Formation of metastable states at small-size defects and pinning in type-II superconductors

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The threshold value for the interaction between small-size defects and a vortex lattice, such that the vortex lattice loses its stability and metastable states can be formed, is found. The size of the region within which the rearrangement of the vortex lattice takes place is very large, for numerical reasons. Therefore, the mutual influence of the defects becomes significant at a comparatively low density of the defects.

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1. INTRODUCTION

A magnetic field penetrates into an ideal type-II superconductor in the form of quantized vortices that form a regular triangular lattice.[1] The flow of current in the volume of the superconductor is accompanied by motion of the vortex lattice as a whole and by dissipation of energy.[2,3]

Defects of various kinds, which are always present in a superconductor, lead to a dependence of the free energy on the position of the lattice relative to the defects and, by the same token, to the possibility of flow of a nondissipative current of finite density. Two essentially different forms of pinning turn out to be possble here—the so-called single-particle and collective types.[4]

Collective pinning arises in the weakest possible interaction, but the density of the critical current turns out to be small in this case. The single-particle pinning arises only in the presence of defects whose interaction with the vortex lattice exceeds some threshold value.[5] In the model with smooth deformations, the threshold value of the interaction of the defect with the vortex lattice turns out to be so high that it cannot be attained for small-size defects.[4,6] However, as was shown by the author,[7] the presence of the defect leads to the development of instability of the vortex lattice near the defect and to the formation of metastable states. The vortex lattice is strongly deformed in the transverse direction at distances of the order of the correlation length $\xi(T)$; therefore, such states cannot be described within the framework of elasticity theory. The loose vortex lattice makes formation of such metastable states relatively easy.

We shall investigate below the vortex lattice near a small-size defect (dimensions of the defect $R \ll \xi(T)$); we shall find the criteria for formation of metastable states on such a defect and calculate the jump in the free energy on going from the metastable to the stable state.

2. EQUATIONS OF STATE OF THE VORTEX LATTICE IN SUPERCONDUCTORS WITH DEFECTS

For the investigation of the possible states of the vortex lattice in superconductors with defects, we use a model in which the defects can be described by the introduction of superconductor parameters that depend on the coordinates.[8,4] In this case, the free energy of the superconductor near the transition temperature T_c can be written in the form[4]

$$\mathscr{F}=\nu\int d^3\mathbf{r}\left\{-\tau|\Delta|^2+\frac{7\zeta(3)}{16\pi^2T^2}|\Delta|^4+\frac{\pi D\eta}{8T}|\partial_-\Delta|^2\right\}+\frac{1}{8\pi}\int d^3\mathbf{r}\,(\mathbf{H}^2-2\mathbf{H}_0\mathbf{H})+\delta\mathscr{F},\tag{1}$$

$$\delta\mathscr{F}=\frac{\pi\nu}{8T}\int d^3\mathbf{r}\,\delta(\eta D)\,|\partial_-\Delta|^2+\nu\int d^3\mathbf{r}\,g_1(\mathbf{r})\,|\Delta|^2,\tag{2}$$

where $\tau = 1 - T/T_c$, $\nu = mp_0/2\pi^2$ is the density of states on the Fermi surface, $\zeta(3)$ is the Riemann zeta function, $\partial_- = \partial/\partial\mathbf{r} - 2ie\mathbf{A}$, H_0 is the external magnetic field, $D = \nu l_{tr}/3$ is the diffusion coefficient,

$$\eta=1-\frac{8T\tau_{tr}}{\pi}\left(\Psi\left(\frac{1}{2}+\frac{1}{4\pi T\tau_{tr}}\right)-\Psi\left(\frac{1}{2}\right)\right),\tag{3}$$

and $\Psi(\mathbf{x})$ is the psi function.

The parameters $g_1(r)$ and $\delta(\eta D)$ determine the local change of the constant of interelectron interaction and of the mean free path of the electrons.

$$g^{-1}(\mathbf{r})=g_m^{-1}+g_1(\mathbf{r}),\tag{4}$$

where g_m is the effective constant of interelectron interaction in the superconducting matrix.

We limit ourselves below to the study of magnetic fields close to the critical field H_{c2}. In this case, Eq. (1) for the free energy reduce to the form

$$\mathscr{F}=\nu\int d^3\mathbf{r}\left\{\frac{7\zeta(3)}{16\pi^2T^2}\left(1-\frac{1}{2\varkappa^2}\right)|\Delta|^4-\frac{e\pi D\eta}{4T}(H_{c2}-H_0)|\Delta|^2+\frac{\pi D\eta}{8T}\left(\frac{\partial\Delta}{\partial z}\frac{\partial\Delta^*}{\partial z}\right)\right\}+\delta\mathscr{F}-\int d^3\mathbf{r}\frac{H_0^2}{8\pi},\tag{5}$$

where

$$\varkappa^2=63\zeta(3)/2\pi^3e^2p^2v^3\tau_{tr}^2\eta^2\tag{6}$$

is the Ginsburg-Landau parameter. The z axis in Eq. (5) is directed along the magnetic field H_0.

