e.g., Ref. 9) and for adiabatically slow variation of $\omega(t)$ (see Ref. 10).

The present paper is devoted to one more case when all the calculations can be carried through to conclusion in analytic form, by using an averaging method.[11] Let $\omega(t) = \omega_0 + \xi(t)$, where ω_0 is the frequency of the unperturbed oscillator. Borrowing the terminology from the theory of parametric amplifiers,[4] we shall call $\xi(t)$ a regenerative signal. Let the regenerative signal $\xi(t)$ be nonmonochromatic, concentrated near a resonant frequency $2\omega_0$. In general form we can write

$$\omega(t) = \omega_0 + \mu(t)\cos\left(2\omega_0 t + \int_0^t \delta(\tau)\,d\tau\right), \tag{1.1}$$

where $\mu(t)$ and $\delta(t)$ are time functions that vary slowly compared with the frequency ω_0. We assume the signal to be weak: $|\mu(t)| \ll \omega_0$. In addition, we set $\delta = 0$, for only in this case can the averaging method yield an analytic solution of the problem. The condition $\delta = 0$ means that the carrier frequency of the regenerative signal is exactly equal to the frequency $2\omega_0$.

We thus obtain an analytic solution under the conditions

$$\tau^{-1} \ll \omega_0/2\pi, \quad |\mu(t)| \ll \omega_0, \quad \delta = 0, \tag{1.2}$$

where τ is the characteristic time of variation of $\mu(t)$.

In the more general case when the condition $\delta = 0$ is not imposed, not even the averaging method can yield an analytic solution, and numerical integration of the equation of motion in terms of the slow variables is necessary (see Ref. 12).

The condition (1.2) notwithstanding, the solution obtained by us in Sec. 2 contains much physical information and permits an investigation of various physical situations that arise when the oscillator is excited. In the present paper we consider one such situation. Let the oscillator be used as a sensor (meter) for an external force $f(t)$. Wishing to change in a required direction the properties of such a sensor, we must regenerate the oscillator by a signal $\xi(t)$. The classical linear oscillator must then satisfy the equation

$$\ddot{x} + \nu\dot{x} + \omega_0^2\left[1 + \frac{2\mu(t)}{\omega_0}\cos\left(2\omega_0 t + \int_0^t \delta(t_1)\,dt_1\right)\right]x = \frac{1}{m}f(t), \tag{1.3}$$

where m is the effective mass of the oscillator and ν is the damping decrement. What regenerative signal must be used, for example, to increase the sensitivity of the sensor or to increase its spectral width? Using the solution obtained by us, we answer in Secs. 3 and 4 these and other similar questions.

Of course, another situation is also possible, wherein the investigated physical phenomenon manifests itself in a change of the oscillator frequency $\omega(t)$, and the external force $f = 0$. Let $\omega(t) = \omega_0 + \varkappa(t) + \xi(t)$, where an external modulation $\mu(t)$ results from the action exerted on the oscillator by the investigated phenomenon and $\xi(t)$ is a regenerative signal applied to improve the properties of the oscillator as a sensor for the external modulation $\varkappa(t)$. If the action imposed on the oscillator by investigated phenomenon is such that

$$\varkappa(t)=2m(t)\cos 2\omega_0 t,$$

and $m(t)$ satisfies conditions similar to (1.2), the solution obtained by us in this paper makes it possible to determine the advantages that can be gained in the measurement of $\varkappa(t)$ by some particular choice of the regenerative signal $\xi(t)$.

By stipulating $\delta=0$ we have confined ourselves to regenerative signals having a perfectly defined carrier frequency. This is a rather strong restriction, but still makes our results widely applicable: When the regenerative signal $\xi(t)$ is specified, it makes sense to formulate a problem in which $\xi(t)$ has all the necessary properties.

Both the external force and the regenerative signal pump the oscillator (increase its energy). In this paper we study the pumping by the regenerative signal only to the extent that it influences the pumping by the external force.

Since the amplitude μ of the regenerative signal can vary with time, this signal is nonmonochromatic. So far, the important question of the influence of the regenerative signal on an external-force signal was considered only for a monochromatic signal (μ and δ constant) on the basis of a well-known analytic solution for this case (see, e.g., Ref. 1 and the literature cited therein). Using the solution obtained in this paper, we shall show that in some cases a nonmonochromatic regenerative signal must be used to improve the properties of the sensor.

In the Appendix we show also that regeneration of the oscillator by a nonmonochromatic signal does not prevent us from performing nondestructive quantum measurements on the oscillator.

