De Almeida–Thouless singularity in a phase transition into the asperomagnetic phase

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It is demonstrated that the line of the de Almeida–Thouless singularity is identical with the line of the phase transition from a state of collinear ferromagnetism to the asperomagnetic state.

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1. INTRODUCTION

It was shown recently[1–3] that in the molecular-field approximation, in magnets with competing ferromagnetic and antiferromagnetic interaction and with continuous symmetry group, a phase transition is possible from the state of collinear ferromagnetism into the asperomagnetic state. The authors of Refs. 2 and 3 state that on the phase diagram there exists a region in which the asperomagnetic state is stable to the de Almeida-Thouless (hereafter AT) singularity.[4,5] With further change of temperature (or of the external magnetic field) an AT singularity arises and there are thus, two phase-transition lines on the phase diagram. This situation seems quite strange. The point is that the asperomagnetic state is in a certain sense a mixed state—a ferromagnet in a direction longitudinal relative to the external magnetic field or the spontaneous moment, and very ordinary spin glass in the transverse direction.

On the other hand, it is known at present that spin glasses are not stable to the AT singularity and there are therefore no spin glasses characterized by only one Edwards-Anderson (hereafter EA) parameter,[6] at least in the molecular-field approximation. It might seem therefore that in the case of a phase transition into the asperomagnetic state the situation should be the same and that the statement made in Refs. 2 and 3, that a region exists in which there is spin glass describable only by the EA parameter, is thus wrong. This is precisely the result obtained in the present paper. We note that this question has already been discussed in Ref. 7. However, the authors of that reference had in essence split the longitudinal and transverse configuration fluctuations and thus, reduced the problem literally to the question of transverse spin glass. In fact, as we shall show below, there is generally speaking no such splitting, i.e., the problem does not reduce to a combination of a longitudinal ferromagnet and transverse spin glass. Therefore the problem was in fact not solved in Ref. 7.

The present paper is devoted to a detailed study of the stability of the asperomagnetic state to the AT singularity. Since the problem is difficult to visualize in the general case, we consider here only a two-component classical spin and assume that $\langle J \rangle = 0$ (J is a random exchange interaction). We assume here that there exists an external magnetic field and it is this which produces the magnetization in the system. It turns out that for our case there are four eigenvalues λ_i that determine the stability of the system (see Refs. 4 and 5). If at least one λ_i is negative, then the system is unstable. We have three eigenvalues that are positive and one that is negative and is of the order of $\lambda_3 \sim -q_\rho^2$, where q_ρ is the EA parameter in the transverse direction.

Thus, the asperomagnetic state, just as any spin glass, is unstable to the AT singularity.

2. BASIC EQUATIONS

We consider the case of a vector spin glass in an external magnetic field. The Hamiltonian is

$$H=-\sum_{ik} J_{ik}\mathbf{S}_i\mathbf{S}_k-\mathbf{h}\sum_i S_i,\quad \langle J_{ik}\rangle=0,\quad \langle J_{ik}^2\rangle=I_{ik},\qquad S^2=pa_0,\tag{1}$$

where $\mathbf{S}$ is the classical p-component spin, $\mathbf{h}$ is the external magnetic field, and J_{ik} is a random exchange integral with Gaussian distribution and random mean value.

The molecular-field equations for such a system can be easily derived in the same manner as was done for Ising spin glass in Ref. 8. They take the following form