Equation (5) leads to strong dispersion of the elastic moduli C_{44} and C_{11}:[9]

$$C_{44}\sim\frac{1}{\mathbf{k}^2},\quad C_{11}-C_{66}=\frac{1}{\mathbf{k}^2(\mathbf{k}^2+k_\Psi^2)}.\tag{7}$$

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The singularity of the type $\mathbf{k}^{-2}$ in these formulas is connected with the neglect in the equation for the current density of terms that lead to screening of the magnetic field at distances of the order of k_h^{-1}. The quantities k_ψ and k_h are determined by the equations[4]

$$k_\psi^2=4e(H_{c2}-B), \quad k_h^2=k_\psi^2/[\beta_A(2\varkappa^2-1)+1], \tag{8}$$

where $\beta_A = \langle|\Delta|^4\rangle/\langle|\Delta|^2\rangle^2$ is a numerical coefficient that depends on the type of the lattice. For a triangular lattice,

$$\beta_A=1.1596. \tag{9}$$

We choose a special gauge of the vector potential A_0 in the form

$$\mathbf{A}_0=H_0(0,\ x,\ 0). \tag{10}$$

The order parameter $\Delta(\mathbf{r})$ in this gauge has the form[10]

$$\Delta=\tilde\Delta_0+\sum_{m=0}^{\infty} C_m(z)\exp[ixy+im\varphi-(x^2+y^2)/2]\rho^m,$$

$$\tilde\Delta_0=C\sum_{n=-\infty}^{\infty}\exp\left[\frac{i\pi}{2}n^2+in\beta+2iy\left(\frac{na}{2}+\alpha a\right)-\left(x-\frac{na}{2}-\alpha a\right)^2\right]. \tag{11}$$

In Eq. (11), we have transformed to the dimensionless variable

$$(\rho,\ z)=(eH)^{1/2}r, \quad a^2=2\pi/3^{1/2}, \tag{12}$$

where ρ is a two-dimensional vector in a plane perpendicular to the magnetic field

$$|C|^2=\frac{1-H_0/H_{c2}}{1-1/2\varkappa^2}\,\frac{8\pi^2T^2\tau^2}{7\cdot 3^{1/4}\zeta(3)\beta_A}. \tag{13}$$

The parameters α and β fix the position of the lattice, and we choose them in the form

$$\alpha=-\frac{1}{4}-\delta x\left(\frac{3^{1/2}}{2\pi}\right)^{1/2}, \quad \beta=\frac{\pi}{2}+\delta y\left(\frac{2\pi}{3^{1/2}}\right)^{1/2}. \tag{14}$$

With such a choice of the parameters α and β, the zero of the unperturbed solution is located at the point $-(\delta x, \delta y)$. In the following, we shall be interested in the small quantities

$$|\delta x|, \quad |\delta y|\ll 1. \tag{15}$$

Following Ref. 7, we introduce the new variables D_m according to the formula

$$C_m=CD_m/\Gamma^{1/2}(m+1), \tag{16}$$

where $\Gamma(x)$ is the Euler gamma function.

Assuming that the defect has a small size $[R\ll\xi(T)]$ and is located at the origin, we get, from the extremum of the free energy

$$\delta\mathcal{F}/\delta D_m^*=0 \tag{17}$$

a set of equations for the coefficients $D_m(z)$

$$\frac{1}{3^{1/4}\beta_A}\Bigg\{\sum_{m_1,m_2=0}^{\infty}\frac{\pi\Gamma(m_1+m_2+1)}{2^{m_1+m_2+1}}$$

$$\times\frac{D_{m_1}D_{m_2}D^*_{m_1+m_2-m}}{(\Gamma(m+1)\Gamma(m_1+1)\Gamma(m_2+1)\Gamma(m_1+m_2-m+1))^{1/2}}$$

$$+\sum_{m_1=0}^{\infty}\Bigg(2I(m,m_1)D_{m_1}+$$

$$+I(m+m_1)\left(\frac{\Gamma(m_1+m+1)}{\Gamma(m+1)\Gamma(m_1+1)}\right)^{1/2}D^*_{m_1}\Bigg)$$

$$+\sum_{m_1,m_2=0}^{\infty}\Bigg[\frac{D_{m_1}D^*_{m_2}}{2^{m_2+m}}M(m_2+m-m_1)$$

$$\times\frac{\Gamma(m_2+m+1)}{(\Gamma(m+1)\Gamma(m_1+1)\Gamma(m_2+1)\Gamma(m+m_2-m_1+1))^{1/2}}$$

$$+\frac{D_{m_1}D_{m_2}}{2^{m_1+m_2+1}}M^*(m_1+m_2-m)\cdot$$

$$\times\frac{\Gamma(m_1+m_2+1)}{(\Gamma(m+1)\Gamma(m_1+1)\Gamma(m_2+1)\Gamma(m_1+m_2-m+1))^{1/2}}\Bigg]\Bigg\}$$

$$-\frac{\pi}{2(1-H_0/H_{c2})}\frac{\partial^2 D_m}{\partial z^2}-\pi D_m \tag{18}$$

$$+\delta_{m,0}\delta(z)\left(D_0+\frac{M(0)}{\pi}\right)\frac{(eH)^{3/2}\int d^3\mathbf{r}\,g_1(\mathbf{r})}{\tau(1-H_0/H_{c2})}$$

$$+\delta_{m,1}\delta(z)\left(D_1+\frac{M(1)}{\pi}\right)\frac{\pi eH}{4T\tau(1-H_0/H_{c2})}(eH)^{3/2}\int d^3\mathbf{r}\,\delta(\eta D)=0.$$

