Influence of Coulomb interaction on the properties of disordered metals

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(Submitted 17 June 1982)
Zh. Eksp. Teor. Fiz. **84**, 168–189 (January 1983)

Coulomb interaction in a metal with impurities is considered. An effective Lagrangian that describes the interaction of the diffusing electrons is obtained. It is shown that the Lagrangian can be renormalized, and the renormalization-group equations are obtained in the two-dimensional case. Expressions that take the Coulomb interaction into account are obtained. The behavior of the state density, for which the doubly logarithmic corrections are significant, is clarified. It is shown that the electronic correlations that determine the conductivity and the state density are quite different in structure. The case is considered when the state density on the Fermi level tends to zero with decreasing temperature and at the same time the conductivity increases without limit.

PACS numbers: 71.45.Gm, 71.55.Dp, 72.15.Nj

§1. INTRODUCTION

In dirty metals, electron-electron repulsion accompanied by scattering by impurites leads to an increase of the resistance with decreasing temperature.[1] In the two-dimensional case ($d = 2$) the resistance increases logarithmically,[2] and the corrections to the state density on the Fermi level increase even more strongly, $\propto -\ln^2 T$. Logarithmic corrections arise also in a large number of other quantities.[3] In the metallic state, the perturbation-theory parameter that determines the scale of these corrections, is[1]

$$g_0 = e^2 \frac{r_\square^0}{2\pi^2} \sim (2\pi\mu\tau)^{-1}, \tag{1.1}$$

where $r_\square^0$ is the resistance of a square film at $T \sim \tau^{-1}$, μ is the Fermi energy, τ is the free-path time, and e is the electron charge. If the impurity density is not too high, we have $g_0 \ll 1$. Despite the smallness of g_0, at sufficiently low temperatures, $T \ll \tau^{-1}$, the corrections become large and the question arises of the behavior of disordered systems in this region.

In the present paper we sum the logarithmic (and doubly logarithmic) corrections of the type $(g_0 \ln T\tau)^n$, which result from Coulomb interaction of the diffuse electrons. In addition, expressions are obtained that take full account of the interaction between the electrons—up to now the screened Coulomb interaction was taken into account only to first order. The corrections discussed are the results of "turning on" of the interaction of singular two-particle propagators having diffusion poles by the Coulomb interaction: $1/(Dq^2 - i\omega)$ (D is the electron diffusion coefficient). Singularities of this type appear either in the particle-hole channel (small momentum and frequency transfers), or in the Cooper channel (small total momentum and small frequency difference). The Cooper propagator alone (without participation of the Coulomb interaction) also makes a logarithmic contribution ($d = 2$) to the conductivity.[4,5] This quantum correction, due to interference produced when the electron is rescattered by impurities, is responsible for the localization of the electron.[6] The physics of disordered Fermi systems includes both localization effects and multiparticle effects connected with the Coulomb interaction. To a certain degree, they can be separated from one another. The point is that the Cooper propagator is quite sensitive to interactions that violate the symmetry with respect to time reversal. Introduction of magnetic impurities[7] or application of a magnetic field[3] suffices to cut off the pole in the Cooper channel. In the present paper the Cooper channel will be purposefully ignored. We shall assume that its contribution is suppressed by the relatively weak magnetic field

$$H > Tc/De. \tag{1.2}$$

The standard crossover technique is not convenient for the questions considered. The point is that the same difficulties arise as when an attempt is made to sum the logarithmic term in the localization problem.[4] An approach is needed in which the integration over the fast momenta of the electron lines is carried out from the very beginning. We are then left with the problem of interaction of diffusion modes, in an important role is played only by large distances exceeding the mean free path. It is thus necessary to construct an effective Lagrangian of diffusion modes. In the localization problem such a Lagrangian was originally obtained with the aid of a functional integral over Bose fields.[8,9] In Ref. 10 the Lagrangian of the diffusion modes was constructed by integration over Fermi fields. In the absence of interaction between electrons, the energy of the diffusing electron is fixed and both methods are on a par.

Since the Coulomb interaction intermixes the electron energies, to obtain correct statistics it is necessary to integrate over the Fermi fields, as in Ref. 10. When describing the interacting diffusing electrons, the Lagrangian of Ref. 10 must be somewhat modified for our purposes. In § 2.1 we derive such a modified Lagrangian. It has the meaning of a functional of the free energy and describes an aggregate of free electrons with energies $\varepsilon < \tau^{-1}$. In § 2.2 we obtain a functional that allows for the Coulomb interaction of electrons in a disordered system, and the Fermi-liquid corrections are taken into account.

It is shown in § 3 that in first order in the small parameter g the obtained functional can be renormalized. The re-

0038-5646/83/010097-12$04.00

normalization-group equations for the two-dimensional case are obtained in the lowest order in g, and these equations take full account of the Coulomb interaction. Besides g and the Coulomb amplitudes, the renormalization-group equations contain also one charge connected with the renormalization of the coefficient of the frequency in the diffusion propagator. As a result, the renormalization-group equation are reconciled with the particle-number conservation condition. The solution of the equations shows that if there is no Zeeman splitting the initial growth of the resistance gives way to a decrease when the temperature is lowered. Therefore the use of the renormalization-group equations in the lowest order in g is valid in this case.

In § 4 we discuss the Einstein relation $\sigma/e^2 = (\partial n/\partial\mu)D_e$, which connects the conductivity σ with the diffusion coefficient of the interacting electrons. It is shown that the quantity $\partial n/\partial\mu$ does not contain diffusion corrections. In contrast to $\partial n/\partial\mu$, the state density[2)] on the Fermi level $\nu_{\varepsilon F}(T)\to 0$ as $T\to 0$. It should be noted that the state density does not enter in the charges of the renormalization group and exerts no influence on the renormalized resistances. In the calculation of $\nu_\varepsilon(T)$ it was necessary to sum doubly logarithmic corrections, which are absent from the conductivity. Thus, the electronic correlations that determine $\nu_\varepsilon(T)$ and $\sigma(T)$ have entirely different forms. As a result, in the case considered in this paper it was possible to have $\nu_{\varepsilon F}(T)\to 0$ and simultaneously $\sigma(T)\to\infty$ as $T\to 0$.

§2. DERIVATION OF THE FREE-ENERGY FUNCTIONAL

1. Free electrons in a disordered system

Our task is to study Coulomb interaction of electrons. To obtain the correct statistics we therefore obtain the Lagrangian of the diffusion model, following Ref. 10, by integrating over the classical Fermi fields (Grassman anticommuting variables[11]). To average over the randomly distributed impurities we use the replica method.[12] We calculate the partition function Z_N of N replicas of the system and average Z_N over the impurities. Then

$$-T\ln\langle\langle Z_N\rangle\rangle\to N\langle\langle F\rangle\rangle \quad \text{as} \quad N\to 0, \tag{2.1}$$

where the double angle brackets denote averaging over the impurities and F is the free energy of the system. Equation (2.1) enables us to find $\langle\langle F\rangle\rangle$ by using standard statistical-physics methods. We express Z_N in terms of a functional integral with respect to the anticommuting variables $\bar\psi$ and ψ:

$$Z_N=\int\exp\Big(\sum_{i=1}^{N} S^i\{\bar\psi^i,\psi^i\}\Big)\Pi\, d\bar\psi^i\, d\psi^i, \tag{2.2}$$

where i is the replica index and S is a functional that has the meaning of action. In the temperature technique

$$S^i=\int dr\sum_{n,\alpha}\Big[\bar\psi_n^{i,\alpha}\Big(i\varepsilon_n+\frac{1}{2m}\Delta+\mu+V(r)\Big)\psi_n^{i,\alpha}\Big]+S^i_{e-e}. \tag{2.3}$$

Here $\varepsilon_n = (2n+1)\pi T$ are the Fermi frequency, α is the spin index, S_e^{i-e} takes into account the electron-electron interaction, and $V(r)$ is the potential of the impurities. When averaging over the impurity locations the potential will be regarded as a random quantity with a Gaussian δ-correlated distribution

$$\langle V(r)V(r')\rangle=\frac{1}{2\pi\nu\tau}\delta(r-r'), \tag{2.4}$$

where ν is the state density at $T\sim\tau^{-1}$. Such a distribution describes small-radius impurities with a weak potential. In the general case all the results remain valid when the constants are redefined.

Averaging Z_N over the distribution (2.4), we obtain

$$\langle\langle Z_N\rangle\rangle=\int e^S\prod_i d\bar\psi^i\, d\psi^i, \tag{2.5}$$

$$S=\int dr\Big\{\sum_{n,i,\alpha}\Big[\bar\psi_n^{i,\alpha}\Big(i\varepsilon_n+\frac{1}{2m}\Delta+\mu\Big)\psi_n^{i,\alpha} + \frac{1}{4\pi\nu\tau}(\bar\psi\psi)^2\Big]+\sum_i S^i_{e-e}\Big\}, \tag{2.6}$$

where

$$(\bar\psi\psi)=\sum_{n,i,\alpha}\bar\psi_n^{i,\alpha}\psi_n^{i,\alpha}.$$

In this section we derive the effective functional for noninteracting electrons in the impurity field. The contribution S_{e-e} will be taken into account in the next section.

