Nonlinear effects in the propagation of sound in a liquid with gas bubbles

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The effect of the nonlinearity of gas bubble oscillations on the damping of a quasimonochromatic wave in the absence of dissipative effects (viscosity, thermal conductivity, etc.) is considered. The temporal problem of the nonlinear damping of sufficiently large amplitude waves is solved, and the magnitude of its energy change is measured. It is shown that a wave packet localized in space always decays completely. The penetration of a sound perturbation that is excited by a harmonic source at the liquid boundary into a gas-liquid medium is also considered.

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1. INTRODUCTION

The addition to a liquid of a small number of randomly distributed gas bubbles changes significantly the dispersion properties of the sound oscillations that propagate in such a medium. Obviously the most interesting characteristic of the gas-liquid mixture is the appearance of sound damping even upon neglect of dissipative processes in the liquid (viscosity, and thermal conductivity). The physical mechanism of such damping is similar to the Landau damping in the physics of plasma and consists in the fact that the wave transfers its energy to resonant bubbles whose characteristic oscillation frequency is the same as the frequency of the wave. The linear theory of the nondissipative damping of sound in a liquid with bubbles was developed in Ref. 1.

So far as nonlinear effects in the propagation of sound in a gas-liquid mixture are concerned, they were studied earlier with neglect of the interaction of the resonant bubbles with the wave. This assumption is valid if, for example, there are only bubbles of one (nonresonant) radius.[2,3] A number of important results have been established within the framework of such a model. A Burgers-Korteweg–de Vries equation is derived and solutions are obtained in the form of stationary waves and acoustic solitons.[4] The results of the theory have been confirmed by experimental investigations.[5]

In the present work we consider the effect of the nonlinearity of oscillations of resonant bubbles on the propagation of sound perturbations. Our approach is based on the assumption that the wavelength λ of the sound disturbance is large in comparison with the distance between the bubbles. This allows us to describe the oscillations in the continuous medium approximation, using microscopic equations averaged over volumes whose linear dimensions are much less than the wavelength but much greater than the distance between particles. Furthermore, we limit ourselves to the study of the case in which the volume content of the bubbles α is small in comparison with the parameter $\varepsilon \equiv p_0/\rho c^2$ that characterizes the compressibility of the pure liquid[1)] (p_0 is the equilibrium pressure, c is the speed of sound in the liquid, ρ is its density). The linear theory[1] predicts in this case that a monochromatic wave decays exponentially with decrement γ_{lin}:

$$\gamma_{\text{lin}} = \pi^2 c^2 \int a g(a) \delta(\omega_0(a) - \omega)\, da, \tag{1.1}$$

where $\omega_0(a)$ denotes the frequency of the linear oscillations of a bubble of radius a and the distribution function $g(a)$ for the bubble radii $g(a)$ is normalized so that $g(a)da$ gives the number of bubbles with radii in the range from a to $a + da$ in a unit volume. If the scatter of bubbles in radius, Δa, is comparable with the characteristic radius a, then $\gamma_{\text{lin}} \sim \omega\alpha/\varepsilon \ll \omega$ in order of magnitude.

The study of nonlinear effects is carried out in the present work for almost monochromatic waves, when the perturbation of the pressure has the form

$$\delta p(\mathbf{r}, t) = \delta p_0(\mathbf{r}, t) \cos(\mathbf{kr} - \omega t + \theta(\mathbf{r}, t)), \tag{1.2}$$

where $\delta p_0(\mathbf{r},t)$ and $\theta(\mathbf{r},t)$ are the amplitude and phase, which are changing slowly in space and time. In Sec. 2, equations are derived by averaging over the rapid oscillations of the frequency ω. They describe the smooth changes of the oscillation amplitude of the resonant bubbles and of the envelope and phase of the wave. The nonlinear damping of a sinusoidal wave (δp_0 and θ do not depend on $\mathbf{r}$) is analyzed in the third section with the help of these equations. This damping is generated in the medium at the time $t = 0$ (such a setup corresponds to the problem solved in Refs. 6 and 7, that of the nonlinear damping of a strong Langmuir wave in a plasma). It is possible to calculate analytically the energy lost to the bubbles in the limit of high values of δp_0 (for the exact criterion, see Sec. 3), when the amplitude of the wave changes insignificantly in the course of the damping. The transition from the nonlinear stage of damping to the linear one with decreasing δp_0 is illustrated by a numerical calculation.