It follows from Eq. (18) that the interaction of the vortex lattice with a small-size defect can be described by two dimensionless parameters $\tilde Z$ and $\tilde Z_1$:

$$\tilde Z=\left(\frac{\pi}{2(1-H_0/H_{c2})}\right)^{1/2}\frac{eH_{c2}}{4T\tau}(eH)^{3/2}\int d^3\mathbf{r}\,\delta(\eta D),$$

$$\tilde Z_1=\left(\frac{3^{1/4}\beta_A}{2\pi(1-H_0/H_{c2})}\right)^{1/2}\frac{(eH)^{3/2}}{\tau}\int d^3\mathbf{r}\,g_1(\mathbf{r}). \tag{19}$$

The matrix elements $M(m)$, $I(m)$ and $I(m,m_1)$ were determined and investigated in Ref. 7:

$$M(m)=\pi\sum_{N=-\infty}^{\infty}\sum_{l=0}\frac{(-)^l\Gamma^{1/2}(m+1)}{2^l\Gamma(l+1)\Gamma(m+1-2l)}\left[\left(\frac{2\pi}{3^{1/2}}\right)^{1/2}(n+2\alpha)\right]^{m-2l}$$

$$\times\exp\left[\frac{i\pi}{2}n^2+in\beta-\frac{\pi}{2\cdot 3^{1/2}}(n+2\alpha)^2\right],$$

$$I(m)=\sum_{N=0}\frac{M(N)M(m-N)}{\pi 2^{m+1}}\left(\frac{\Gamma(m+1)}{\Gamma(N+1)\Gamma(m-N+1)}\right)^{1/2},$$

$$I(m,m_1)=\sum_{N=0}\frac{M(N)M^*(m_1+N-m)}{\pi 2^{m_1+N+1}} \tag{20}$$

$$\times\frac{\Gamma(m_1+N+1)}{(\Gamma(m+1)\Gamma(m_1+1)\Gamma(N+1)\Gamma(m_1+N-m+1))^{1/2}}.$$

A structural transition first arises in the vicinity of the point $(\delta x, \delta y) = 0$, when the zero of the unperturbed solution coincides with the defect. In the vicinity of the point $(\delta x, \delta y) = 0$, Eq. (18) breaks up into four weakly coupled subsystems:[7]

$$\{1+6K\},\{6K,\ 6K+2\},\ \{3+6K, 5+6K\},\{4+6K\}, K=0,\ 1,\ 2,\ldots \tag{21}$$

A structural transition arises in the subsystem $\{6K, 6K+2\}$ and induces weak transitions in the other subsystems. Near the transition point, only the coupling with the subsystem $\{1+6K\}$ turns out to be significant. As has already been shown in Ref. 7, the matrix elemenst $M(m)$ contain the phase factor

$$\exp\left[\frac{i}{2}\left(\frac{2\pi}{3^{1/2}}\right)^{1/2}\delta y + i\delta x\delta y\right],$$

which we can easily eliminate if we transform to the new variable

$$D_m(\delta x, \delta y) = \exp\left[\frac{i}{2}\left(\frac{2\pi}{3^{1/2}}\right)^{1/2}\delta y + i\delta x\delta y\right]\tilde{D}_m(\delta x, \delta y). \qquad (22)$$

We shall assume in what follows that this substitution has been made.

We now proceed to the investigation of two special cases, one in which the defect is a region with a changed value of the constant of the interelectron interaction ($\tilde{Z}_1 \neq 0$), and the other in which the mean free path of the electrons changes inside a small region of the superconductor ($\tilde{Z} \neq 0$).

3. FORMATION OF METASTABLE STATES ON A DEFECT WITH CHANGED VALUE OF THE INTERELECTRON INTERACTON

In the case that we have considered ($\tilde{Z}=0$, $\tilde{Z}_1 \neq 0$) the critical value of the parameter $\tilde{Z}_1^c$, at which the metastable state is first formed on the defect, is determined from the condition for the appearance of a nonzero solution of the set of equaitons

$$-\frac{\pi}{2(1-H_0/H_{c2})}\frac{\partial^2 D_m}{\partial z^2} + \frac{1}{3^{1/4}\beta_A}\sum_{m_1} S(m, m_1) D_{m_1}$$

$$+\left(\frac{2\pi}{3^{1/4}\beta_A(1-H_0/H_{c2})}\right)^{1/2} Z_1 \delta(z)\delta_{m,0} D_0 = 0, \qquad (23)$$

$$m = \{6K, 6K+2\}, \quad K = 0, 1, 2, \ldots,$$

where the matrix S is determined by the set of equations (18) and is equal to

$$S(m, m_1) = 2I(m, m_1) + I(m+m_1)\left(\frac{\Gamma(m+m_1+1)}{\Gamma(m+1)\Gamma(m_1+1)}\right)^{1/2} - \pi\beta_A\cdot 3^{1/4}\delta_{m,m_1}; \qquad (24)$$

$$m, m_1 = \{6K, 6K+2\}, \quad K = 0, 1, 2, \ldots$$

The matrix $S(m, m_1)$ must be taken at the point $(\delta x, \delta y) = 0$. We denote by λ_i the eigenvalues of the matrix S and by $F(K, i)$ the corresponding normalized eigenvectors. Then the point of instability is determined from the condition

$$\det\{\lambda_i^{1/2} F(k, i) + \delta_{k,1} Z_1^c F(1, i)\} = 0. \qquad (25)$$