2. BASIC EQUATIONS AND OSCILLATOR ENERGY

The evolution of the oscillator density matrix $\rho(t)$ satisfies the equation

$$\frac{\partial}{\partial t}\rho(t)+\frac{i}{\hbar}[H,\rho(t)]=I_c, \tag{2.1}$$

where H is the Hamiltonian of a linear oscillator with frequency (1.1). Introducing, as usual, the operators a and a^+ and averaging only the resonant terms, we write the Hamiltonian in the form

$$H=\hbar\omega_0\left(a^+a+\frac{1}{2}\right)+\frac{\hbar\mu(t)}{4}\left\{a^{+2}\exp\left[-i\left(2\omega_0 t+\int_0 \delta(t_1)dt_1\right)\right]\right.$$

$$\left.+a^2\exp\left[i\left(2\omega_0 t+\int_0 \delta(t_1)dt_1\right)\right]\right\}-f(t)\left(\frac{\hbar}{2m\omega_0}\right)^{1/2}(a^++a),$$

$$f(t)=f_c(t)\cos\omega_0 t+f_s(t)\sin\omega_0 t,$$

where $f(t)$ is the external force, while f_c and f_s are functions of the time t that are slow compared with ω_0. The interaction of the oscillator with the thermostat is described by the collision integral I_c (see Refs. 7 and 13):

$$I_c=\frac{\nu}{2}[(N+1)(2a\rho a^+-a^+a\rho-\rho a^+a)$$

$$+N(2a^+\rho a-aa^+\rho-\rho aa^+)],$$

where ν is the effective collision frequency (the reciprocal relaxation time), $N=(e^{\hbar\omega_0/T}-1)^{-1}$ is the average number of quanta in the oscillator in the state of thermodynamic equilibrium with temperature T. It is convenient to solve Eq. (2.1) in the representation of the Wigner function for the density matrix[14]

$$W(\alpha)=\frac{1}{\pi}\int d^2\eta\exp(\eta^*\alpha-\eta\alpha^*)\,\mathrm{Sp}[\rho\exp(\eta a^+-\eta^* a)],$$

$$d^2\eta=d(\mathrm{Re}\,\eta)\,d(\mathrm{Im}\,\eta),$$

where α and η are complex numbers. Let

$$\mathrm{Re}\,\alpha=(m\omega_0/2\hbar)^{1/2}x(u,v),\quad \mathrm{Im}\,\alpha=p(u,v)/(2\hbar m\omega_0)^{1/2}.$$

Writing down Eq. (2.1) in the representation of the Wigner functions and averaging the obtained equation over the period $2\pi/\omega_0$ we obtain a Fokker-Planck equation for the function $W(t,u,v)$ in terms of the slow variables u and v:

$$\frac{\partial W}{\partial t}-\left(N+\frac{1}{2}\right)\frac{\nu}{4}\left(\frac{\partial^2 W}{\partial u^2}+\frac{\partial^2 W}{\partial v^2}\right)$$

$$+\frac{\partial W}{\partial u}\left(-\frac{\nu u}{2}-\frac{\mu(t)+\delta(t)}{2}v\right)$$

$$+\frac{\partial W}{\partial v}\left(-\frac{\nu v}{2}-\frac{\mu(t)-\delta(t)}{2}u\right)-\frac{f_c(t)}{2(2\hbar m\omega_0)^{1/2}}\frac{\partial W}{\partial u}$$

$$-\frac{f_s(t)}{2(2\hbar m\omega_0)^{1/2}}\frac{\partial W}{\partial v}-\nu W=0. \tag{2.2}$$

The calculation procedure that makes it possible to transform from (2.1) to (2.2) is described in greater detail in Ref. 7. The physical meaning of the slow variables u and v is clear from expression (2.4) below. The analytic solution of (2.2) can be easily obtained if an analytic solution is found for the equation of the characteristics:

$$\frac{d}{dt}\begin{pmatrix}u(t)\\ v(t)\end{pmatrix}=\frac{1}{2(2\hbar m\omega_0)^{1/2}}\begin{pmatrix}-f_c(t)\\ f_s(t)\end{pmatrix}$$

$$+\frac{1}{2}\begin{bmatrix}-\nu & \mu(t)+\delta(t)\\ \mu(t)-\delta(t) & -\nu\end{bmatrix}\begin{pmatrix}u(t)\\ v(t)\end{pmatrix}. \tag{2.3}$$

We note that Eq. (2.3) arises also in a simpler formulation of the problem, when the equation (1.3) is solved for a classical oscillator. Indeed, setting

$$x=\left(\frac{2\hbar}{m\omega_0}\right)^{1/2}(u\cos\omega_0 t+v\sin\omega_0 t),$$

$$\dot{x}=-\left(\frac{2\hbar\omega_0}{m}\right)^{1/2}(u\sin\omega_0 t-v\cos\omega_0 t) \tag{2.4}$$

and using the averaging method, we reduce (1.3) to the form (2.3). We obtain the solution (2.3) by a standard method, stipulating that the matrix $\hat{A}(t)$ in the square brackets of (2.3) satisfy the commutation condition

$$\hat{A}(t_1)\hat{A}(t_2)-\hat{A}(t_2)\hat{A}(t_1)=0 \tag{2.5}$$

at all t_1 and t_2. We can then find a constant matrix $\hat{O}$ such that the matrix $\hat{O}^{-1}\hat{A}\hat{O}$ is diagonal, after which the solution of (2.3) entails no difficulty. The condition (2.5) is satisfied for all t_1 and t_2 if $\delta(t)=C\mu(t)$, where C is a constant. This relation holds, for example, in the well-investigated case when μ and δ are independent of time (see, e.g., Ref. 11). We consider another case, when $\delta = 0$ and it must be assumed that $C = 0$. The solution of the homogeneous equation (2.3) takes in this case the form