We separate in the impurity term the regions of interest to us, with small momentum transfer $k<(v_F\tau)^{-1}=k_0$. Since we are investigating the magnetic case, when the Cooper correlations are suppressed, the corresponding momentum-transfer region is not considered:

$$\frac{1}{4\pi\nu\tau}\int(\bar\psi\psi)^2\,dr$$

$$=\frac{1}{4\pi\nu\tau}\frac{1}{V}\sum_{\substack{p_1,p_2;\\ k<k_0}}[(\bar\psi_{p_1}\psi_{p_1+k})(\bar\psi_{p_2}\psi_{p_2-k})+(\bar\psi_{p_1}\psi_{p_2})(\bar\psi_{p_2+k}\psi_{p_1+k})], \tag{2.7}$$

and V is the volume. To reveal the slow variables we express the exponential of (2.7) with the aid of Gaussian integrals. The first term is separated by integrating with respect to the slow variable $\mu(r)$:

$$\exp\Big[\frac{1}{4\pi\nu\tau V}\sum_{p,k}(\bar\psi_{p_1}\psi_{p_1+k})(\bar\psi_{p_2}\psi_{p_2-k})\Big]$$

$$=\int\exp\Big\{-\int\Big[\frac{1}{2}\mu^2(r)$$

$$-\frac{1}{(4\pi\nu\tau)^{1/2}}\mu(r)(\bar\psi(r)\psi(r))\Big]dr\Big\}\Pi\, d\mu$$

$$\times\Big[\int\exp\Big\{-\frac{1}{2}\int\mu^2(r)\,dr\Big\}\Pi\, d\mu\Big]^{-1}. \tag{2.8}$$

It follows from (2.8) that the field $\mu(r)$ leads only to small and smooth changes of the chemical potential μ; we shall therefore not consider hereafter the first term of (2.7). To split the second member of the impurity term (2.7) it is necessary to integrate over the matrix variables[3)] $Q_{nm}^{ij}(r)$:

$$\exp\left[\frac{1}{4\pi\nu\tau V}\sum_{p,k}(\bar\psi_{p_1}\psi_{p_2})(\bar\psi_{p_2+k}\psi_{p_1+k})\right]$$

$$=\int\exp\left\{-\int\left[\frac{\pi\nu}{4\tau}\operatorname{Sp}\hat Q^2-\frac{i}{2\tau}(\bar\psi\hat Q\psi)\right]dr\right\}\Pi\,dQ$$

$$\times\left[\int\exp\left(-\int\frac{\pi\nu}{4\tau}\operatorname{Sp}\hat Q^2\,dr\right)\Pi\,dQ\right]^{-1}, \tag{2.9}$$

where

$$(\bar\psi\hat Q\psi)=\sum\psi_n{}^iQ_{nm}{}^{ij}\psi_m{}^j,\quad \operatorname{Sp}\hat Q^2=\sum Q_{nm}{}^{ij}Q_{mn}{}^{ji}.$$

The matrix Q is chosen to be Hermitian: $Q_{nm}{}^{ij}=(Q_{mn}{}^{ji})^*$. We integrate $\langle\langle Z_N\rangle\rangle$ with allowance for (2.9) over the Fermi fields $\bar\psi$ and ψ and we use the fact that the matrix $\hat Q$ varies slowly in space. As a result we obtain $\langle\langle Z_N\rangle\rangle$ in the form of a functional integral over $\hat Q$:

$$\langle\langle Z_N\rangle\rangle=\int\exp(-\mathscr{F}\{\hat Q\}/T)\,\Pi\,dQ,$$

$$\mathscr{F}\{\hat Q\}=T\int\left\{-\operatorname{Sp}\ln\left[i\hat\varepsilon+\left(\frac{1}{2m}\Delta+\mu\right)\hat I+\frac{i}{2\tau}\hat Q(r)\right]\right. \tag{2.10}$$

$$\left.+\frac{\pi\nu}{4\tau}\operatorname{Sp}\hat Q^2(r)\right\}dr, \tag{2.11}$$

where $\hat\varepsilon$ is a matrix made up of the Fermi frequencies:

$$\hat\varepsilon=\varepsilon_n\delta_{nm}\delta_{ij}\delta_{\alpha\beta},\quad \varepsilon_n=(2n+1)\pi T, \tag{2.12}$$

and $\hat I$ is a unit matrix

$$I_{nm}{}^{ij}=\delta_{nm}\delta_{\alpha\beta}\delta_{ij}. \tag{2.13}$$

In the integral (2.11) an important role is played by the minima of $\mathscr{F}\{\hat Q\}$. The position of a minimum is determined from the equation

$$\hat Q=\frac{i}{\pi\nu}\int\hat G(p)\frac{dp}{(2\pi)^d}, \tag{2.14}$$

$$\hat G^{-1}(p)=i\hat\varepsilon-\left(\frac{p^2}{2m}-\mu\right)\hat I+\frac{i}{2\tau}\hat Q. \tag{2.15}$$

The solution of (2.14) is the matrix $\hat Q=\hat\Lambda$, where

$$\Lambda_{nm}{}^{ij}=\operatorname{sign}\varepsilon_n\delta_{nm}\delta_{\alpha\beta}\delta_{ij}. \tag{2.16}$$

Substituting $\hat Q=\hat\Lambda$ in (2.15), we obtain for $G(p)$ the standard expression

$$G(p)=\left(i\varepsilon_n-\frac{p^2}{2m}+\mu+\frac{i}{2\tau}\operatorname{sign}\varepsilon_n\right)^{-1}.$$

We note that if the ε-dependence in (2.11) is disregarded, homogeneous unitary transformations (rotations) of the matrix $\hat\Lambda$

$$\hat Q=\hat U^{-1}\hat\Lambda\hat U,\quad \hat Q^2=\hat I,\quad \operatorname{Sp}\hat Q=0 \tag{2.17}$$

($\hat U$ is a unitary matrix) do not change at all the free energy $\mathscr{F}$. Expanding $\mathscr{F}\{Q\}$ in terms of $\hat\varepsilon$ and in terms of small deviations of $\hat Q$ from the solution (2.16), we obtain

$$\mathscr{F}\{Q\}=F_\Lambda-T\int\frac{dk}{(2\pi)^d}\left\{\frac{1}{4\tau^2}\int\frac{dp}{(2\pi)^d}\operatorname{Sp}\hat G_p\delta\hat Q_k\hat G_{p+k}\delta\hat Q_{-k}\right.$$

$$\left.-\frac{\pi\nu}{2\tau}\operatorname{Sp}\delta\hat Q_k\,\delta\hat Q_{-k}\right\}-iT\int\operatorname{Sp}(\hat\varepsilon\hat G(p))\frac{dp}{(2\pi)^d}. \tag{2.18}$$

It is necessary to separate the longitudinal changes of $\hat Q$, in which the conditions (2.17) are violated, from the transverse ones, when these conditions are satisfied and

$$\hat Q\delta\hat Q+\delta\hat Q\hat Q=0. \tag{2.19}$$

The longitudinal variations make a large contribution to the free energy, so that these fluctuations can be neglected. As a result, it follows from (2.17)–(2.19) that the problem of electron diffusion in the impurity field reduces to a matrix nonlinear model

$$\mathscr{F}_D=T\frac{\pi\nu}{4}\int[D\operatorname{Sp}(\nabla\hat Q)^2-4\operatorname{Sp}(\hat\varepsilon\hat Q)]dr, \tag{2.20}$$

where

$$\hat Q(r)=\hat U^{-1}(r)\hat\Lambda\hat U(r). \tag{2.21}$$

The last term in (2.20) violates the symmetry of the functional $\mathscr{F}_D$ with respect to the rotations $\hat Q$.[8] If an analogy is made with a Heisenberg magnet, the matrix $\hat\varepsilon$ is the analog of the magnetic field: $\hat\varepsilon$ fixes the equilibrium position $\hat Q=\hat\Lambda$. But at low frequencies the matrix $\hat Q$ is not strictly fixed, and transverse deviations of $\hat Q$ from the equilibrium positions are substantial. We shall show that these fluctuations are diffusion modes. To this end we write the unitary matrix $\hat U$ in (2.21) in the form[10]

$$\hat U=\exp\left(\frac{\hat W}{2}\right),$$

$$W_{\varepsilon\varepsilon'}=\begin{array}{c|c} & \begin{array}{cc}\varepsilon'>0 & \varepsilon'<0\end{array}\\ \end{array}\left(\begin{array}{cc}0 & B\\ -B^+ & 0\end{array}\right)\begin{array}{l}\varepsilon>0\\ \varepsilon<0\end{array},\quad \hat Q=\hat\Lambda\exp\hat W \tag{2.22}$$

Expanding $\mathscr{F}_D$ in power of B and retaining only the quadratic terms, we obtain

$$\mathscr{F}_0=T\frac{\pi\nu}{2}\int\sum_{\substack{n>0\\ n'<0}}[D\nabla B^{ij}_{nn'}\nabla B^{+ji}_{n'n}+(\varepsilon_n-\varepsilon_{n'})B^{ij}_{nn'}B^{+ji}_{n'n}]dr. \tag{2.23}$$

Consequently, the correlator $\langle BB^+\rangle_0$ is the propagator of the diffusion mode:

$$\langle B^{ij}_{\varepsilon_1\varepsilon_2}B^{+kl}_{\varepsilon_3\varepsilon_4}\rangle_0=\frac{2}{\pi\nu}\frac{I^{il}_{\varepsilon_1\varepsilon_4}I^{jk}_{\varepsilon_2\varepsilon_3}}{Dk^2+\Omega},\quad \Omega=\varepsilon_1-\varepsilon_2>0; \tag{2.24}$$

since we are using the temperature technique, there is no factor $(-i)$ in front of the frequency in the diffusion propagator.

In the derivation of the free-energy functional (2.20) of the diffuse electrons we followed mainly Ref. 10. In that paper (just as in Refs. 8 and 9) was considered an interaction via the impurity field between only two levels with energies $\varepsilon=\pm\omega/2$. This was technically carried out by introducing a doubled set of replica indices. In contrast to Refs. 8–10, account if taken in the functional $\mathscr{F}_D$ of the aggregate of all the levels with energies $|\varepsilon|<\tau^{-1}$, and only one set of replicas was used.