The solution of the set of equations (23) at the point $\tilde{Z}_1^c$ is proportional to the vector

$$\mathcal{F}(K, z) = \sum_i C_i F(K, i)\exp(-|z|, t_i),$$

$$(D_0, D_2^*, D_6, D_8^* \ldots) = Y\{\mathcal{F}(K, z)\}, \qquad (26)$$

where

$$t_i = [2\lambda_i(1-H_0/H_{c2})/\pi 3^{1/4}\beta_A]^{1/2}, \qquad (27)$$

$\{C_i\}$ is a vector normalized to unity.

As was pointed out in Ref. 7, the relatively easy onset of instability is connected with the anomalously small value of the first eigenvalue of the matrix S. For the determination of the numerical value of the quantity $\tilde{Z}_1^c$ we have restricted the matrix to the size (10×10). The critical value of $\tilde{Z}_1^c$, the eigenvalues λ_i, the vector C_i and the first eigenvector $F(k, 1)$ are equal to

$$Z_1^c = -0.89; \quad \{\lambda_i\} = \{0.04;\ 0.278;\ 0.406\ldots\},$$

$$\{C_i\} = \{0.741;\ -0.504;\ 0.295;\ 0.187;\ -0.197;\ 0.115\ldots\},$$

$$\{F(K, 1)\} = \{0.233; -0.306; -0.548; 0.535; -0.228;\ 0.217;\ 0.3;\ -0.264;\ 0.057;\ -0.053\}. \qquad (28)$$

Using Eq. (19), we rewrite the criteria (38) for formation of the metastable states on the defect in a more illustrative form:

$$(eH_{c2})^{1/2}\int d^3r g_1(\mathbf{r}) = -1.8\tau(1-H_0/H_{c2})^{1/4}, \qquad (29)$$

where

$$eH_{c2} = \frac{4}{\pi}\frac{T_c - T}{D\eta}.$$

It follows from Eq. (29) that the metastable states are easily formed only at $g_1 < 0$, i.e., for defects that have a higher transition temperature than the T_c of the matrix.

In the case of a small supercriticality, the form of the perturbtion of the homogeneous state is determined by Eq. (26). We find the amplitude Y as a function of the displacement $(\delta x, \delta y)$ and the value of the jump in the free energy. For this, we must consider the nonlinear terms in Eq. (18). We define as follows:

$$|Y|^2\sum_{i,j} U_K^{ij}\exp(-|z|(t_i+t_j)) = -\frac{1}{3^{1/4}\beta_A}\sum_{m_1,m_2}\left(\frac{D_{m_1}D_{m_2}^*}{2^{m_2+m}} M(m_2+m-m_1)\frac{\Gamma(m_2+m+1)}{\Gamma^{1/2}(m_2+m-m_1+1)} + \frac{D_{m_1}D_{m_2}}{2^{m_1+m_2+1}}M^*(m_1+m_2-m)\frac{\Gamma(m_1+m_2+1)}{\Gamma^{1/2}(m_1+m_2-m+1)}\right\}\frac{1}{(\Gamma(m+1)\Gamma(m_1+1)\Gamma(m_2+1))^{1/2}}, \qquad (30)$$

where $m_1, m_2 = 0, 2, 6, 8, \ldots$, $m = 1+6(K-1)$, $K = 1, 2, 3, \ldots$.

From the set of equations (18), we obtain the following for $m = 1+6(K-1)$, $K = 1, 2, 3\ldots$ with account of Eq. (30)

$$-\frac{\pi}{2}\frac{1}{1-H_0/H_{c2}}\frac{\partial^2 D_m}{\partial z^2} + \sum_{m_1} W(m, m_1) D_{m_1} = |Y|^2\sum_{i,j} U_K^{ij}\exp(-|z|(t_i+t_j)). \qquad (31)$$

In Eq. (31), $m, m_1 = 1, 7, 13,...$. The matrix W is equal to

$$W(m, m_1)=\frac{1}{3^{1/4}\beta_A}\left\{2I(m, m_1)+I(m+m_1)\times\left(\frac{\Gamma(m_1+m+1)}{\Gamma(m+1)\Gamma(m_1+1)}\right)^{1/2}\right\}-\pi\delta_{m,m_1}. \tag{32}$$

$$D_m=\sum_j \alpha_j(z)Y(K,j),\quad m=1+6(K-1),\quad K=1,2,3\ldots, \tag{33}$$

where $Y(K,j)$ is the normalized eigenfunction of the matrix W corresponding to the jth eigenvalue μ_j;