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = e^{-\nu t/2} \begin{pmatrix} \operatorname{ch} J & \operatorname{sh} J \\ \operatorname{sh} J & \operatorname{ch} J \end{pmatrix} \begin{pmatrix} u(0) \\ v(0) \end{pmatrix}, \quad J = \frac{1}{2}\int_0^t \mu(t_1)\, dt_1. \tag{2.6}$$

We note that when $\nu = \delta = 0$ the substitution $t \to J$ leads immediately to (2.6). Now the solution of (2.2) for the Wigner function can be found in standard fashion by taking the Fourier transform with respect to the variables u and v (see, e.g., Ref. 15). We write this solution with the aid of a Green's function,

$$W(t, u, v) = \int du_1\, dv_1 G(t, u, v | v_1, u_1) W(0, u_1, v_1),$$

$$G = \frac{1}{2\pi (4a^2 - b^2)^{1/2}} \exp\left[-\frac{a(f^2 + d^2) + b\, df}{4a^2 - b^2} \right], \tag{2.7}$$

where

$$a = \frac{\nu}{8}\left(N + \frac{1}{2}\right)\left[e^{-2\Lambda_1(t)} \int_0^t e^{2\Lambda_1(\tau)} d\tau + e^{-2\Lambda_2(t)} \int_0^t e^{2\Lambda_2(\tau)} d\tau \right],$$

$$b = \frac{\nu}{4}\left(N + \frac{1}{2}\right)\left[e^{-2\Lambda_1(t)} \int_0^t e^{2\Lambda_1(\tau)} d\tau - e^{-2\Lambda_2(t)} \int_0^t e^{2\Lambda_2(\tau)} d\tau \right],$$

$$d = i\left\{ -u + \frac{u_1}{2}\left(e^{-\Lambda_1(t)} + e^{-\Lambda_2(t)}\right) + \frac{v_1}{2}\left(e^{-\Lambda_1(t)} - e^{-\Lambda_2(t)}\right) \right.$$
$$+ \frac{1}{4(2\hbar m\omega_0)^{1/2}}\left[e^{-\Lambda_1(t)} \int_0^t e^{\Lambda_1(\tau)} (f_c(\tau) + f_s(\tau))\, d\tau \right.$$
$$\left.\left. - e^{-\Lambda_2(t)} \int_0^t e^{\Lambda_2(\tau)} (f_c(\tau) - f_s(\tau))\, d\tau \right]\right\},$$

$$f = i\left\{ -v + \frac{u_1}{2}\left(e^{-\Lambda_1(t)} - e^{-\Lambda_2(t)}\right) + \frac{v_1}{2}\left(e^{-\Lambda_1(t)} + e^{-\Lambda_2(t)}\right) \right.$$
$$- \frac{1}{4(2\hbar m\omega_0)^{1/2}}\left[e^{-\Lambda_1(t)} \int_0^t e^{\Lambda_1(\tau)} (f_c(\tau) + f_s(\tau))\, d\tau \right.$$
$$\left.\left. - e^{-\Lambda_2(t)} \int_0^t e^{\Lambda_2(\tau)} (f_c(\tau) - f_s(\tau))\, d\tau \right]\right\},$$

$$\Lambda_{1,2} = \frac{1}{2}\left(\nu t \pm \int_0^t \mu(\tau)\, d\tau \right).$$

Knowing the Wigner function, we can find all the characteristics of the quantum system, for example the level population p_n (to calculate p_n it is convenient to use the procedure of Ref. 7). In the present paper we are interested in a relatively rough but but palpable characteristic of the system—its total energy

$$E(t) = \hbar\omega_0 \int (u^2 + v^2) W(t, u, v)\, du\, dv.$$

The calculations yield

$$E(0) = E_T = \hbar\omega_0 (N + 1/2),$$

$$E(t) = \frac{\hbar\omega_0}{2}\left(N + \frac{1}{2}\right)\left[e^{-2\Lambda_1(t)} \left(1 + \nu \int_0^t e^{2\Lambda_1(\tau)} d\tau\right) + e^{-2\Lambda_2(t)} \left(1 + \nu \int_0^t e^{2\Lambda_2(\tau)} d\tau\right) \right] \tag{2.8}$$
$$+ \frac{1}{16m}\left[e^{-\Lambda_1(t)} \int_0^t e^{\Lambda_1(\tau)} (f_c(\tau) + f_s(\tau))\, d\tau + e^{-\Lambda_2(t)} \int_0^t e^{\Lambda_2(\tau)} (f_s(\tau) - f_c(\tau))\, d\tau \right]^2.$$