$$q_0=\langle m_1^2\rangle_x,\qquad q_\rho=\frac{1}{p-1}\langle m_\rho^2\rangle_x,$$
$$G_0=\langle(\mathbf{S}_1-\mathbf{m}_1)^2\rangle_{T,x},\qquad G_\rho=\frac{1}{p-1}\langle(\mathbf{S}_\rho-\mathbf{m}_\rho)^2\rangle_{T,x},$$
$$\mathbf{m}_{1,\rho}=\langle\mathbf{S}_{1,\rho}\rangle_T,\qquad \langle A\rangle_T=\frac{\int d\mathbf{S}A e^{-H_0/T}\delta(S^2-pa_0)}{\int d\mathbf{S}e^{-H_0/T}\delta(S^2-pa_0)},$$
$$\langle B\rangle_x=\int\frac{d\mathbf{x}}{(2\pi)^{p/2}}B(\mathbf{x})e^{-x^2/2},\tag{2}$$
$$\frac{H_0}{T}=DS_1^2-\mathbf{z}_1\mathbf{S}_1-\mathbf{z}_\rho\mathbf{S}_\rho,\qquad D=\frac{2I_0}{T^2}(G_\rho-G_0),$$
$$\mathbf{z}_1=\frac{\mathbf{h}}{T}+\frac{(4I_0q_0)^{1/2}}{T}\mathbf{x}_1,\qquad \mathbf{z}_\rho=\frac{(4I_0q_\rho)^{1/2}}{T}\mathbf{x}_\rho,$$
$$\mathbf{S}_1=(\mathbf{S}\mathbf{n})\mathbf{n},\quad \mathbf{S}_\rho=\mathbf{S}-\mathbf{S}_1,\quad \mathbf{n}=\mathbf{h}/h,\quad \mathbf{x}_1=(\mathbf{x}\mathbf{n})\mathbf{n},\quad \mathbf{x}_\rho=\mathbf{x}-\mathbf{x}_1,$$
$$q_0+G_0+(p-1)(q_\rho+G_\rho)=pa_0,\qquad I_0=\sum_k I_{ik}.$$

Equations (2) were written for four quantities, $q_{1,\rho}$ and $G_{1,\rho}$, which are connected by a single relation. There are therefore three independent variables, which can be conveniently chosen to be q_1, q_ρ, and D. These equations contain, as usual[8], two types of averaging—over the temperature at a

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fixed local dimensionless molecular field $z_1 + \mathbf{z}_\rho = \mathbf{z}$ and over the local molecular field—longitudinal $\mathbf{z}_1$ and transverse $\mathbf{z}_\rho$. The vectors $\mathbf{x}_1$ and $\mathbf{S}_1$ in (2) are one-dimensional and act in a subspace parallel to the external magnetic field, while the vectors $\mathbf{x}_\rho$ and $\mathbf{S}_\rho$ are $(p-1)$-dimensional and act in a subspace orthogonal to the external field.

It is easy to show that, at high, temperatures q_ρ and consequently also $\mathbf{m}_\rho$ are equal to zero and Eq. (2) become much simpler. The temperature at which a nonzero q_ρ appears is given by

$$\frac{4I_0}{T_c^2}\langle G_\perp{}^2\rangle_x=1, \qquad G_\perp=\frac{1}{p-1}\langle S_\rho{}^2\rangle_T. \tag{3}$$

This is in fact the temperature of the phase transition into the asperomagnetic state.

The AT singularity can be obtained in the following manner. Let

$$G_{\alpha\beta}=\langle(S_\alpha-m_\alpha)(S_\beta-m_\beta)\rangle_T,$$
$$m_\alpha=\langle S_\alpha\rangle_T, \qquad T_{\gamma\delta}{}^{\alpha\beta}=\langle G_{\alpha\beta}G_{\gamma\delta}\rangle_x, \tag{4}$$

where the averaging over T and $\mathbf{x}$ is carried out just as in (2). Let μ_i be the eigenvalues of the operator $\hat{T}$. Then all μ_i should satisfy the condition

$$\lambda_i=1-4I_0\mu_i/T^2>0, \tag{5}$$

and the system is then stable. If (5) is not satisfied for at least one eigenvalue λ_i, the system is unstable. This is in fact the AT instability. The criterion (5) is obtained obviously with the aid of summation of the ladder diagrams

$$\hat{K}=\hat{E}+\frac{4I_0}{T^2}\hat{T}+\ldots=\frac{\hat{E}}{\hat{E}-\frac{4I_0}{T^2}\hat{T}},$$
$$(\hat{E})_{\gamma\delta}{}^{\alpha\beta}=\delta_{\alpha\beta}\delta_{\gamma\delta}. \tag{6}$$

The matrix $\hat{K}$ is, as is well known, one of the generalized susceptibilities and its becoming infinite means approach to a certain phase-transition line. It is known (see, e.g., Ref. 9) that this is a phase transition into a state with degeneracy. If the criterion (5) is satisfied, this means that there is no degeneracy and the system can be described simply by the EA parameter. If, however, the criterion (5) is not satisfied, this is evidence of the presence of degeneracy, which at the present time nobody can describe.