$$\alpha_\nu(z)=|Y|^2\sum_{i,j}\frac{(U_K{}^{ij}Y(K,\nu))}{\mu_\nu-(\lambda_i{}^{1/2}+\lambda_j{}^{1/2})^2/3^{1/4}\beta_A}\times\left[\exp(-|z|(t_i+t_j))-\frac{t_i+t_j}{T_\nu}\exp(-|z|T_\nu)\right], \tag{34}$$

where

$$T_\nu=(2\mu_\nu(1-H_0/H_{c2})/\pi)^{1/2}. \tag{35}$$

Using Eqs. (26) and (33) for the coefficients D_m, we can obtain a cubic equation for the quantity Y by the standard method. To this end, the set of equations (18) must be mulitplied on the left by the column (26) and integrated with the respect to z. As a result, we get

$$0.164|Y|^2Y+1.03\delta\tilde{Z}_1Y+2.84(\delta x+i\delta y)-18|Y|^2(\delta x+i\delta y)=0. \tag{36}$$

The last term in Eq. (36) is small in the supercriticality parameter $\delta\tilde{Z}_1$ and is needed only for an estimate of the region of applicability of Eq. (36). For its calculation, it is first necessary to find the matrix $W_1(m, m_1)$ ($m = 0, 2, 6, 8, ...,$ $m_1 = 1, 7, 13, ...$), which is proportional to the small parameter $(\delta x + i\delta y)$. This matrix is easily found from Eqs. (18) and we shall not write it out.

Setting

$$\delta x+i\delta y=\rho e^{i\varphi},\qquad Y=\tilde{Y}e^{i\varphi}, \tag{37}$$

we obtain for the quantity Y at the point of discontinuity $Y_{\rm dis}$ and in the final state $Y_{\rm fin}$

$$\tilde{Y}_{\rm dis}=1.45(-\delta\tilde{Z}_1)^{1/2}+36.6\rho_{\rm dis},\qquad \rho_{\rm dis}=0.35(-\delta\tilde{Z}_1)^{3/2};\qquad \tilde{Y}_{\rm fin}=-2.9(-\delta\tilde{Z}_1)^{1/2}+36.6\rho_{\rm dis}. \tag{38}$$

The jump in the free energy in the transition from the metastable state to the stable state can be found from Eq. (5):

$$\delta\left(\frac{\mathcal{F}-\mathcal{F}_0}{\nu/(eH)^{3/2}}\right)=\frac{8\pi^2T^2\tau^2(1-H_0/H_{c2})^2}{7\cdot3^{1/4}\zeta(3)\beta_A(1-1/2\varkappa^2)}\times\left(\frac{2\pi}{3^{1/4}\beta_A(1-H_0/H_{c2})}\right)^{1/2}\delta(0.26\delta\tilde{Z}_1\tilde{Y}^2+2.13\rho\tilde{Y}). \tag{39}$$

Using the equations (38) for the quantity $\tilde{Y}$ in the initial and final states, we reduce Eq. (39) for the jump in the free energy to the form

$$\delta\left(\frac{\mathcal{F}-\mathcal{F}_0}{\nu/(eH)^{3/2}}\right)=-\frac{8\pi^2T^2\tau^2}{7\zeta(3)}\cdot\frac{(1-H_0/H_{c2})^{3/2}}{1-1/2\varkappa^2}6.48(-\delta\tilde{Z}_1)^2. \tag{40}$$

Comparing the two terms in Eq. (38) for $\tilde{Y}_{\rm dis}$, we find the region of applicability of Eqs. (36), (38) and (39):

$$|\delta\tilde{Z}_1/\tilde{Z}_1|<0.13. \tag{41}$$

4. FORMATION OF METASTABLE STATES ON INHOMOGENEITIES OF THE MEAN FREE PATH OF THE ELECTRONS

The study of metastable states formed on inhomogeneities of the mean free path of the electrons ($Z \neq 0, Z^1 = 0$) is a more difficult problem, since the development of the instability takes place against a background of a finite perturbation of the homogeneous state by the defect. The smallness of the first eigenvalue of the matrix S will be of major importance for us.

As follows from the set (18), the presence of the defect ($\tilde{Z} \neq 0$) leads to the appearance of non-vanishing values of D_m with $m = 1, 7, 13, ...$ With accuracy up to terms of second order, we find

$$D_m=\sum_\nu Y(K,\nu)Q(\nu,z),$$

$$Q(\nu,z)=BY(1,\nu)\frac{\exp(-|z|T_\nu)}{\mu_\nu^{1/2}}+\sum_{i,j}\frac{B^2}{\mu_\nu-(\mu_i^{1/2}+\mu_j^{1/2})^2}\times(K_{K_1}{}^{ij}Y(K_1,\nu))\left(\exp(-|z|(T_i+T_j))-\frac{\mu_i^{1/2}+\mu_j^{1/2}}{\mu_\nu^{1/2}}\exp(-|z|T_\nu)\right);$$

$$m=1+6(K-1),\quad K=1,2,3\ldots \tag{42}$$

The matrices K_K^{ij} are determined in the following way:

$$K_K{}^{ij}=-\frac{1}{3^{1/4}\beta_A}\frac{Y(1,i)Y(1,j)}{(\mu_i\mu_j)^{1/2}}\sum_{K_1,K_2}Y(K_1,i)Y(K_2,j)\times\left\{\frac{M(m_2+m-m_1)}{2^{m+m_2}}\frac{\Gamma(m_2+m+1)}{\Gamma^{1/2}(m+m_2-m_1+1)}+\frac{M^*(m_1+m_2-m)}{2^{m_1+m_2+1}}\times\frac{\Gamma(m_1+m_2+1)}{\Gamma^{1/2}(m_1+m_2-m+1)}\right\}\frac{1}{(\Gamma(m+1)\Gamma(m_1+1)\Gamma(m_2+1))^{1/2}} \tag{43}$$

where

$$m=1+6(K-1),\quad m_1=1+6(K_1-1),\quad m_2=1+6(K_2-1),\quad K, K_1, K_2=1,2,3\ldots$$

The quantity B is connected with the constant $\tilde{Z}$ by the relation

$$B=-\frac{Z}{1+Z\sum_\nu Y^2(1,\nu)/\mu_\nu^{1/2}}\times\left\{\frac{M(1)}{\pi}+B^2\sum_{ij\nu}\frac{Y(1,\nu)(K_{K_1}^{ij}Y(K_1,\nu))}{\mu_\nu^{1/2}(\mu_\nu^{1/2}+\mu_i^{1/2}+\mu_j^{1/2})}\right\}. \tag{44}$$

As was noted above, the instability develops in the subsystem $m = \{0, 2, 6, 8,...\}$. At the point of instability there appears a nonzero solution of the homogeneous set of equations.

$$-\frac{\pi}{2(1-H_0/H_{c2})}\frac{\partial^2 D_m}{\partial z^2}+\frac{1}{3^{1/4}\beta_A}\sum_{m_1} S(m,m_1)D_{m_1}+\frac{1}{3^{1/4}\beta_A}\sum_{\nu,m_1} Q(\nu,z)E^\nu(m,m_1)D_{m_1}=0; \tag{45}$$

$$m,\ m_1=0,\ 2,\ 6,\ 8\ldots$$

The matrices $E^2(m, m_1)$ in Eq. (45) are defined as follows:

$$E^\nu(m,m_2)=\sum_{K_1} Y(K_1,\nu)\left\{\frac{M(m_2+m-m_1)}{2^{m_2+m}}\frac{\Gamma(m_2+m-1)}{\Gamma^{1/2}(m+m_2-m_1+1)}+\frac{M^*(m_1+m_2-m)\Gamma(m_1+m_2+1)}{2^{m_1+m_2}\Gamma^{1/2}(m_1+m_2-m+1)}+\frac{M(m_1+m-m_2)}{2^{m_1+m}}\frac{\Gamma(m_1+m+1)}{\Gamma^{1/2}(m+m_1-m_2+1)}\right\}\times\frac{1}{(\Gamma(m+1)\Gamma(m_1+1)\Gamma(m_2+1))^{1/2}}, \tag{46}$$

where

$m_1 = 1 + 6(K - 1)$, $K = 1, 2, 3...$, m, $m_2 = 0, 2, 6, 8...$

We are interested in the solution of the set of equations (45) that corresponds to the lowest value of the parameter $\tilde{Z}$. The basic contribution to this solution is made by the eigenfunction $F(K,1)$ of the matrix S. The corrections can be found from perturbation theory. As a result, we get

$$(D_0,\ D_2^*,\ D_6,\ D_8^*\ldots)=Y\{\mathcal{F}(K,\ z)\};$$

$$\mathcal{F}(K,z)=F(K,1)\left\{\exp(-|z|t_1)+B\sum_i A_i\exp(-|z|(t_1+T_i))\right. \tag{47}$$

$$+B^2\sum_{i,\nu}\frac{Y(1,\nu)}{\mu_\nu^{1/2}}A_iR(1,\nu)\frac{\exp(-|z|(t_1+T_i+T_\nu))}{(\lambda_1^{1/2}+\gamma_\nu^{1/2}+\gamma_i^{1/2})^2-\lambda_1}$$

$$+B^2\sum_{i=2}\sum_{\nu,\nu_1}R(i,\nu)R(i,\nu_1)\frac{Y(1,\nu)Y(1,\nu_1)}{(\mu_\nu\mu_{\nu_1})^{1/2}}\frac{1}{(\lambda_1^{1/2}+\gamma_{\nu_1}^{1/2})^2-\lambda_i}$$

$$\times\left(\frac{\exp(-|z|(t_1+T_\nu+T_{\nu_1}))}{(\lambda_1^{1/2}+\gamma_\nu^{1/2}+\gamma_{\nu_1}^{1/2})^2-\lambda_1}-\frac{\lambda_1^{1/2}+\gamma_{\nu_1}^{1/2}}{\lambda_i^{1/2}}\frac{\exp(-|z|(t_i+T_\nu))}{(\lambda_i^{1/2}+\gamma_\nu^{1/2})^2-\lambda_1}\right)$$

$$+B^2\sum_{i,j,\nu}\frac{R(1,\nu)}{\mu_\nu-(\mu_i^{1/2}+\mu_j^{1/2})^2}(K_{K_1}Y(K_1,\nu))$$