3. REGENERATION BY A NONMONOCHROMATIC SIGNAL

We have obtained an analytic expression for $E(t)$ under the assumption that $\mu(t)$ is a small function of the time, slowly varying over the period $2\pi/\omega_0$, but otherwise arbitrary. We now assume that $\mu(t)$ is a random function and, averaging the obtained expressions over the realizations of μ, obtain in a statistical formulation a solution that is closest to the real experimental conditions. We note that the traditional statistical methods are restricted to the assumption that either the entire signal $\xi(t)$ (Refs. 2 and 5) or its phase $\delta(t)$ (Ref. 1) is delta-correlated. We, however, do not need this assumption.

We note also that the assumption that $\mu(t)$ is random is not mandatory for our formulation of the problem. It is perfectly possible for the optimum change of the properties of the oscillator as a sensor for an external force to take place under the influence of a regular regenerative signal, for example when $\mu(t) = 2M\cos\Delta t$ [in this case, of course, we must assume that $M \ll \omega_0$ and $\Delta \ll \omega_0$; see conditions (1.2)]. The actual type of regenerative signal that must be used to measure the force f depends on the type of the force f, on those characteristics of this force which are measured in the particular experiment, etc. We assume $\mu(t)$ to be a random function, since the advantages of regeneration by a nonmonochromatic signal are clearly evident when the force $f(t)$ is of definite form (see Sec. 4 below).

We consider now the expression (2 .8) for the energy. We assume that the amplitudes $f_{s,c}$ of the force $f(t)$ are such that

$$\langle f_i(t) f_k(t') \rangle = \delta_{ik} \int \frac{d\omega}{2\pi} F(\omega) e^{i\omega(t-t')},$$
$$\langle f_i(t) \rangle = 0, \quad i = c, s,$$

where $F(\omega)$ is the external-forces spectrum centered about the resonant frequency ω_0. The angle brackets denote averaging over the realizations. We represent $\mu(t)$ in the form

$$\mu(t) = \mu_0 + \mu_1(t),$$

where $\mu_1(t)$ will be regarded as a random Gaussian process, i.e.,

$$\langle \mu_1(t) \mu_1(t') \rangle = \int \frac{d\omega}{2\pi} M(\omega) e^{i\omega(t-t')},$$

$$\langle \mu_1(t) \rangle = 0.$$

We assume also that f_i and μ_1 are statistically independent. We average expression (2.8) for $E(t)$ over the realizations of f_i and μ_1 (see Chap. 1 of Ref. 3 in this connection). In the steady state as $t\to\infty$ we obtain $E = E_T + \delta E_{sp} + \delta E_f$, where E_T is the thermal energy, δE_{sp} is the so-called statparametric energy,[7] and δE_f is the energy obtained from the external force. In this case

$$\delta E_f = \int \frac{d\omega}{2\pi} F(\omega) l(\omega); \tag{3.1}$$

$l(\omega)$ here is the contour of the oscillator that describes the reaction of the oscillator to the external force. The appearance of δE_{sp} is due to the simultaneous influence of the thermostat and (or) of the quantum fluctuations, on the one hand, and of the parametric excitation on the other. A classical oscillator in a thermostat with zero temperature has $\delta E_{sp} = 0$. In our case calculations yield

$$\delta E_{sp}/E_T = \nu \int_0^\infty d\xi \exp\left[-\nu\xi + 2\int \frac{d\Omega}{2\pi} \frac{M(\Omega)}{\Omega^2} \sin^2\frac{\Omega\xi}{2}\right] \operatorname{ch}(\mu_0\xi) - 1. \tag{3.2}$$

The contour $l(\omega)$ turns out to equal

$$l(\omega) = {}^1/_2 (l_1(\omega) + l_2(\omega)),$$

$$l_i(\omega) = \frac{1}{m} \int_0^\infty \int_0^\infty d\xi_1 d\xi_2 \exp\left[-\xi_1\left(\frac{\nu_i}{2} - i\omega\right) - \xi_2\left(\frac{\nu_i}{2} + i\omega\right)\right.$$
$$\left. + \frac{1}{2}\int \frac{d\Omega}{2\pi} \frac{M(\Omega)}{\Omega^2}\left(2\sin^2\frac{\Omega\xi_1}{2} + 2\sin^2\frac{\Omega\xi_2}{2} - \sin^2\frac{\Omega(\xi_1-\xi_2)}{2}\right)\right],$$
$$\nu_{1,2} = \nu \mp \mu_0; \qquad i = 1, 2. \tag{3.3}$$

We consider first the behavior of the oscillator under the action of only parametric excitation, i.e., when $f(t) = 0$. The condition for the convergence of the integral (3.2) determines the region of parametric stability, when $\delta E_{sp}(t\to\infty) < \infty$. It can be easily seen that the oscillator remains stable if

$$\nu > \mu_0 + M(0)/2, \tag{3.4}$$

where $M(0)$ is the spectral density of μ_1 at zero frequency. In the case when (3.4) is not satisfied, stability is lost and the oscillator energy increases in time without limit.