3. STUDY OF THE AT SINGULARITY

As seen from (4), to study the AT singularity it is necessary first to calculate the correlators m_α and $G_{\alpha\beta}$ for the single-node Hamiltonian H_0 written out in (2). The Hamiltonian H_0 contains preferred vectors $\mathbf{z}_1$ and $\mathbf{z}_\rho$. If we introduce the single-node partition function

$$Z=\int d\mathbf{S}e^{-H_0/T}\delta(S^2-pa_0), \tag{7}$$

it can be easily shown that $Z(z)$ is a function of only the moduli of the vectors $\mathbf{z}_1$ and $\mathbf{z}_\rho$. It can then be readily shown that the correlators $G_{\alpha\beta}$ and m_α take the following form

$$\mathbf{m}=\mathbf{n}_1m_1+\mathbf{n}_\rho\mathbf{m}_\rho\equiv\mathbf{m}_1+\mathbf{m}_\rho,$$
$$G_{\alpha\beta}=(\delta_{\alpha\beta}-n_{1\alpha}n_{1\beta}-n_{\rho\alpha}n_{\rho\beta})G_2$$
$$+n_{1\alpha}n_{1\beta}G_3+n_{\rho\alpha}n_{\rho\beta}G_1+(n_{1\alpha}n_{\rho\beta}+n_{\rho\alpha}n_{1\beta})G_4, \tag{8}$$
$$m_1=\frac{\partial\ln Z}{\partial z_1},\quad m_\rho=\frac{\partial\ln Z}{\partial z_\rho},\quad G_1=\frac{\partial m_\rho}{\partial z_\rho},\quad G_2=\frac{m_\rho}{z_\rho},\quad G_3=\frac{\partial m_1}{\partial z_1},$$
$$G_4=\frac{\partial m_\rho}{\partial z_1}=\frac{\partial m_1}{\partial z_\rho},\qquad \mathbf{n}_1=\frac{\mathbf{z}_1}{z_1},\qquad \mathbf{n}_\rho=\frac{\mathbf{z}_\rho}{z_\rho}.$$

It can be seen from (8) that the local-magnetization vector $\mathbf{m}$ does not necessarily have to be directed along the vector $\mathbf{z}=\mathbf{z}_1+\mathbf{z}_\rho$. Furthermore, G_3 is obviously a longitudinal correlator with respect to a direction singled out by the external magnetic field. We note that the unit vector $\mathbf{n}_1$ can differ in sign from the vector $\mathbf{n}$, but the quadratic combination $n_{1\alpha}n_{1\beta}$ is obviously equal to $n_\alpha n_\beta$. Further, in $(p-1)$-dimensional subspace orthogonal to the external-magnetic-field vector there is also a preferred vector $\mathbf{z}_\rho$, therefore this subspace contains two correlators—longitudinal G_1 and transverse G_2. Finally, there is an interference correlator G_4, which entangles both subspaces. If G_4 were equal to zero, the problem could be fully split into longitudinal and transverse problems and reduce to the problem of Ref. 7. But $G_4\neq0$ and allowance for it yields, as we shall see, the same order in q_ρ as the other terms.

If we average $G_{\alpha\beta}$ over $\mathbf{x}$, we obtain

$$\langle G_{\alpha\beta}\rangle_x=G_0n_\alpha n_\beta-G_\rho(\delta_{\alpha\beta}-n_\alpha n_\beta),$$
$$G_0=\langle G_3\rangle_x,\qquad G_\rho=\frac{1}{p-1}\{\langle G_1\rangle_x+(p-2)\langle G_2\rangle_x\}. \tag{9}$$

Obviously, G_0 and G_ρ are precisely those correlators which were defined in (2). It is interesting to note that G_4 has dropped out of (9).