$$\times\left.\left(\frac{\exp(-|z|(t_1+T_i+T_j))}{(\lambda_1^{1/2}+\gamma_j^{1/2})^2-\lambda_1}-\frac{\mu_i^{1/2}+\mu_j^{1/2}}{\mu_\nu^{1/2}}\frac{\exp(-|z|(t_i+T_\nu))}{(\lambda_1^{1/2}+\gamma_\nu^{1/2}]^2-\lambda_1}\right)\right\}$$

$$+B\sum_\nu\frac{Y(1,\nu)}{\mu_\nu^{1/2}}\sum_{i=2}\frac{F(k,i)R(i,\nu)}{(\lambda_1^{1/2}+\gamma_\nu^{1/2})^2-\lambda_i}(\exp(-|z|(t_1+T_\nu))-\frac{\lambda_1^{1/2}+\gamma_\nu^{1/2}}{\lambda_i^{1/2}}\exp(-|z|t_i)), \tag{48}$$

where

$$\gamma_i=\mu_i3^{1/4}\beta_A;\quad R(i,j)=\sum_{K,K_1}F(K,i)E^j(m(K),m(K_1))F(K_1,1);$$

$$m(1)=0,\quad m(2)=2,\quad m(3)=6,\quad m(4)=8\ldots;$$

$$A_i=\frac{Y(1,i)}{\mu_i^{1/2}}\frac{R(1,i)}{(\lambda_1^{1/2}+\gamma_i^{1/2})^2-\lambda_1}. \tag{49}$$

The critical value $\tilde{Z}^c$ at which the metastable states first appear is found from the condition of the vanishing of the jump in the derivative at $z = 0$ of the expression in the curly brackets of Eq. (48). Numerical evaluation gives the following values for $\tilde{Z}^c$ and B:

$$B=1,37,\qquad \tilde{Z}^c=0,893. \tag{50}$$

The criterion (50) for the formation of metastable states can be written more illustratively in the form

$$(eH)^{3/2}\int d^3r\delta(\eta D)/\eta D=2,24(1-H_0/H_{c2})^{1/2}. \tag{51}$$

We now proceed to the calculation of the free-energy jump in the transition from the metastable to the stable state. For this we must first find in the quantities D_m the term that is proportional to $|Y|^2$. From the set of Eqs. (18), with account of (42) and (47), we find

$$D_m=\sum_\nu Y(K,\nu)Q(\nu,z)+|Y|^2\sum_\nu Y(K,\nu)\left[\frac{(Y(K_1,\nu))U_{K_1}}{\mu_\nu-4\lambda_1/3^{1/4}\beta_A}\times\left(\exp(-2|z|t_1)-\left(\frac{4\lambda_1}{\gamma_\nu}\right)^{1/2}\exp(-|z|T_\nu)\right)-\frac{\tilde{Z}Y(1,\nu)}{\mu_\nu^{1/2}}\frac{\exp(-|z|T_\nu)}{1+\tilde{Z}\sum_{\nu_1}Y^2(1,\nu_1)/\mu_{\nu_1}^{1/2}}\times\sum_{\nu_1}\frac{Y(1,\nu_1)(U_{K_1}Y(K_1,\nu_1))}{\mu_{\nu_1}^{1/2}(\mu_{\nu_1}^{1/2}+(4\lambda_1/3^{1/4}\beta_A)^{1/2})}\right]+\ldots, \tag{52}$$

where

$$U_K=-\frac{1}{3^{1/4}\beta_A}\sum_{K_1K_2}\frac{F(K_1,1)F(K_2,1)}{(\Gamma(m+1)\Gamma(m_1+1)\Gamma(m_2+1))^{1/2}}\times\left\{\frac{M(m_2+m-m_1)}{2^{m_2+m}}\frac{\Gamma(m_2+m+1)}{\Gamma^{1/2}(m+m_2-m_1+1)}+\frac{M^*(m_1+m_2-m)}{2^{m_1+m_2+1}}\frac{\Gamma(m_1+m_2+1)}{\Gamma^{1/2}(m_1+m_2-m+1)}\right\}. \tag{53}$$

In Eq. (53), $m_1 = m(K_1)$, $m_2 = m(K_2)$, and the function $m(k)$ is determined by Eq. (49). In Eq. (52) we keep only the first term of the expansion. The remaining terms are omitted by us because of their cumbersome nature. In the numerical calculations, we have kept terms up to the second order of smallness inclusive.