Let Δ_μ be the characteristic width of the spectrum $M(\Omega)$ of the quantity $\mu_1(t)$, as well as a smooth function in the interval Δ_μ. For fast fluctuations of μ_1, when $\Delta_\mu \gg \nu - M(0)/2$, (3.2) assumes the simpler form

$$E_T + \delta E_{sp} = E_T \nu \frac{\nu - M(0)/2}{(\nu - M(0)/2)^2 - \mu_0^2}. \tag{3.5}$$

4. COMPARISON OF CONTOURS AND DISCUSSION

Let now the oscillator be acted upon also by an external force. The energy received by the oscillator from the external force is determined by the contour (3.3). In the presence of an external force the parametric instability threshold does not change and is determined as before by expression (3.4). Assume that we are near the threshold, i.e., let $\nu - \mu_0 - M(0)/2 \ll \nu$. The contribution from l_2 can then be neglected and $l(\omega) \approx \frac{1}{2} l_1(\omega)$. We consider two limiting cases. In the case of fast fluctuations, when the spectrum of μ_1 is so broad that

$$\Delta_\mu/M(0) \gg 1, \qquad \frac{\Delta_\mu}{\nu - \mu_0 - M(0)/2} \gg 1, \tag{4.1}$$

we obtain from (3.3) for the controur $l(\omega)$ the expression

$$l(\omega) = \frac{1}{2m} \frac{\nu - \mu_0 - M(0)/4}{\nu - \mu_0 - M(0)/2} \left[\frac{(\nu - \mu_0 - M(0)/4)^2}{4} + \omega^2\right]^{-1}. \tag{4.2}$$

For slower fluctuations, when

$$\Delta_\mu/M(0) \ll 1, \qquad \frac{\Delta_\mu}{\nu - \mu_0 - M(0)/2} \gg 1, \tag{4.3}$$

we have

$$l(\omega) = \frac{1}{2m} \frac{(2\pi)^{1/2} \exp[-4\omega^2/M(0)\Delta_\mu]}{(M(0)\Delta_\mu)^{1/2}(\nu - \mu_0 - M(0)/2)}. \tag{4.4}$$

We present also a known expression for the contour if the regeneration is produced by a monochromatic signal ($\mu_1 = 0, \mu_0 > 0$)

$$l(\omega) = \frac{1}{2m} \frac{1}{(\nu - \mu_0)^2/4 + \omega^2}. \tag{4.5}$$

When $\mu_1(t)$ is a randon δ-correlated function of time, its spectrum $M(\Omega)$ has a width $\Delta_\mu = \infty$. This case is equivalent in a certain sense to the case of fast fluctuations, since (4.2) contains only $M(0)$ but not Δ_μ. The case of slower fluctuations, however, is way outside the framework of the assumed δ correlation. Indeed, the conditions (4.3) limit Δ_μ both from below and from above; the result (4.4) contains Δ_μ and does not admit of the limiting transition $\Delta_\mu\to\infty$. Regeneration by a monochromatic signal, which yields the contour (4.5), differs qualitatively from the regeneration by a nonmonochromatic signal, which gives a contour (4.2) or (4.4), in the following respect. We approach the instability threshold, for example, because of the adiabatically slow increase of μ_0. In the case of regeneration by a monochromatic signal, the threshold value μ_c of μ_0 is equal to ν. When $\mu_0\to\mu_c = \nu$ the central part of the contour (4.5) increases without limit, but the wings do not increase. Indeed, according to (4.5) we have $l(\omega) = 1/2m\omega^2$ at $\omega^2 \gg (\nu - \mu_0)^2$. Therefore the width of the contour (4.5), equal to $|\mu_c - \mu_0|$ at half-maximum, tends to zero as $\mu_0\to\mu_c$. In the case (4.1) or (4.3) the threshold value of μ_c for μ_0 is somewhat lower than in the first case: $\mu_c = \nu - M(0)/2$. More important, however, is another fact. As $\mu\to\mu_c$, both the central part of the contour $l(\omega)$ and its wings increase without limit; the contour width does not decrease but equals $M(0)/4$ in case (4.1) and $[\ln 2\cdot M(0)\Delta_\mu]^{1/2}$ in case (4.3) (see Figs. 1a and 1b).