We note now that at $p=2$ the term with G_2 drops out of (8) as a result of the orthogonality condition

$$n_{1\alpha}n_{1\beta}+n_{\rho\alpha}n_{\rho\beta}=\delta_{\alpha\beta}. \tag{10}$$

Physically this is quite understandable, since at $p=2$ the $(p-1)$-dimensional space is one-dimensional and there exists no transverse correlator G_2 in this space.

In the general case of arbitrary p, the expression for $T_{\gamma\delta}^{\alpha\beta}$ is quite cumbersome and its analysis is difficult. We consider therefore only the case $p=2$. If we direct the first axis along the external magnetic field and the second perpendicular to it, we find the following simple expressions for the matrix elements $T_{\gamma\delta}^{\alpha\beta}$:

$$T_{11}{}^{11}=\langle G_3{}^2\rangle_x,\quad T_{12}{}^{12}=T_{21}{}^{21}=\langle G_4{}^2\rangle_x,$$
$$T_{22}{}^{11}=T_{11}{}^{22}=\langle G_1G_3\rangle_x,\quad T_{12}{}^{21}=T_{21}{}^{12}=\langle G_4{}^2\rangle_x,\quad T_{22}{}^{22}=\langle G_1{}^2\rangle_x. \tag{11}$$

The remaining matrix elements are equal to zero. From this we obtain directly the following four eigenvalues of the matrix $\hat{T}$:

$$\mu_{1,2}=\langle G_1G_3\rangle_x\pm\langle G_4{}^2\rangle_x,$$
$$\mu_{3,4}={}^1/_2\{\langle G_1{}^2\rangle_x+\langle G_3{}^2\rangle_x\pm[(\langle G_1{}^2\rangle_x-\langle G_3{}^2\rangle_x)^2+4\langle G_4{}^2\rangle_x{}^2]^{1/2}\}. \tag{12}$$

As we shall see below, at small q_μ we have $\langle G_4^2\rangle_x\sim q_\rho$, and the remaining quantities are finite. We therefore obtain for $\mu_{3,4}$

$$\mu_3=\langle G_1^2\rangle_x+\frac{\langle G_4^2\rangle_x^2}{\langle G_1^2\rangle_x-\langle G_3^2\rangle_x}$$
$$\mu_4=\langle G_3^2\rangle_x-\frac{\langle G_4^2\rangle_x^2}{\langle G_1^2\rangle_x-\langle G_3^2\rangle_x}. \quad (13)$$

It is easily seen that only μ_3 can be of interest to us. Indeed, above the phase-transition point, where $G_4=0$. We have

$$\mu_1=\mu_2=\langle G_1G_3\rangle_x,\ \ \mu_3=\langle G_1^2\rangle_x, \quad (14)$$
$$\mu_4=\langle G_3^2\rangle_x.$$

On the other hand, at $p=2$ we have from (8) and (10) in this region, recognizing that $n_{1\alpha}n_{1\beta}=n_\alpha n_\beta$,

$$G_{\alpha\beta}=(\delta_{\alpha\beta}-n_\alpha n_\beta)G_1+n_\alpha n_\beta G_3. \quad (15)$$

Since $\mathbf{z}_\rho$ is zero in this region, there is only one singled-out vector $\mathbf{h}$, and G_1 is the transverse susceptibility while G_3 is the longitudinal one. On the other hand, the transverse susceptibility exceeds the longitudinal one, i.e.,

$$G_1>G_3,$$
$$\langle G_1^2\rangle_x>\langle G_1G_3\rangle_x>\langle G_3^2\rangle_x. \quad (16)$$