In the supercritical region, at values of $\tilde{Z}$ close to $\tilde{Z}^c$, the form of the solution is, as before, determined by Eqs. (47) and (52), and it is necessary to find only the value of Y. For this purpose, we multiply the subset $\{6K, 6K + 2\}$ of the set of equations (18) by the vector $(D_0, D_2, D_6,...)^*$ and integrate over z. As a result of the numerical calculation, we obtain a cubic equation for the quantity Y:

$$0.183|Y|^2Y-0.66\tilde{Z}Y+4.07(\delta x+i\delta y)-38.3|Y|^2(\delta x+i\delta y)=0. \quad (54)$$

We set

$$\delta x+i\delta y=\rho e^{i\varphi}, \qquad Y=\tilde{Y}e^{i\varphi}. \quad (55)$$

From Eq. (54), with account of Eq. (55), we find the value of Y at the point of the discontinuity $Y_{\rm dis}$ and in the final state

$$\tilde{Y}_{\rm dis}=1.05(\delta\tilde{Z})^{1/2}+69.7\rho_{\rm dis}, \quad \rho_{\rm dis}=0.103(\delta\tilde{Z})^{3/2};$$
$$\tilde{Y}_{\rm fin}=-2.1(\delta\tilde{Z})^{1/2}+69.7\rho_{\rm dis}. \quad (56)$$

The jump in the free energy is determined by Eq. (5) and is equal to

$$\delta\left(\frac{\mathcal{F}-\mathcal{F}_0}{\nu/(eH)^{3/2}}\right)=\frac{8\pi^2T^2\tau^2(1-H_0/H_{c2})^2}{7\cdot3^{1/4}\zeta(3)\beta_A(1-1/2\varkappa^2)}$$
$$\times\left(\frac{2\pi}{3^{1/4}\beta_A(1-H_0/H_{c2})}\right)^{1/2}\delta(-0.15\delta\tilde{Z}\tilde{Y}^2+3.05\rho\tilde{Y}). \quad (57)$$

Substituting the value of Y in the initial and final states in (57), we obtain

$$\delta\left(\frac{\mathcal{F}-\mathcal{F}_0}{\nu/(eH)^{3/2}}\right)=-\frac{8\pi^2T^2\tau^2(1-H_0/H_{c2})^{3/2}}{7\zeta(3)(1-1/2\varkappa^2)}\cdot1.97(\delta\tilde{Z})^2. \quad (58)$$

From (56), we find the region of the supercriticality parameter $\delta\tilde{Z}$ in which the Eqs. (54), (56) and (58) are valid:

$$|\delta\tilde{Z}/\tilde{Z}|<0.15. \quad (59)$$

On the edge of the region of applicability, the term proportional to ρ in Eq. (56) becomes of the same order as the principal term.

At low density of the defects, we can neglect their mutual influence. In this approximation, the density of the critical current j_c is proportional to the defect density n. The presence of a transport curent leads to the appearance of a mean force acting on the vortex latice. The condition of the vanishing of the total force acting on the vortex lattice allows us to express the critical current density j_c in terms of the jump of the free energy:[10]

$$j_cB=n\delta\mathcal{F}/\tilde{a}, \quad (60)$$

where n is the density of the defects, $\delta\mathcal{F}$ is the jump of the free energy at a single defect, $\tilde{a}$ is the mean distance between jumps at a single defect. For small values of ρ_{av},

$$\tilde{a}=\frac{3^{1/2}a^2}{4\rho_{\rm dis}}(eH)^{1/2}=\frac{\pi}{2\rho_{\rm dis}(eH)^{1/2}}, \quad (61)$$

where a is the period of the vortex lattice.

From Eqs. (60) and (61) we find the expression for the critical current density of the pinning:

$$j_cB=2n\delta\mathcal{F}\rho_{\rm dis}(eH)^{1/2}/\pi. \quad (62)$$

Equations (40), (58) and (62) completely determine the critical current density of the pinning near the threshold. However, as follows from Eqs. (41), (59) saturation is very quickly reached in the parameter of interaction of the vortex lattice with the defect. The character of the solution at large values of the interaction parameters $\tilde{Z}$ and $\tilde{Z}_1$ has not been studied. If we assume that saturation of the pinning forces does occur when the values of $\delta\tilde{Z}$ and $\delta\tilde{Z}_1$ determined by Eqs. (41) and (59) are reached, in this case

$$j_cB\sim\tau(1-H/H_{c2})^{3/2}. \quad (63)$$

5. CONCLUSION

Small-size defects, whose interaction with the vortex lattice prevents the entry of the core of the vortex into the defect, destroy the metastable state of the vortex lattice localized near the defect. The transverse size of such a state is of the order of the correlation length $\xi(T)$. However, for numerical reasons, the radius of the state in the transverse direction turns out to be comparatively large: $\sim 6\xi(T)$. In the longitudinal direction, because of the numerically small value of the first eigenvalue of the matrix s, a very slow falloff takes place in the deviation of the vortex

$$\sim\exp\{-0.09|z|(1-H_0/H_{c2})^{1/2}/\xi(T)\}.$$

By the same token, the state that is formed has the shape of a strongly elongated cigar. The effective volume of the metastable state turns out to be anomalously large: $\sim 2500\xi^3(T)/(1-H_0/H_{c2})^{1/2}$. This circumstance can have a decisive value, since even at low density of the defects an overlap of states arises and it is necessary to take into account the mutual influence of the impurities. This means that in the pure form, single-particle pinning can be realized only at very low defect density. The presence of several defects in the volume necessary for the formation of the metastable state cannot only sharply lower the threshold condition (29) and (51), but also significantly change the dependence of the critical current density (63) on the temperature and on the value of the external magnetic field. Evidently, the anomalously large volume necessary for the creation of the metastable state can explain the strongly nonlinear dependence of the critical current density of the pinning on the density of the defects.[11]

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