Assume that the oscillator is acted upon by an external force whose spectrum does not contain harmonics near the resonance frequency ω_0 (see Fig. 1c). We regenerate the oscillator to increase its sensitivity as an external-force sensor. If the regeneration is produced by a monochromatic signal, the energy δE_f obtained from the external source will not increase without limit when the instability threshold is approached [see Eqs. (3.1) and (4.5)]. If, however, the regeneration is by a nonmonochromatic signal, this energy increases

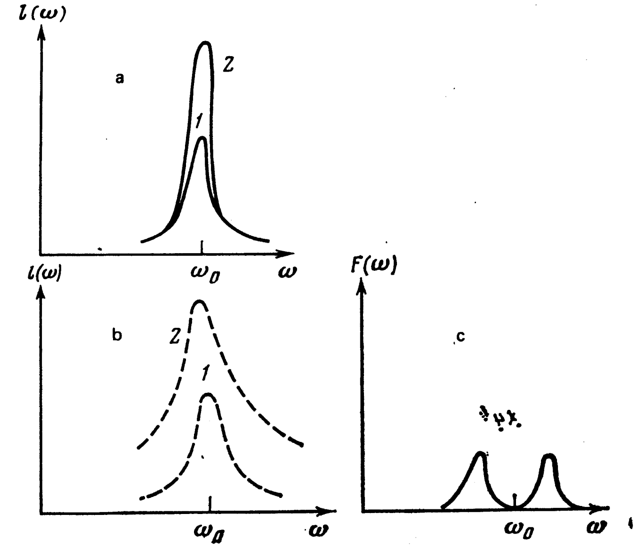

FIG. 1. Plots of contours and spectrum of external force, centered about the frequency ω_0: a) the contour $I(\omega)$ for regeneration by a monochromatic signal; b) contour $I(\omega)$ for regeneration by a nonmonochromatic signal [case of fast fluctuations, (4.1)]. Plots a and b are constructed under the condition that μ_0 on curve 2 is larger than μ_0 on curve 1. In the case of curve 2 the oscillator is closer to the instability threshold than in the case of curve 1; c) external-force spectrum containing no harmonics near the resonance frequency.

without limit[1)] [see Eqs. (3.1) and (4.3) or (4.4) and Figs. 1 and 2].

The oscillator as an external-force sensor operates better the higher the ratio:

$$R=\delta E_f/(E_T+\delta E_{sp}). \tag{4.6}$$

The quantity R is a measure of the "signal/noise" ratio, the noise being due to the external force and to thermal and quantum fluctuations of the oscillator. The ratio R has a complicated dependence on the spectrum of the external force and on the regeneration method, inasmuch as regeneration increases both δE_f and δE_{sp}.

Let, unlike in the spectrum shown in Fig. 1c, $F(\omega)$ have no anomalous smallness at $\omega=0$. We introduce the characteristic width Δ_f of $F(\omega)$ and consider, for the sake of argument, the case of a narrow spectrum, when Δ_f is small (see below). It follows then from (3.1), (3.5), (4.2), (4.4), and (4.5) that the regeneration always increases the ratio R, but the degree of this increase depends on the form of the spectrum of the regenerative signal. If the regeneration is produced by a monochromatic signal, $\mu_1=0$ and $\Delta_f \ll \nu$, as μ_0 approaches the threshold value we have in order of magnitude

$$R(\mu_0\to\mu_c)/R_0\sim\nu/\Delta_f, \tag{4.7}$$

where R_0 is the ratio R in the absence of regeneration, i.e., at $\mu_0=\mu_1=0$. If the regeneration is by a nonmonochromatic signal and $\Delta_f \ll \nu$ and $\Delta_f \ll M(0)$, we have

$$R(\mu_0\to\mu_c)/R_0\sim\nu/M(0). \tag{4.8}$$

Since the ratio (4.7) is larger than (4.8), regeneration by a monochromatic signal is more effective. The reason is that at $\omega=0$ the contour (4.5) increases like $(\mu_c-\mu_0)^{-2}$ as $\mu_0\to\mu_c$, and the contours (4.2) and (4.4) increase more slowly than $(\mu_c-\mu_0)^{-2}$ If, however the external-force spectrum $F(\omega)$ is anomalously small at $\omega=0$, as in Fig. 1c, R can be increased only by regeneration with a nonmonochromatic signal. Indeed, as $\mu_0\to\mu_c$, at the statparametric energy $\delta E_{sp}=\infty$. At the same time, in regeneration by a monochromatic signal, δE_f remains finite and the ratio $R\to 0$. If, however, the regeneration is by a nonmonochromatic signal and $\Delta_f \ll \nu$, $M(0)$, the energy δE_f, as well as δE_{sp}, tends to infinity as $\mu_0\to\mu_c$. As a result, the estimate (4.8), according to which the ratio R is considerably increased in the case $M(0) \ll \nu$, remains in force.