2. Allowance for Coulomb interaction

We return to Eq. (2.6) and take into account the Coulomb interaction. We separate in S^i_{e-e} the regions of small momentum and frequency transfers, just as in the consideration of the impurity term (2.7), the magnetic field [see (1.12)] allows us to ignore the small-transfer region corresponding to the Cooper channel. Then

$$S^{i}_{e-e}=-\frac{T}{2V}\sum\{\Gamma\bar{\psi}^{i,\alpha}_{n_1}(p_1)\bar{\psi}^{i,\beta}_{n_2}(p_2)\psi^{i,\beta}_{n_2-m}(p_2-k)\psi^{i,\alpha}_{n_1+m}(p_1+k)$$
$$+\Gamma_2\bar{\psi}^{i,\alpha}_{n_1}(p_1)\bar{\psi}^{i\,\cdot}_{n_2}(p_2)\psi^{i,\beta}_{n_1+m}(p_1+k)\psi^{i,\alpha}_{n_2-m}(p_2-k)\}. \tag{2.25}$$

Here Γ and Γ_2 are the static amplitudes of the electron-electron interaction; their connection with the Fermi-liquid constants is discussed below.

We introduce the matrix fields $\hat{X}$ and $\hat{Y}$:

$$X_{mn}{}^{ij}=X_{\alpha\beta}(m-n)\delta_{ij},\quad Y_{mn}{}^{ij}=Y(m-n)\delta_{ij}\delta_{\alpha\beta}$$

and write the exponential of S^{i}_{e-e} in the form of a Gaussian integral over these fields:

$$\exp S^{i}_{e-e}\{\psi\}=\frac{\int\exp(S^{i}_{e-e}\{\psi,\hat{X},\hat{Y}\}-S_2{}^{i}\{\hat{X}\}-S_2{}^{i}\{\hat{Y}\})\Pi\,dX\,dY}{\int\exp(-S_2{}^{i}\{\hat{X}\}-S_2{}^{i}\{\hat{Y}\})\Pi\,dX\,dY}, \tag{2.26}$$

where

$$S^{i}_{e-e}\{\psi,\hat{X},\hat{Y}\}=T\int dr[i\Gamma^{1/2}(\bar{\psi}\hat{Y}\psi)+\Gamma_2{}^{1/2}(\bar{\psi}\hat{X}\psi)], \tag{2.27}$$

$$S_2{}^{i}\{\hat{X}\}=\frac{T}{2}\int\sum_{m,\alpha}X_{\alpha\beta}{}^{i}(m)X_{\beta\alpha}{}^{i}(-m)\,dr,$$
$$S_2{}^{i}\{\hat{Y}\}=\frac{T}{2}\int\sum_{m}Y^{i}(m)Y^{i}(-m)\,dr. \tag{2.28}$$

We integrate $\langle\langle Z_N\rangle\rangle$ over the Fermi fields with allowance for the fact that the matrices $\hat{Q}$, $\hat{X}$, and $\hat{Y}$ vary slowly in space. We obtain the free-energy functional

$$\mathscr{F}\{\hat{Q},\hat{X},\hat{Y}\}$$
$$=T\int\Big\{-\mathrm{Sp}\ln\Big[i\hat{\varepsilon}+\Big(\frac{1}{2m}\Delta+\mu\Big)\hat{I}+\frac{i}{2\tau}\hat{Q}+iT\Gamma^{1/2}\hat{Y}+T\Gamma_2{}^{1/2}\hat{X}\Big]$$
$$+\frac{\pi\nu}{4\tau}\mathrm{Sp}\,\hat{Q}^2\Big\}dr+T\sum_{i}(S_2{}^{i}\{\hat{X}\}+S_2{}^{i}\{\hat{Y}\}). \tag{2.29}$$

We expand Sp ln [...] in powers of $\hat{X}$ and $\hat{Y}$. The small parameter $T\tau\ll1$ allows us to restrict the expansion to terms linear in $\hat{X}$ and $\hat{Y}$. In analogy with the procedure used in the derivation of the functional $\mathscr{F}_D\{\hat{Q}\}$, We integrate over the fields Q by the saddle-point method. As a result we obtain

$$\mathscr{F}\{\hat{Q},\hat{X},\hat{Y}\}=\mathscr{F}_D\{\hat{Q}\}-\pi\nu T\int[T\Gamma^{1/2}\,\mathrm{Sp}(\hat{Y}\hat{Q})$$
$$-iT\Gamma_2{}^{1/2}\,\mathrm{Sp}(\hat{X}\hat{Q})]dr+T\sum_{i}(S_2{}^{i}\{\hat{X}\}+S_2{}^{i}\{\hat{Y}\}), \tag{2.30}$$

where the matrices $\hat{Q}$ satisfy (2.21). Integrating over the fields $\hat{X}$ and $\hat{Y}$ we arrive at the final expression

$$\mathscr{F}\{\hat{Q}\}=T\frac{\pi\nu}{4} \tag{2.31}$$
$$\times\int[D\,\mathrm{Sp}(\nabla\hat{Q})^2-4\,\mathrm{Sp}(\hat{\varepsilon}\hat{Q})-\nu\Gamma(\hat{Q}\gamma_1\hat{Q})+\nu\Gamma_2(\hat{Q}\gamma_2\hat{Q})]dr.$$

Here γ_1 and γ_2 are fourth-rank tensors that differ in their spin structure:

$$(\hat{Q}\gamma_1\hat{Q})=2\pi T\sum_{n,i}Q^{\alpha\alpha}_{n_1n_2}Q^{\beta\beta}_{n_3n_4}\delta(n_1+n_3,n_2+n_4)\delta^{i},$$
$$(\hat{Q}\gamma_2\hat{Q})=2\pi T\sum_{n,i}Q^{\alpha\beta}_{n_1n_2}Q^{\beta\alpha}_{n_3n_4}\delta(n_1+n_3,n_2+n_4)\delta^{i}. \tag{2.32}$$

The symbol δ^{i} indicates that all the replica indices of the matrices $\hat{Q}$ coincide, and $\delta(n,m)=\delta_{nm}$. The functional $\mathscr{F}$ describes the interacting electrons with energies $\varepsilon<\tau^{-1}$.

So far we have completely ignored the energy region $\tau^{-1}<\varepsilon\lesssim\mu$. Allowance for this part of the spectrum leads to Fermi-liquid renormalizations in (2.31). Let the Green's function of the electron have near the Fermi level the form

$$G=a\Big(i\varepsilon_n-v_Fp+\frac{i}{2\tau}\,\mathrm{sign}\,\varepsilon_n\Big)^{-1}.$$

Then, in the case $d=2$ and 3

$$D=\frac{v_F{}^2\tau}{d},\qquad\nu=\frac{m^*}{2\pi}\Big(\frac{k_F}{\pi}\Big)^{d-2}, \tag{2.33}$$

where, as usual, $v_F=k_F/m^*$, $m^*/m=1+F_1/3$ (F_1 are the standard coefficients of the Landau theory). We separate from the amplitude Γ the statically screened Coulomb interaction Γ_0

$$\Gamma=\Gamma_0+\Gamma_1,\quad\Gamma_0(k)=\frac{1}{(1+F_0)^2}\frac{v_c(k)}{1+v_c(k)\pi(\Omega=0)}. \tag{2.34}$$

Here $v_c(k)$ is the Coulomb potential, π ($\Omega=0$) is the static part of the polarization operator, and the factor $1/(1+F_0)^2$ is due to allowance for the vertices (Fig. 1). By virtue of the known Ward identity we have $\pi(\Omega=0)=\partial n/\partial\mu$. The value of $\partial n/\partial\mu$ is determined by the joint contribution of all the electrons and is equal to

$$\pi(\Omega=0)=\partial n/\partial\mu=2\nu/(1+F_0). \tag{2.35}$$

Figure 2a shows one of the diagrams that contributes to Γ_1. These diagrams cannot be separated by cutting only one Coulomb line. The amplitude Γ_2 (Fig. 2b) describes scattering through a large angle. We show now how Γ_1 and Γ_2 are connected with the quasiparticle scattering amplitude Γ^{k} (Ref. 13)

$$\Gamma_1\delta_{\alpha\gamma}\delta_{\beta\delta}-\Gamma_2\delta_{\alpha\delta}\delta_{\beta\gamma}=a^2\Gamma^{k}_{\alpha\beta;\,\gamma\delta},$$

and note that the following relation holds:

$$1-2\nu\Gamma_1+\nu\Gamma_2=1/(1+F_0). \tag{2.36}$$

Thus the effective functional $\mathscr{F}$ of the free energy of the

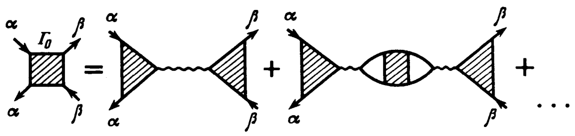


FIG. 1.

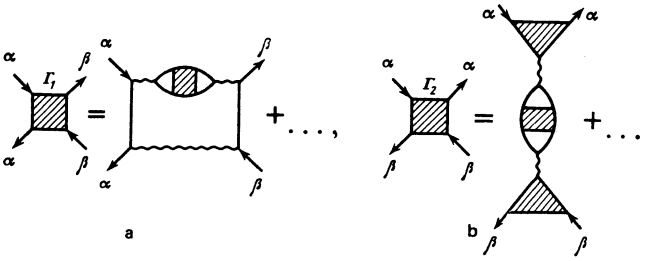


FIG. 2.

interacting diffuse electrons is completely determined.