Since, as can be easily understood, G_1 coincides in this region with $G_\perp$ from (3), it is clear that the critical mode is λ_3, and for all the remaining λ_i near T_c we have

$$\lambda_i=1-4I_0\mu_i/T^2>0,\quad i=1,\ 2,\ 4. \quad (17)$$

We therefore need consider just λ_3 near T_c. To this end it is necessary to solve the system of equations (2), find q_0, g_ρ, and D, and calculate μ_3 in (13). It is of course impossible to carry out this program in explicit form. It turns out, however, that this is not necessary. The answer can be obtained by studying all these quantities in implicit form. It is perfectly clear that the critical variable is q_ρ. We therefore consider the equation for q_ρ, assuming q_0 and D to be parameters that depend, of course, on the temperature. We consider the equation for q_ρ in implicit form. To this end we expand the logarithm of the single-node partition function introduced in (7), as well as its derivatives with respect to z_ρ^2:

$$\ln Z(\mathbf{z})=d+{}^1/_2az_\rho^2-{}^1/_4bz_\rho^4+{}^1/_6cz_\rho^6,$$
$$m_1=d'+{}^1/_2a'z_\rho^2-{}^1/_4b'z_\rho^4+{}^1/_6c'z_\rho^6,$$
$$m_\rho=az_\rho-bz_\rho^3+cz_\rho^5, \quad (18)$$
$$G_1=a-3bz_\rho^2+5cz_\rho^4,\quad G_2=a-bz_\rho^2+cz_\rho^4,$$
$$G_3=d''+{}^1/_2a''z_\rho^2-{}^1/_4b''z_\rho^4+{}^1/_6c''z_\rho^6,$$
$$G_4=a'z_\rho-b'z_\rho^3+c'z_\rho^5,$$

where a, b, c, and d are functions of z_1^2, q_0, D, and the remaining parameters of the problem. The primes in (18) denote differentiation with respect to z_1. It can be seen from (18) that $\langle G_4^2\rangle_x\sim q_\rho$. The equation for q_ρ at $p=2$ can be easily obtained from (2). It is of the following form:

$$1-\frac{4I_0}{T^2}\langle a^2\rangle_{x_1}+6\left(\frac{4I_0}{T^2}\right)^2q_\rho\langle ab\rangle_{x_1}-15\left(\frac{4I_0}{T^2}\right)^3q_\rho^2\langle b^2+2ac\rangle_{x_1}=0. \quad (19)$$

In the derivation of (19) we have used the fact that $\mathbf{z}_\rho$ in (2) depends only on $\mathbf{x}_\rho$, and the Gaussian distribution function with respect to $\mathbf{x}$ breaks up into distribution functions with respect to $\mathbf{x}_\rho$ and $\mathbf{x}_1$. We have therefore averaged over $\mathbf{x}_\rho$ in (19), and the averaging over x_1 remained. A similar procedure can be used also when averaging in (13). We then obtain, using Eq. (19)

$$\lambda_3=1-\frac{4I_0}{T^2}\mu_3=-\left(\frac{4I_0}{T^2}\right)^3q_\rho^2\left\{12\langle b^2\rangle_{x_1}+\frac{\langle(a')^2\rangle_{x_1}^2}{\langle a^2\rangle_{x_1}-\langle(d'')^2\rangle_{x_1}}\right\}. \quad (20)$$

Equation (20) is the final answer. It can be seen from (18) that the conditions (16) denote that at $T\geqslant T_c$, where $\mathbf{z}_\rho=0$,

$$\langle a^2\rangle_{x_1}>\langle(d'')^2\rangle_{x_1}. \quad (21)$$

Obviously, this relation remains valid near T_c. It is then seen from (20) that

$$\lambda_3<0. \quad (22)$$

This is precisely the result we wanted to obtain. It can be seen from (20) that if we discard the second term in the curly brackets we obtain exactly the same answer as for a single-component spin glass. This answer agrees with the result of Ref. (7). The second term, however, is generally speaking not small compared with the first, except for the case when the external field is weak. In the latter case this term can be shown to be proportional to $(h/T)^2$. We note that this term, which describes the coupling of the longitudinal and transverse configuration fluctuations, increases the negative contribution to λ_3. This means that the interaction of the longitudinal and transverse fluctuations strengthens the AT singularity and apparently leads in final analysis to a stronger degeneracy than would obtain without this interaction.

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