Depending on the spectrum of the regenerative signal, the oscillator can have a Lorentz contour (4.2), (4.5) or a Gaussian contour (4.4), or else a contour of more complicated shape. Let, e.g., $\mu_0=0$ and

$$M(\Omega)=M[\delta(\Omega-\Delta)+\delta(\Omega+\Delta)]. \tag{4.9}$$

The contour $I(\omega)$ is then given by the integral (3.3) in which it is possible to integrate with respect to $d\Omega$. Figure 3 shows the contour $I(\omega)$ for the case $\Delta=4\nu$ and $M=160\nu^2$. The nontrivial character of the contour in Fig. 3 demonstates by way of example the extensive possibilities afforded by regeneration with a nonmonochromatic signal.

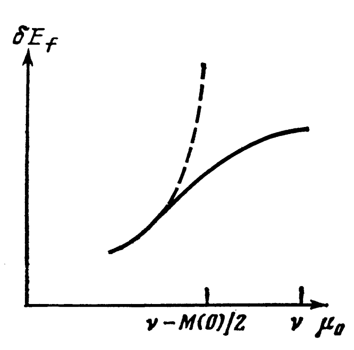

FIG. 2. Energy received by oscillator from the external force in the steady state, as a function of the amplitude of the constant part of the regenerative signal. The solid and dashed curves correspond respectively to the monochromatic and nonmonochromatic cases. The difference between the threshold values of μ_0 in these cases is equal to $M(0)/2$.

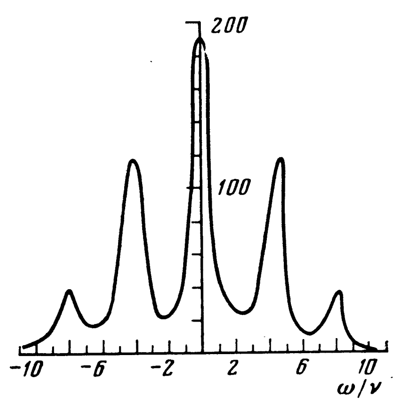

FIG. 3. Contour $I(\omega)$ for regeneration by a nonmonochromatic signal with spectrum (4.9). The ordinates and abscissas are respectively the ratios $I(\omega)/m\omega_0^2\nu^2$ and ω/ν.

The external force $f(t)$ has no harmonics at the carrier frequency ω_0, i.e., $F(0)=0$, when $f(t)$ is an amplitude-modulated sinusoid of frequency ω_0. This case is frequently encountered, for example, in information transmission. To record such a force it is necessary to use nonmonochromatic regeneration of the oscillator while the width $M(0)/4$ of the contour $l(\omega)$ in the case of fast fluctuations and the width $[\ln 2\cdot M(0)\Delta_\mu]^{1/2}$ in the case of slower fluctuations must be not less than the width of the external-force spectrum.

Regeneration by a nonmonochromatic signal may be useful also when it is desirable to record an external force with an unknown spectrum. This problem arises, for example, in gravitation-wave experiments. Indeed, it follows from the foregoing that if the external force has a relatively high spectral density at the frequency ω_0, it is necessary to use regeneration with a monochromatic signal in order to increase the "signal/noise" ratio. If this density is small, however, a nonmonochromatic signal must be used. Since the external-force spectrum is unknown, it is reasonable to use first one and then the other method of regeneration. It must be emphasized here that the width Δ_f of the external-force spectrum should in our calculation be much smaller than the carrier frequency ω_0 but can exceed greatly ν, μ_0, $M(0)$, and all other frequencies of this type. In all cases when the width Δ_f is large in the indicated sense and is not contained in some expression, it can be assumed that this expression remains valid as to order of magnitude also at $\Delta_f \sim \omega_0$.

APPENDIX

NONDESTRUCTIVE PARAMETRIC QUANTUM OPERATOR

A continuously measureable nondestructive quantum operator is an operator $Q(t)$ that satisfies in the Heisenberg representation the condition of continuous measurement[16,17]:

$$[Q(t), Q(t')]=0.$$

For a linear system with one degree of freedom the operator $Q(t)$, made up of coordinates and momenta of degree not higher than second and taken at one instant of time, can be represented in the general case in the form[8]

$$Q_1(t)=\Big[\eta_1(t)+\eta_2(t)x(0)+C_1\eta_2(t)p(0)+\eta_3(t)x^2(0)$$
$$+C_1^2\eta_3(t)p^2(0)+\Big(C_2\eta_2(t)+2C_1\eta_2(t)\int_0 \theta\, d\tau\Big)x(0)p(0)\Big]C_3, \qquad \text{(A.1)}$$

$$\theta\equiv\dot\eta_3/\eta_2-\dot\eta_2\eta_3/\eta_2^2,$$
$$Q_2(t)=[\eta_1(t)+\eta_2(t)\,(x(0)+C_1p(0)+C_3x^2(0)$$
$$+C_3C_2p^2(0)+C_4x(0)p(0))]C_5; \qquad \text{(A2)}$$

here η_1 to η_5 are arbitrary functions of the time, C_1 to C_5 are constants, and $C_1^2\neq C_2$ in (A.2); furthermore, $x(0)$ and $p(0)$ are operators in the Schrödinger representation. To obtain the explicit form of the operators $Q(t)$ it is necessary to substitute in (A.1) and (A.2) the operators $x(0)$ and $p(0)$ expressed in terms of the operators $x(t)$ and $p(t)$ in the Heisenberg representation.