Using $\mathscr{F}$, we obtain the polarization operator $\pi(k,\Omega)$ in the Fermi-liquid approximation. To this end it is necessary to expand $\mathscr{F}$ up to second order in B [see (2.23)–(2.24)] and calculate the ladder diagrams shown in Fig. 3b:

$$\pi(k,\ \Omega)=\frac{\partial n}{\partial\mu}\left[1-\frac{1}{1+F_0}\frac{\pi^2\nu T\sum\limits_n\langle B^{11}_{\varepsilon_n\varepsilon_n-|\Omega|}B^{+11}_{\varepsilon_n-|\Omega|\varepsilon_n}\rangle_0}{1+(-2\nu\Gamma_1+\nu\Gamma_2)\pi^2\nu T\sum\limits_n\langle B^{11}_{\varepsilon_n\varepsilon_n-|\Omega|}B^{+11}_{\varepsilon_n-|\Omega|\varepsilon_n}\rangle_0}\right],$$

where $\langle BB^+\rangle_0$ is given by (2.24). The coefficient 2 in front of Γ_1 is of spin origin. Summing over the frequencies we obtain, taking (2.36) into account,

$$\pi(k,\Omega)=\frac{\partial n}{\partial\mu}\frac{Dk^2}{Dk^2+(1-2\nu\Gamma_1+\nu\Gamma_2)|\Omega|}. \tag{2.37}$$

We note that $\pi(\Omega;\ k=0)=0$, which is a mathematical formulation of the particle-number conservation condition. Summation of the ladder diagrams leads to the factor $(1-2\nu\Gamma_1+\nu\Gamma_2)$ in front of $|\Omega|$ in the denominator of (2.37). As a result, the diffusion coefficient in a Fermi liquid is

$$D_{FL}=D/(1-2\nu\Gamma_1+\nu\Gamma_2)=(1+F_0)D. \tag{2.38}$$

The continuity equation connects the conductivity σ with $\pi(k,\Omega)$:

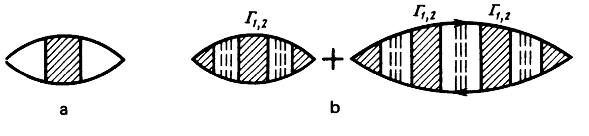
$$\frac{\sigma}{e^2}=\frac{\Omega}{k^2}\pi(k,\Omega). \tag{2.39}$$

This leads to the Einstein relation

$$\frac{\sigma}{e^2}=\frac{\partial n}{\partial\mu}D_{FL}. \tag{2.40}$$

Equations (2.38) and (2.40) duplicate well the known results of the Landau theory in the presence of impurities. In conclusion we rewrite, with allowance for (2.35) and (2.40), the parameter g_0 [see (1.1)] that determines the scale of the corrections in a metal with impurities:

$$g_0=e^2/\sigma\cdot 2\pi^2=1/(2\pi)^2\nu D. \tag{2.41}$$



a b

FIG. 3. Diagrams for polarization operator: a—static, b—dynamic part.

§3. RENORMALIZATION GROUP

1. Separation of the "fast" diffusion modes

To study the logarithmic corrections that arise when account is taken of the Coulomb interaction in a two-dimensional metal with impurities, we use the renormalization-group method. In contrast to the case of free electrons, the functional that takes into account the Coulomb interaction contains a large number of renormalizable quantities (charges):

$$S=-\frac{\pi\nu}{4}\int[D\,\mathrm{Sp}(\nabla\hat{Q})^2-\nu(\Gamma_0+\Gamma_1)\times(\hat{Q}\gamma_1\hat{Q})+\nu\Gamma_2(\hat{Q}\gamma_2\hat{Q})-4z\,\mathrm{Sp}(\hat{\varepsilon}\hat{Q})]dr. \tag{3.1}$$

Compared with (2.31), we have introduced in S the factor z. The point is that in contrast to the case of free electrons, the coefficient of the last term in the integral (3.1) is altered by the renormalization. Besides z, the charges in (3.1) are the dimensionless quantities g, $\nu\Gamma_0$, $\nu\Gamma_1$, and $\nu\Gamma_2$ (we shall find it convenient to retain the division of Γ into Γ_0 and Γ_1). The small parameter in the theory is only the quantity g:

$$g=1/(2\pi)^2\nu D\ll 1. \tag{3.2}$$

It will be shown below that in first order in g the theory considered is renormalizable, and exact equations of the renormalization group are contained in this order.

The matrix $\hat{Q}W$ in (3.1) satisfies the conditions

$$\hat{Q}^2=\hat{I},\quad \mathrm{Sp}\,\hat{Q}=0,\quad \hat{Q}=\hat{Q}^+. \tag{3.3}$$

These matrices can be written in the form

$$Q=U^{-1}\Lambda U, \tag{3.4}$$

where U is a unitary matrix. It is assumed in Eq. (3.1) for S that $|\varepsilon|$, $Dk^2<\tau^{-1}$. Following the renormalization-group method, 14 we integrate e^S with respect to the variables Q in the layer

$$\lambda\tau^{-1}<|\varepsilon|<\tau^{-1},\quad \lambda\tau^{-1}<Dk^2<\tau^{-1}, \tag{3.5}$$

where $0<\lambda<1$. To this end we express the matrix U as a product of a fast and slow parts[15,10]

$$U=U_0\tilde{U}, \tag{3.6}$$

where U_0 and $\tilde{U}$ are unitary matrices, U_0 fast and $\tilde{U}$ slow. Substituting (3.6) in (3.1) we reduce the functional S to the form

$$S=-\frac{\pi\nu}{4}\int\{D(\mathrm{Sp}(\nabla Q_0)^2+2\,\mathrm{Sp}([\nabla Q_0,Q_0]A)+\mathrm{Sp}[Q_0,A]^2)-\nu(\Gamma_0+\Gamma_1)(\tilde{U}^+Q_0\tilde{U}\gamma_1\tilde{U}^+Q_0\tilde{U})+\nu\Gamma_2(\tilde{U}^+Q_0\tilde{U}\gamma_2\tilde{U}^+Q_0\tilde{U})-4z\,\mathrm{Sp}(\hat{\varepsilon}\tilde{U}^+Q_0\tilde{U})\}dr, \tag{3.7}$$

$$Q_0=U_0{}^+\Lambda U_0,\quad A=\nabla\tilde{U}\tilde{U}^+=-\tilde{U}\nabla\tilde{U}^+. \tag{3.8}$$

If we now integrate over the fast variables Q_0, we obtain $\tilde{S}$, which describes the slow fluctuations:

$$\tilde{S}\{\tilde{Q}\}=\ln\int e^S\Pi\,dQ_0, \tag{3.9}$$

where $\tilde{Q}=\tilde{U}^+\Lambda\tilde{U}$ is a slow variable.

Before we proceed to calculate the integral with respect to Q_0, we note that Q is a nonlocal variable

$$Q_{nm}\neq Q(n-m), \qquad (3.10)$$

where n and m are indices corresponding to Matsubara frequencies. What is essential is the dependence on each of the energy indices separately, and not on the difference $\varepsilon_n - \varepsilon_m$. In particular, if at least one of the indices of the matrix $\tilde{U}_{nm}$ is in the integration band, then $\tilde{U}$ must be set equal to the unit matrix:

$$\tilde{U}_{nm}{}^{ij}=I_{nm}{}^{ij}, \text{ if } \lambda\tau^{-1}<|\varepsilon_n|<\tau^{-1} \text{ or } \lambda\tau^{-1}<|\varepsilon_m|<\tau^{-1}. \qquad (3.11)$$

The physical meaning of this condition is the following. As already noted, ε in (3.1) is the analog of the magnetic field in a magnet. At large ε_n the slow matrix $\tilde{Q}_{nm}$ must be "aligned with the field": $\tilde{Q}_{nm} = \Lambda_{nm}$ and consequently, Eq. (3.11) holds.

When integrating with respect to Q_0 it is convenient to use the parametrization[10] (2.22) for the matrix U_0:

$$U_0=\exp(W/2), \quad W=-W^+, \quad \Lambda W=-W\Lambda. \qquad (3.12)$$

It is assumed that if all the arguments of W are outside the integration region, then $W=0$:

$$W_{nm}(k)=0, \text{ if } \{Dk^2, \varepsilon_n, \varepsilon_m\}<\lambda\tau^{-1}. \qquad (3.13)$$

To find the renormalization-group equations in first order in g and S it suffices to retain the terms quadratic in W. We write down the last term of the functional (3.7)

$$\mathrm{Sp}\,(\hat{\varepsilon}Q)\approx\mathrm{Sp}\,(\hat{\varepsilon}\tilde{Q})+{}^1/_2\,\mathrm{Sp}\,(\hat{\varepsilon}\tilde{U}^+\Lambda WW\tilde{U}), \qquad (3.14)$$

$$\mathrm{Sp}(\hat{\varepsilon}\tilde{U}^+\Lambda WW\tilde{U})=\sum_{\varepsilon_n>\lambda\tau^{-1}}\varepsilon_n(\Lambda WW)_{nn}+\sum_{\varepsilon_n<\lambda\tau^{-1}}\varepsilon_n(\tilde{U}^+\Lambda WW\tilde{U})_{nn}. \qquad (3.15)$$

Proceeding similarly with the remaining terms of the functional (3.7), we obtain

$$\begin{aligned} S\{Q\}=S\{\tilde{Q}\}&+S_0\{W\}+S_1\{W,\tilde{U}\}+S_2\{W,\tilde{U}\} \\ &+S_{int}\{W,\tilde{U}\}+S_\varepsilon\{W,\tilde{U}\}, \end{aligned} \qquad (3.16)$$

where

$$S_0\{W\}=\frac{\pi\nu}{4}\int[D\,\mathrm{Sp}(\nabla W)^2+2z\,\mathrm{Sp}(\hat{\varepsilon}\Lambda W^2)]\,dr, \qquad (3.17)$$

$$S_1=\frac{\pi\nu}{2}\int D\,\mathrm{Sp}([\nabla W, W]A)\,dr, \qquad (3.18a)$$

$$S_2=-\frac{\pi\nu}{2}\int D[\mathrm{Sp}(\Lambda A\Lambda W^2A)+\mathrm{Sp}(\Lambda WA)^2]\,dr, \qquad (3.18b)$$

$$\begin{aligned} S_{int}=&\frac{\pi\nu}{4}\int(\tilde{U}^+\Lambda W\tilde{U}[(\Gamma_0+\Gamma_1)\gamma_1-\Gamma_2\gamma_2]\tilde{U}^+\Lambda W\tilde{U})\,dr \\ &+\frac{\pi\nu}{4}\int(\tilde{U}^+\Lambda W^2\tilde{U}[(\Gamma_0+\Gamma_1)\gamma_1-\Gamma_2\gamma_2]\tilde{Q})\,dr, \end{aligned} \qquad (3.18c)$$

$$S_\varepsilon=\frac{\pi\nu}{2}\int z\,\mathrm{Sp}(\hat{\varepsilon}\tilde{U}^+\Lambda W^2\tilde{U})\,dr. \qquad (3.18d)$$

The determination of the renormalized functional $\tilde{S}\{\tilde{Q}\}$ reduces thus to averaging of several expressions containing W, with a weight $\exp(S\{W\})$. We shall designate such an averaging by $\langle\ldots\rangle_0$. A convenient form of writing down the correlator $\langle WW\rangle_0$ is [cf. (2.24)]:

$$\begin{aligned} &\langle W_{mn}^{\alpha\beta}(k)\,W_{rt}^{\gamma\delta}(-k)\rangle_0 \\ &=-\frac{2}{\pi\nu}\left(\frac{I_{mt}{}^{\alpha\delta}I_{nr}{}^{\beta\gamma}-\Lambda_{mt}{}^{\alpha\delta}\Lambda_{nr}{}^{\beta\gamma}}{2}\right)\frac{1}{Dk^2+z|\varepsilon_n-\varepsilon_m|}. \end{aligned} \qquad (3.19)$$

The expression in the parentheses in the right-hand side of (3.19) is a projection operator—for the existence of a diffusion propagator it is necessary that one of the electron lines have a positive frequency and the other a negative one.

2. Renormalization of the charge g

The renormalization of the charge g is determined by those terms of the functional integral (3.9) which contain the gradients of the matrices $\tilde{U}$. If the Coulomb interaction is not turned on the charge g is not renormalized in the considered magnetic case,[8,10] since

$${}^1/_2\langle S_1{}^2\rangle_0+\langle S_2\rangle_0\to 0 \text{ as } N\to 0.$$

With account taken of the Coulomb interaction, we confine ourselves initially to the first order in the amplitudes Γ and Γ_2. The renormalization is then determined by the following terms of the integral (3.9):

$$\delta\tilde{S}_g=\langle S_1S_{int}\rangle_0+{}^1/_2\langle S_1{}^2S_{int}\rangle_0+\langle S_2S_{int}\rangle_0+\langle S_{int}\rangle_0. \qquad (3.20)$$

Let us explain the presence of the last term, since it does not contain explicitly the gradients of $\tilde{U}$. When calculating $\langle S_{\text{int}}\rangle_0$ the second term of (3.18c) will not be considered, for in final analysis its contribution is $\sim\mathrm{Sp}\,\hat{Q}=0$. Separating the region of the fast frequencies and taking (3.11) into account, we obtain after simple transformations

$$\begin{aligned} \langle S_{int}\rangle_0=&2\pi T\sum_{|\varepsilon_r|>\lambda\tau^{-1}}\tilde{U}_{mn}^{+ij}(k)\,\tilde{U}_{pm}{}^{li}(-k)\,\frac{I_{np}{}^{jl}I_{rr}{}^{ii}-\Lambda_{np}{}^{jl}\Lambda_{rr}{}^{ii}}{2} \\ &\times\frac{\nu\Gamma(m-r;q)-2\nu\Gamma_2(m-r;q)}{D(q+k)^2+z|\varepsilon_n-\varepsilon_r|} \end{aligned} \qquad (3.21)$$

(the factor 2 in front of the amplitude Γ_2 is the result of summation over the spin projections). If we neglect the dependence on the slow variables in the Coulomb amplitudes and in the diffusion propagator against the background of the fast ones, the unitary matrices $\tilde{U}$ and $\tilde{U}^+$ can be multiplied. Then the functional dependence on the matrices $\tilde{U}$ in (3.21) vanishes, and the reveal this dependence it is necessary to expand in terms of the slow variable. Expanding the diffusion propagator in terms of the momentum k and recognizing that

$$-\sum_k k^2\tilde{U}^+(k)\,\tilde{U}(-k)=\int\mathrm{Sp}(\mathbf{AA})\,dr,$$

we obtain

$$\langle S_{int}\rangle_0\approx\text{const}+\frac{1}{4\pi}(J_1-J_2)\int\mathrm{Sp}(\mathbf{AA})\,dr, \qquad (3.22)$$

where J_1 and J_2 are logarithmic integrals:

$$\begin{aligned} J_1&=\int_{\lambda\tau^{-1}}\frac{\nu\Gamma_0+\nu\Gamma_1-2\nu\Gamma_2}{(Dq^2+z\Omega)^2}\,2Dq\,dq\,d\Omega, \\ J_2&=\int_{\lambda\tau^{-1}}\frac{\nu\Gamma_0+\nu\Gamma_1-2\nu\Gamma_2}{(Dq^2+z\Omega)^3}\,4D^2q^3\,dq\,d\Omega. \end{aligned} \qquad (3.23)$$

Calculating the remaining terms in (4.20) we obtain

$$\delta S_g=\frac{1}{16\pi}J_2\int\mathrm{Sp}(\nabla\tilde{Q})^2\,dr, \qquad (3.24)$$

where account is taken of the fact that

$$\mathrm{Sp}(A\Lambda A\Lambda-A^2)={}^1/_2\mathrm{Sp}(\nabla Q)^2. \tag{3.25}$$

Equation (3.24) corresponds to

$$\delta D=-\frac{1}{(2\pi)^2\nu}J_2,\qquad \delta g=g^2J_2. \tag{3.26}$$

We note that only the sum (3.20), rather than each individual term, reduces to the combination (3.25) and duplicates $(\nabla Q)^2$.

In the calculation of δS_g we have neglected the contribution of several diagrams. Figure 4 shows the diagrams corresponding to different methods of pairing the matrices W in the second term of (3.20). In diagram 4a the integration is carried out both over the momentum and over the frequency, while the contribution of the second diagram was already taken into account in the derivation of (3.24). On the contrary, in diagram 4b all the frequencies are fixed by the arguments of the matrices A and only the momentum is integrated. The result is a power-law divergence at small momenta. Besides 4b there exist also several terms with divergences of this type. We shall not stop to consider them for the following reason. In contrast to diagrams ot type 4a, which renormalize $(\nabla Q)^2$, the contribution of the diagram 4b has the same structure as those terms of S which describe the electron interaction:

$$\delta S_{(4b)}\sim\frac{1}{4\pi}\frac{1}{\lambda\tau^{-1}}2\pi T\sum\nu\Gamma_i\mathbf{A}_{n_1n_2}(k)\mathbf{A}_{n_3n_4}(-k)$$
$$\times\ \delta(n_1+n_3,n_2+n_4). \tag{3.27}$$

Comparing (3.27) with the Coulomb terms we can verify that the contribution of the discussed diagram 4b contains the small parameter

$$gDk^2/\lambda\tau^{-1}\ll 1.$$

The diagrams of type 4b can therefore be disregarded.

Confining ourselves to first order in g, we now take full account of the coulomb interaction. In the renormalization process, each integration with respect to the fast momenta $d^2\mathbf{q}$ corresponds to the appearance of a charge g. On the contrary, integration over the frequencies is by itself not connected with the appearance of the small parameter g, so that the order of the normalization-group equations is determined simply by the number of integrations over the momenta. It is clear from the foregoing that all the diagrams in the functional integral (3.9) take in the lowest order in g the form of a ring in which a fast pulse passes.

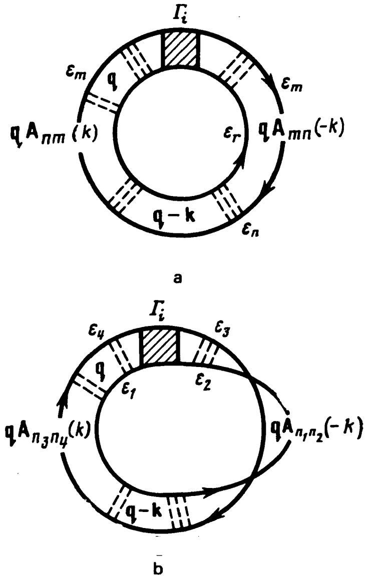


FIG. 4.

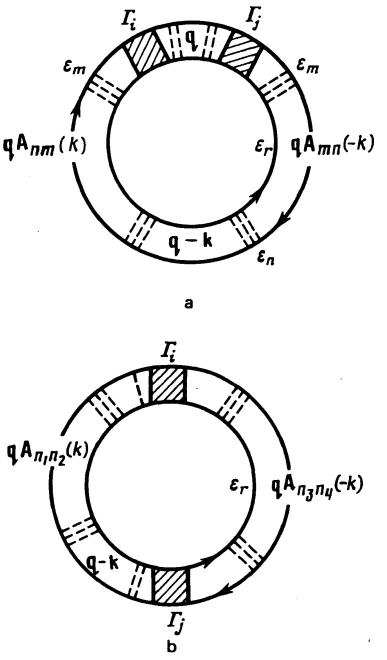


FIG. 5

Figure 5 shows possible elaborations of the diagram 4a within the framework of ring diagrams. After integrating with respect to the fast frequency ε_r the diagram in Fig. 5b turns out to be identical with diagram 4b and, for the reasons discussed above, the contribution of diagram 5b can be neglected. Examination of the remaining diagrams shows that when the charge g is renormalized the full account of the Coulomb interaction reduces to replacement of the Coulomb amplitudes in (3.26) and (3.23) by a sum of ladder diagrams (Fig. 6). We note, to avoid misunderstanding, that taking these diagrams into account is not equivalent to renormalization of the Coulomb amplitudes. In the calculation of the polarization operator (2.37) it was shown that summation of the ladder diagrams of this type changes the coefficient in front of the frequency in the denominator of the diffusion propagator. We introduce in this connection the following notation for the propagators that appear when the diagrams 6 are summed:

$$\mathscr{D}_0=(Dk^2+z|\Omega|)^{-1},\quad \mathscr{D}_i=(Dk^2+z_i|\Omega|)^{-1},\quad i=1,\ 2, \tag{3.28}$$