We consider an oscillator without a thermostat with a Hamiltonian H (see Sec. 2). Solving the Heisenberg equation and using in this case (2.6), we find that

$$\begin{pmatrix} x(0) \\ p(0) \end{pmatrix}=(y_{ik})\begin{pmatrix} x(t) \\ p(t) \end{pmatrix}+(b_{ik}),$$

where

$$y_{11}=\mathrm{Re}[(1+i)\,\mathrm{ch}\,(J+i\omega_0 t)],$$

$$y_{12}=\Big(-\frac{1}{m\omega_0}\Big)\mathrm{Re}[(1-i)\,\mathrm{sh}\,(J+i\omega_0 t)],$$

$$y_{21}=\Big(-\frac{1}{m\omega_0}\Big)\mathrm{Re}[(1+i)\,\mathrm{ch}\,(J+i\omega_0 t)],$$

$$y_{22}=\mathrm{Re}[(1-i)\,\mathrm{ch}\,(J+i\omega_0 t)],$$

$$(b_{ik})=\frac{1}{2}\frac{1}{(2\hbar m\omega_0)^{1/2}}\begin{pmatrix} \mathrm{ch}\,J & -\mathrm{sh}\,J \\ -\mathrm{sh}\,J & \mathrm{ch}\,J \end{pmatrix}$$

$$\times\int_0^t d\tau\begin{pmatrix} \mathrm{ch}\,T & \mathrm{sh}\,T \\ \mathrm{sh}\,T & \mathrm{ch}\,T \end{pmatrix}\begin{pmatrix} -f_s(\tau) \\ f_c(\tau) \end{pmatrix},$$

$$T\equiv\int_\tau^t \mu(\tau')\,d\tau', \qquad J=-\frac{1}{2}\int_0^t \mu(\tau)\,d\tau.$$

These formulas solve our problem in principle. The result, however, is extremely unwieldy, and we present it therefore only for the case when there are no external forces:

$$Q_i(t)=R_i^1(t)+R_i^x(t)x(t)+R_i^p(t)p(t)+R_i^{xx}(t)x^2(t)$$
$$+R_i^{pp}p^2(t)+R_i^{xp}x(t)p(t), \quad i=1, 2.$$

The functions $R_1(t)$ for the operator $Q_1(t)$ are given by

$$R_1^x=\eta_2(t)\,(y_{11}+C_1y_{12}), \quad R_1^p=\eta_2(t)\,(y_{12}+C_1y_{22}),$$

$$R_1^{xx}=\eta_3(t)\,(y_{11}^2+C_1^2y_{21}^2)+y_{11}y_{21}\eta_2(t)\Big(C_2+2C_1\int_0^t \theta\, d\tau\Big),$$

$$R_1^{pp}=\eta_3(t)\,(y_{12}^2+y_{22}^2C_1^2)+y_{12}y_{22}\eta_2(t)\Big(C_3+2C_1\int_0^t \theta\, d\tau\Big),$$

$$R_1^{xp}=2\eta_3(t)\,(y_{11}y_{12}+C_1^2y_{22}y_{11})$$
$$+(y_{11}y_{22}+y_{12}y_{21})\,\eta_2(t)\Big(C_3+2C_1\int_0 \theta\, d\tau\Big).$$

For the operator $Q_2(t)$ we have

$$R_2^x=\eta_2(t)\,(y_{11}+C_1y_{21}), \quad R_2^p=\eta_2(t)\,(y_{12}+C_1y_{22}),$$
$$R_2^{xx}=\eta_2(t)\,(y_{11}^2C_3+y_{21}^2C_3C_2+C_4y_{11}y_{21}),$$
$$R_2^{pp}=\eta_2(t)\,(y_{12}^2C_3+y_{22}^2C_3C_2+C_4y_{12}y_{22}),$$
$$R_2^{xp}=\eta_2(t)\,(2C_3(y_{11}y_{12}+C_2y_{21}y_{22})+C_4(y_{11}y_{22}+y_{12}y_{21})).$$

In the limiting case $J \ll 1$ the expressions obtained are equivalent to the corresponding expressions in Ref. 8, where the same question was analyzed by perturbation theory at $J \ll 1$. Our analysis dispenses with this restriction.

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[1)] More accurately, so long as the mathematical model used in this paper is applicable.

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