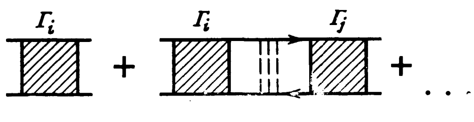


FIG. 6

where $z_1 = z - 2\nu\Gamma_1 + \nu\Gamma_2$ and $z_2 = z + \nu\Gamma_2$. The renormalization of the charge g can then be represented in the form

$$\delta g = g^2 \int_{\lambda\tau^{-1}} 4(\nu\tilde{\Gamma}_0\mathscr{D}_0\mathscr{D}_1^2 + \nu\Gamma_1\mathscr{D}_0\mathscr{D}_1\mathscr{D}_2 - 2\nu\Gamma_2\mathscr{D}_0^2\mathscr{D}_2) D^2 q^3\, dq\, d\Omega, \tag{3.29}$$

where

$$\tilde{\Gamma}_0 = \frac{1}{(1+F_0)^2}\frac{v_c(q)}{1+v_c(q)\pi(\Omega, q)},$$

$$\pi(\Omega, q) = \frac{2\nu}{1+F_0}\frac{Dq^2}{Dq^2+(z-2\nu\Gamma_1+\nu\Gamma_2)|\Omega|};$$

$$\tilde{\Gamma}_0 \approx \frac{1}{2(1+F_0)\nu}\frac{\mathscr{D}_1^{-1}}{Dq^2}. \tag{3.30}$$

The last term in (3.29) corresponds to diagrams 6 when $\Gamma_i = \Gamma_j = \ldots = \Gamma_2$, and the amplitudes Γ_0 and Γ_1 are not included. After summing over the spin projections this term acquires the factor 2. The contribution of the dynamically screened Coulomb interaction is separated in the first term of (3.29). As already noted in connection with Eq. (2.37), conservation of the total number of particles corresponds to $\pi(\Omega; q = 0) = 0$ so that Dq^2 in denominator of (3.30) does not have any term whatever with $|\Omega|$. As a result, the term with Γ_0 is more singular than the remaining terms in (3.29). It is precisely to separate this singularity that the Coulomb amplitude Γ was subdivided in the form $\Gamma_0 + \Gamma_1$.

Equation (3.29) corresponds to the following renormalization-group equation:

$$\frac{dg}{d\xi} = \frac{g^2(\xi)}{2}\left[\frac{2}{1+F_0} f_1(z, z_1) + (2\nu\Gamma_1) f_2(z, z_1, z_2) - 2(2\nu\Gamma_2) f_2(z, z, z_2)\right], \tag{3.31}$$

where $\xi = \ln\lambda^{-1}$ and

$$f_1(a, b) = \frac{\ln(a/b)}{a-b}, \quad f_2(a, b, c) = \frac{2b}{b-c} f_1(a, b) - \frac{2c}{b-c} f_1(a, c). \tag{3.32}$$

The functions f_1 and f_2 are symmetric in permutation of the arguments, with

$$f_1(1, 1) = f_2(1, 1, 1) = 1 \quad \text{и} \quad f_2(a, b, 0) = 2f_1(a, b).$$

The functions f_1 and f_2 appear as the result of integration of products of different propagators $\mathscr{D}$.

We turn now to Eqs. (3.23) and consider the elaboration of the combination J_1 when account is taken of diagrams 6. Replacing in J_1 the Coulomb amplitudes by sums of the corresponding ladder diagrams, we obtain

$$J_1 = \int (\nu\tilde{\Gamma}_0\mathscr{D}_1^2 + \nu\Gamma_1\mathscr{D}_1\mathscr{D}_2 - 2\nu\Gamma_2\mathscr{D}_0\mathscr{D}_2)\, 2Dq\, dq\, d\Omega. \tag{3.33}$$

The singularity of the amplitude Γ_0 leads to the appearance of a doubly logarithmic dependence of $\tilde{J}_1$ on λ:

$$J_1 \approx \frac{1}{2(1+F_0)z_1}\frac{1}{2}\ln^2\lambda.$$

For the construction of the normalization-group equation it is very important that the combination $\tilde{J}_1$, which contains the square of the logarithm, is completely cancelled out in the final expression (3.31) even though it appears in the intermediate stages of the calculation.

3. Renormalization of the parameter z and of the Coulomb amplitudes

The renormalization of the coefficient z is determined by two contributions. The first is the result of averaging of the product of S_ε and S_{int}; an example of the diagram is shown in Fig. 7. The sum of diagrams 7 can be easily calculated:

$$\delta S_z^{(1)} = -\pi\nu g z J_1 \operatorname{Sp}(\hat{\varepsilon}Q). \tag{3.34}$$

The second contribution stems from the expansion of (3.21) in terms of the slow frequency. It is necessary to expand in (3.21) in terms of the frequency ε_m, for if the expansion is in terms of the second slow frequency ε_n the unitary matrices $\tilde{U}$ and U^+ are multiplied and the functional dependence on $\tilde{U}$ drops out. Replacing in (3.21) the sum over ε_r by an integral and taking into account the augmentation of the amplitudes Γ_i by the ladder diagrams, we obtain after integrating by parts

$$\delta S_z^{(2)} = \pi\nu g\left(\sum \varepsilon_m \tilde{U}^{+ij}_{mn}\Lambda_{np}{}^{jl}\tilde{U}_{pm}{}^{li}\right)\Big[zJ_1 + \int (\nu\tilde{\Gamma}_0\mathscr{D}_1^2\mathscr{D}_0^{-1} + \nu\Gamma_1\mathscr{D}_1\mathscr{D}_2\mathscr{D}_0^{-1} - 2\nu\Gamma_2\mathscr{D}_2)\Big|_{\varepsilon_r = \lambda\tau^{-1}} 2Dq\, dq\Big]. \tag{3.35}$$

As a result, just as in the derivation of (3.31), the combination $\tilde{J}_1$ is exactly cancelled out and the renormalization-group equation for the parameter z takes the form

$$\frac{1}{g}\frac{dz}{d\xi} = -\left(\frac{1}{2(1+F_0)} + \nu\Gamma_1 - 2\nu\Gamma_2\right). \tag{3.36}$$

We proceed now to consider the renormalization-group equations for the Coulomb amplitudes:

$$\frac{1}{g}\frac{d\nu\Gamma_1}{d\xi} = \nu\Gamma_2 - \Phi(\nu\Gamma_2, z, z_2), \tag{3.37}$$

$$\frac{1}{g}\frac{d\nu\Gamma_2}{d\xi} = \left(\frac{1}{2(1+F_0)} + \nu\Gamma_1\right) - 2\Phi(\nu\Gamma_2, z, z_2). \tag{3.38}$$

The first terms in the right-hand side of the equations are due to the averaging of S_{int} over W. The logarithmic integration

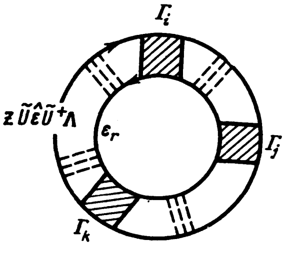


FIG. 7

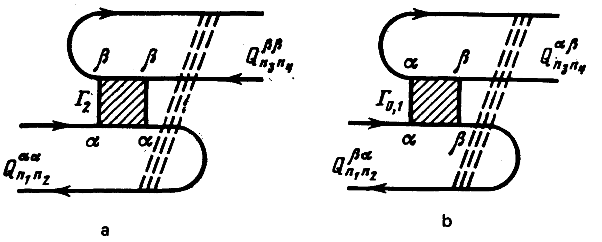


FIG. 8.

is then carried out only with respect to the momentum (see the diagram in Fig. 8). For example,

$$-\frac{\pi\nu}{4}\Gamma_2\langle \bar{U}^+\Lambda W\bar{U}\gamma_2\bar{U}^+\Lambda W\bar{U}\rangle_0$$

$$=\frac{\pi\nu}{4}\Gamma_2\left[\frac{1}{2}\bar{U}^+\Lambda\bar{U}\gamma_1\bar{U}^+\Lambda\bar{U}+\text{const}\right]\int\frac{4q\,dq}{(2\pi)^2\nu(Dq^2+\Omega)};$$

which corresponds to the first term in (3.37).

The function Φ in Eqs. (3.37) and (3.38) is of the form

$$\Phi=\frac{(\nu\Gamma_2)^2}{z_2}-2(\nu\Gamma_2)^2f_1(z,z_2)+\frac{(\nu\Gamma_2)^3}{2z_2}f_2(z,z,z_2)$$

$$-\frac{(\nu\Gamma_2)^3}{2z}f_2(z_2,z_2,z)+(\nu\Gamma_2)^4\frac{1/z+1/z_2-2f_1(z,z_2)}{(z-z_2)^2}. \quad (3.39)$$

We shall not derive Φ here; we confine ourselves only to the necessary clarifications. The five terms of the function Φ correspond to diagrams a–e in Fig. 9 in the case of Eq. (3.37) and to diagrams a–e of Fig. 10 in the case of (3.38). Diagrams 9 and 10 are paired: the diagrams of each pair differ in the manner of opening the ring inside which the fast pulse passes. The factor 2 in front of the function Φ in (3.38) is the result of summation over the spin projections. In Figs. 9 and 10 are shown only skeleton diagrams without elaborating the inner amplitudes by adding ladder diagrams; in addition, the diffusion propagators can be differently arranged in diagrams a–e. We note that all the diagrams with exception of the pair *a* are constructed with account taken of the second term in S_{int} [see (3.18a)].

Let us explain, finally, why Figs. 9, 10, and 11 contain only the amplitude Γ_2. The point is that an important role is played by the spin structure of the tensors $\gamma_{1,2}$: if one or several amplitudes $\Gamma_2\gamma_2$ is replaced by $\Gamma_0\gamma_1$ (or $\Gamma_1\gamma_1$) then the diagrams of the pair cancel each other. In particular, for this reason we have

$$d\Gamma_0(\xi)/d\xi=0. \quad (3.40)$$

The presence of paired diagrams leads to cancellation of the doubly logarithmic corrections when the renormalization-group equations are constructed. Figure 11 shows diagrams, redrawn in more conventional form, of the pair b after replacement of the amplitude Γ_2 by Γ_0. The diagrams 11 demonstrate the cancellation of the elaborations of the single-particle Green's functions containing doubly logarithmic Green's functions by the augmentations of the vertex.

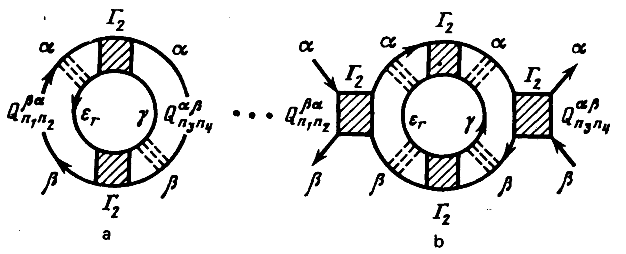


FIG. 10. These diagrams differ from those in Fig. 9 by the method of opening the ring; the diagrams b, c, and d have been left out.

4. Integration of the renormalization-group equations

The quantity z_1 is the first integral of the system of equations of the normalization group. In fact, by virtue of (3.36)–(3.38)

$$dz_1/d\xi=0 \quad (3.41)$$

and when account is taken of relation (2.36)

$$z_1(\xi)\equiv z(\xi)-2\nu\Gamma_1(\xi)+\nu\Gamma_2(\xi)=1/(1+F_0). \quad (3.42)$$

We introduce new variables

$$v_1=\nu\Gamma_1+1/2(1+F_0),\qquad v_2=\nu\Gamma_2, \quad (3.43)$$

in which

$$z=2v_1-v_2,\qquad z_2=2v_1. \quad (3.44)$$

We rewrite the remaining renormalization-group equations in terms of these variables ($\xi=\ln\lambda^{-1}$):

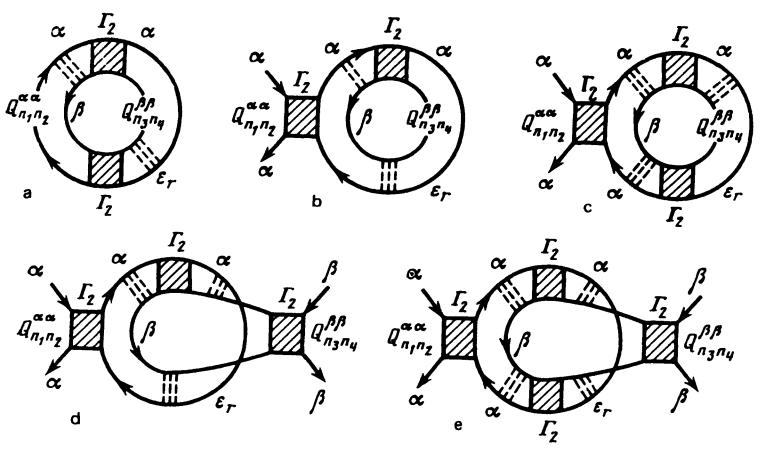


FIG. 9.

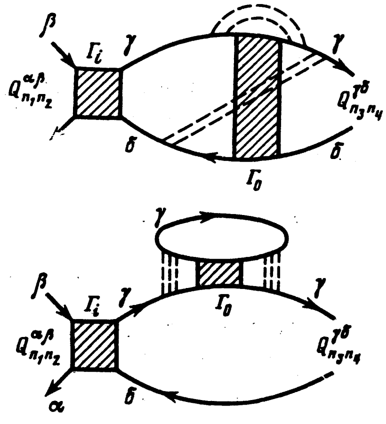


FIG. 11.

$$\frac{dg}{d\xi}=g^2\left(4-6\frac{v_1}{v_2}\ln\frac{2v_1}{2v_1-v_2}\right), \tag{3.45a}$$

$$\frac{dv_1}{d\xi}=g\left(v_2-\frac{v_2^2}{2v_1}+2v_2\ln\frac{2v_1}{2v_1-v_2}\right), \tag{3.45b}$$

$$\frac{dv_2}{d\xi}=g\left(v_1-\frac{v_2^2}{v_1}+4v_2\ln\frac{2v_1}{2v_1-v_2}\right). \tag{3.45c}$$

The system obtained is homogeneous and can be integrated in quadratures by introducing $w = v_2/v_1$. The trajectories of the solutions of Eqs. (3.45) are of the form

$$g(w)=\frac{g_0r(w)}{r(w_0)},\quad w_0=\left.\frac{2\nu\Gamma_2}{1+\nu\Gamma_2}\right|_0,$$

where g_0 and w_0 are the initial values of g and w, while a plot of $r(w)$ is shown in Fig. 12. The function $r(w)$ has a maximum at $w\approx0.9$ and vanishes at $w\approx1.6$.

The solution g as a function of the logarithmic variable ξ is of the form

$$g(\xi)=\frac{g_0R(\eta)}{R(\eta_0)},\quad \eta=\frac{g_0\xi}{R(\eta_0)}+\eta_0, \tag{3.46}$$

where η_0 is uniquely connected with w_0, and the function $R(\eta)$ is shown in Fig. 13. It is interesting that the initial growth of $g(\xi)$ is replaced by the zero-charge asymptotic relation

$$g(\xi)=1/2\xi\quad\text{at}\quad\xi\gg1/g_0. \tag{3.47}$$

Thus, if $g_0\ll1$ we have $g(\xi)\ll1$ for all ξ, including the region of the maximum. Therefore the use of the renormalization group equations in the lowest order in g is therefore legitimate.

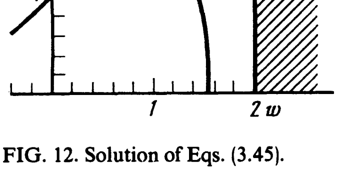


FIG. 12. Solution of Eqs. (3.45).

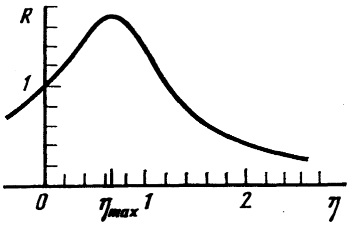


FIG. 13. Dependence of the resistance on the logarithmic variable; $\eta_{\max}\approx0.67$.

§4. CONDUCTIVITY, COMPRESSIBILITY, AND STATE DENSITY

1. The Einstein relations

We consider $\partial n/\partial\mu$, which determines the compressibility of the electron gas:

$$\partial n/\partial\mu=2T\sum_{\varepsilon}\int\frac{\partial}{\partial\mu}\mathcal{G}(\varepsilon,p)\frac{dp}{(2\pi)^d}. \tag{4.1}$$

Calculation shows that the diffusion corrections to $\partial n/\partial\mu$ cancel out. The reason for this is that of importance for diffusion corrections are small momentum transfers near the Fermi surface. In this case differentiation with respect to μ in the integral (4.1) is equivalent to differentiation with respect to the momentum. As a result, the quantity $\partial n/\partial\mu$ is determined only by the Fermi-liquid corrections:

$$\partial n/\partial\mu=2\nu/(1+F_0). \tag{4.2}$$

The quantity $\partial n/\partial\mu$ is equal to the static limit of the polarization operator. The dynamic part $\pi(k,\Omega)$ is described by diagrams of type 3b. In the renormalization of $\pi(k,\Omega)$ the correction to the external triangular vertices and to the Green's functions cancel one another [see (3.40) and Fig. 11], therefore

$$\pi(k,\Omega)=\frac{2\nu}{1+F_0}\left(1-\frac{1}{1+F_0}\frac{\Omega}{D(\xi)k^2+[z(\xi)-2\nu\Gamma_1(\xi)+\nu\Gamma_2(\xi)]|\Omega|}\right). \tag{4.3}$$

With allowance for (3.42), we obtain

$$\pi(k,\Omega)=\frac{\partial n}{\partial\mu}\frac{D_e(\xi)k^2}{D_e(\xi)k^2+|\Omega|}, \tag{4.4}$$

where the diffusion coefficient of the interacting electrons is

$$D_e=(1+F_0)D(\xi).$$

The renormalization-group equations obtained in §3 are thus reconciled with the particle-number-conservation condition, since (4.4) satisfies the requirement $\pi(k = 0) = 0$.

From (4.4) and the continuity equation [see (2.39)] follows the Einstein relation

$$\frac{\sigma(\xi)}{e^2}=\frac{\partial n}{\partial\mu}D_e(\xi). \tag{4.5}$$

It is seen thus that the connection between the charge $g(\xi)$ and the film resistance $r_\square$ is preserved in the renormalization:

$$r_\square(\xi)=\frac{2\pi^2}{e^2}g(\xi). \tag{4.6}$$

The solution (3.46) shows that when the temperature is lowered the initial increase of the resistance gives way to a decrease.

2. Density of states

The density of states $\nu_\varepsilon(T)$ does not enter in the charges that determine the renormalization process, and the constant ν [$\nu = \nu_\varepsilon(T)$ at $T \sim \tau^{-1}$] is eliminated from the renormalization-group equations by a suitable change of variables.

The quantity $\nu_\varepsilon(T)$ directly measured in tunnel experiments is given by the expression

$$\nu_\varepsilon(T) = -\frac{1}{\pi}\,\mathrm{Im}\int \mathcal{G}^R(\varepsilon, p)\frac{dp}{(2\pi)^d}. \qquad (4.7)$$

In terms of the variables Q, expression (4.7) takes the form

$$\nu_\varepsilon(T) = \nu(\hat{\Lambda}\hat{Q})_{\varepsilon\varepsilon} \qquad (4.8)$$

and its calculation is analogous to the determination of the Debye-Waller factor

$$(\hat{\Lambda}\hat{Q}) = (\Lambda\bar{U}^+\Lambda e^W\bar{U}) \approx (\Lambda\bar{U}^+\Lambda \exp({}^1/_2\langle WW\rangle)\bar{U}). \qquad (4.9)$$

The quantity $\langle WW\rangle$ is calculated in the samy way as $\delta S_z^{(1)}$ [see (3.34) and Fig. 7], and is determined by the combination $\tilde{J}_1 \approx \frac{1}{4}\xi^2, \xi = \ln(\tau^{-1}/\max \varepsilon, T)$. As a result, the state density decreases rapidly with decreasing temperature:

$$\nu_\varepsilon(T) = \nu \exp(-g\tilde{J}_1) \approx \nu \exp(-g\xi^2/4). \qquad (4.10)$$

In the region $\xi \gtrsim g_0^{-1}$ it is necessary to take into account the renormalization of the parameter g. The renormalization-group equation for the quantity $\nu(\xi)$ is of the form

$$d\nu(\xi)/d\xi = -{}^1/_2 g(\xi)\xi\nu(\xi). \qquad (4.11)$$

We must therefore use in place of (4.10) the running value of the parameter $g(\xi)$:

$$\nu(\xi) = \nu \exp\left(-\int_0^\xi \frac{g(\eta)\eta}{2}\,d\eta\right). \qquad (4.12)$$

In the region of the zero-charge asymptotic form $g(\xi) = 1/2\xi$ we find

$$\nu_\varepsilon(T) \sim (\max \varepsilon, T)^{1/4}. \qquad (4.13)$$

The state density thus tends to zero as the Fermi level is approached, although not as rapidly as in accord with Eq. (4.10).

§5. CONCLUSION

We have derived here renormalization-group equations for a disordered metal in a magnetic field; these equations take full account of the Coulomb interaction. The Coulomb interaction of the electrons makes the properties of a two-dimensional metal in a magnetic field quite unusual. When the temperature is lowered, the initial growth of the resistance gives way to a tendency to zero (see Fig. 13), and at the same time the state density on the Fermi level drops to zero [see (4.10) and (4.13)]. The coexistence of these seemingly mutually exclusive properties is due to the fact that, as shown in §§3 and 4, the electron correlations that determine the conductivity and the state density are quite different. When the conductivity is calculated the corrections to the Green's functions and to the vertices cancel one another to a great degree. Therefore the correlation effects that cause the decrease of the state density do not influence the resistance. In contrast to $\nu_\varepsilon(T)$, there are no diffusion corrections at all in the quantity $\partial n/\partial\mu$ which determines the static limit of the polarization operator. It is important to note that it is precisely $\partial n/\partial\mu$ which connects the conductivity with the diffusion coefficient of the interacting electrons in the Einstein relation (see §4.1). The use of the density of states $\nu_{\varepsilon F}(T)$ in this relation in place of $\partial n/\partial\mu$ is in error.

The equations derived in the present paper show that when the Coulomb amplitudes are renormalized account must be taken of the amplitude Γ_2 that describes large-angle scattering, for otherwise the Coulomb interaction is not renormalized. We note that even if the unrenormalized value of the amplitude Γ_2 is zero, a nonzero Γ_2 arises in the renormalization process.

McMillan[16] has attempted to describe the metal-insulator transition by using similarity theory. He did not derive renormalization-group equations, and the scheme proposed by him has the following shortcomings:

1) his renormalization-group equations do not contain Γ_2;

2) these equations contain a state density shown to have no effect on the renormalization of the resistance and of the Coulomb amplitudes;

3) he used ν_ε in place of $\partial n/\partial\mu$ in the Einstein relation. As a result, the conductivity and the diffusion coefficient vanished in the critical region in accord with different laws.

Let us explain why the temperature dependence of the resistance takes in this case the form shown in Fig. 13. To this end we write down the resistance correction calculated in Ref. 2 in first order in the screened Coulomb interaction:

$$\delta g = {}^1/_2 g_0^2(2-2F)\ln(1/T\tau). \qquad (5.1)$$

Here $F = 2\nu\Gamma_2$ describes the contribution of the Hartree diagrams, and the coefficient 2 in front of F is of spin origin. Because of the additional spin factor, the negative contribution due to Γ_2 increases more rapidly in the course of the renormalization than the positive one and causes in final analysis the growth of the resistance to give way to a decrease.

Equation (5.1) can be obtained from (3.31) by putting in the latter $f_1 = f_2 = 1$ and $\nu\Gamma_1 = F_0 = 0$. When full account is taken of the Coulomb interaction, the correction to the conductivity is described by the following expression [see (3.31) or (3.45a) at $\xi = 0$]:

$$\delta\sigma = \frac{e^2}{2\pi^2}\left[4 - 3\frac{1+\nu\Gamma_2}{\nu\Gamma_2}\ln(1+\nu\Gamma_2)\right]\ln T\tau, \qquad (5.2)$$

which must be kept in mind when reducing the experimental data. We note that (5.2) does not coincide with (5.1), even if $F \ll 1$:

$$\delta\sigma = \frac{e^2}{4\pi^2}(2-1.5F)\ln T\tau, \qquad F = 2\nu\Gamma_2 \ll 1.$$

We have so far disregarded the Zeeman splitting, which cuts off the pole in the diffusion propagator with opposite projections of the electron spin,[17,18] if

$$g_L eH/mc > T$$

(g_L is the Landé factor). The spin splitting can be taken into account by slightly modifying the equations obtained above. It is found then that with decreasing temperature the resistance increases without any anomalies whatever and the close-coupling regime is realized. We shall discuss this case elsewhere.

Thus, the analysis in the present paper is valid in the temperature region

$$eDH/c > T > g_L eH/mc \tag{5.3}$$

(the first inequality was discussed in §1, see (1.2)). This region is not small, since $mD = \varepsilon_F \tau \gg 1$. For the solution shown in Fig. 13 to be valid all the way to the maximum, however, we must have

$$g_0 \ln (mD/g_L) > \varkappa, \qquad \varkappa = (\eta_{max} - \eta_0) R(\eta_0). \tag{5.4}$$

The quantity $\varkappa$ is determined only by the value of the parameter $w_0 = 2\nu\Gamma_2/(1+\nu\Gamma_2)$; $\varkappa \approx 0.67$, when $\nu\Gamma_2 = 0$ and $\varkappa \approx 0.25$ at $\nu\Gamma_2 = 0.5$. Since it is assumed that $g_0 \ll 1$, for a region with decreasing resistance to exist we must have $g_L \ll 1$. There are a number of alloys, e.g., $\mathrm{Al}_x\mathrm{Ga}_{1-x}\mathrm{As}$, whose composition can be chosen such that $g_L \approx 0$. In addition, in p-Si and p-Ge deformed along the [111] direction, for symmetry reasons, $g_L^{\perp} \equiv 0$ (Ref. 19). This means that if the magnetic field is applied in a plane perpendicular to the [111] axis there is no Zeeman splitting. An experimental realization of the case considered in the present paper is undoubtedly of interest.

The author thanks B. L. Al'tshuler, A. G. Aronov, K. B. Efetov, A. I. Larkin, and P. A. Lee for helpful discussions and for interest in the work.

The author is deeply grateful to D. E. Khmel'nitskiĭ for numerous helpful conversations in question touched upon in the article.

[1] We put $\hbar = 1$ throughout.

[2] We have in mind the single-particle state density [see (4.7)] measured in tunnel experiments, and not the total state density that determines the heat capacity.

[3] The matrix subscripts always correspond to Matsubara frequencies, and in the superscripts the spin index is combined with the replica index.

[1] B. L. Al'tshuler and A. G. Aronov, Zh. Eksp. Teor. Fiz. **77**, 2028 (1979) [Sov. Phys. JETP **50**, 968 (1979)].
[2] B. L. Altshuler, A. G. Areonov, and P. A. Lee, Phys. Rev. Lett. **44**, 1288 (1980).
[3] B. L. Altshuler, D. E. Khmel'nitskii, A. I. Larkin, and P. A. Lee, Phys. Rev. **B22**, 5142 (1980).
[4] L. P. Gor'kov, A. I. Larkin, and D. E. Khmel'nitskiĭ, Pis'ma Zh. Eksp. Teor. Fiz. **30**, 248 (1979) [JETP Lett. **30**, 228 (1979)].
[5] E. Abrahams and T. V. Ramakrishnan, J. Non-Cryst. Sol. **35**, 15 (1980).
[6] E. Abrahams, P. W. Anderson, P. C. Licciardello, and T. V. Ramakrishnan, Phys. Rev. Lett. **42**, 673 (1979).
[7] P. A. Lee, J. Non-Cryst. Sol. **35**, 21 (1980).
[8] F. Wegner, Zeit. Physik **B35**, 207 (1980).
[9] L. Schäfer and F. Wegner, Zeit. Physik **B38**, 113 (1980).
[10] K. B. Efetov, A. I. Larkin, and D. E. Khmel'nitskiĭ, Zh. Eksp. Teor. Fiz. **79**, 1120 (1980) [Sov. Phys. JETP **52**, 568 (1980)].
[11] F. A. Berezin, Second-Quantization Method [in Russian], Fizmatgiz, 1965.
[12] S. F. Edwards, J. Phys. C., Sol. St. Phys. **8**, 1660 (1975).
[13] L. D. Landau, Zh. Eksp. Teor. Fiz. **35**, 97 (1958) [Sov. Phys. JETP **8**, 70 (1959)].
[14] K. G. Wilson and J. Kogut, Phys. Rep. **C12**, 75 (1974).
[15] A. M. Polyakov, Phys. Lett. **B59**, 79 (1975).
[16] W. L. McMillan, Phys. Rev. **B24**, 2739 (1981).
[17] A. Kawabata, J. Phys. Soc. Jpn. **50**, 2461 (1981).
[18] P. A. Lee and T. V. Ramakrishnan, Preprint, 1981.
[19] G. L. Bir and G. E. Pikus, Symmetry and Strain-Induced Effects in Semiconductors, Wiley, 1975, §39 (Table 39.1).

Translated by J. G. Adashko