In the fourth section is considered the evolution of the envelope of a wave packet. In contrast to a sinusoidal perturbation, a localized packet is found to be damped to the end even in the nonlinear regime.

The fifth section is devoted to study of penetration, into a liquid with gas bubbles, of an acoustic perturbation excited on the boundary of the medium by a harmonic source. A feature of this problem is that even at an arbitrarily small amplitude δp_0 the nonlinear effects play a decisive role. We obtain here the penetration rate and the width of the wave front.

The influence of dissipative effects (viscosity, heat conduction) are neglected throughout the paper. The conditions

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for such an approximation are discussed in the concluding sixth section.

2. BASIC EQUATIONS

For the description of wave propagation in a liquid with gas bubbles, we use a set of equations from Ref. 2:

$$\frac{\partial^2\delta\rho}{\partial t^2}-\Delta\delta p=0,\quad \delta\rho=\delta p/c^2-4\pi\rho\int a^2\xi\, g(a)\,da, \tag{2.1}$$

where $\delta\rho$ denotes the perturbation of the density of the mixture, δp the perturbation of the presure, $\xi=R-a$ is the departure of the actual radius R of the bubble from its equilibrium value a.

The first of Eqs. (2.1) is a consequence of the linearized equations of the hydrodynamics of an ideal liquid, while the second takes into account the fact that the perturbation of the density of the mixture arises both as a consequence of the compressibility of the liquid and from the change in the volume of the bubble. Here and elsewhere it is assumed that the perturbation of the pressure is small in comparison with the equilibrium pressure p_0 and, and as a consequence, the amplitude of the oscillations of the bubbles is also small, $\xi\ll a$.

The system (2.1) should be supplemented by an equation connecting the deviation ξ with the pressure in the wave δp. Under neglect of dissipative processes, this equation has the following form:[8]

$$R\frac{d^2R}{dt^2}+\frac{3}{2}\left(\frac{dR}{dt}\right)^2=\frac{p_g-p_0-\delta p}{\rho}, \tag{2.2}$$

where p_g is the gas pressure inside the bubble. For definiteness, we shall assume the oscillations to be adiabatic; then $p_g=p_0(a/R)^{3\gamma}$, where γ is the adiabatic coefficient of the gas. Expanding in (2.2) in terms of the small parameter ξ/a with accuracy to terms $(\xi/a)^2$, we obtain

$$\ddot\xi+\omega_0^2\xi+\frac{\delta p}{\rho a}=a\omega_0^2\left[\frac{3}{2}(\gamma+1)\frac{\xi^2}{a^2}-\frac{9\gamma^2+18\gamma+11}{6}\frac{\xi^3}{a^3}+\frac{3}{2}\frac{\dot\xi^2}{\omega_0^2a^2}\left(\frac{\xi}{a}-1\right)\right], \tag{2.3}$$

where $\omega_0=(3\gamma p_0/\rho a^2)^{1/2}$ is the frequency of linear oscillations. Account of nonlinear terms in the right side of Eq. (2.3) leads to a nonlinear frequency shift $\Delta\omega_n$ which depends on the amplitude of the linear oscillations ξ_0. With the help of standard techniques (see, for example, Ref. 9), it is not difficult to find that, in the first nonvanishing approximation,

$$\Delta\omega_n=-(6\gamma^2-3\gamma-2)\,\omega_0\xi_0^2/16a^2. \tag{2.4}$$

In the problems considered below on the evolution of quasimonochromatic waves, the dependence of δp on the time has the form (1.2) and the basic role in the interaction with the wave is played by the resonant bubbles, whose characteristic frequency of oscillation is close to ω. Under the action of the pressure δp, the amplitude of the oscillations of such bubbles changes slowly in comparison with ω_0^{-1}, which enables us to average Eq. (2.3) over the rapid oscillations with frequency ω_0. This is most simply done if we introduce the Hamiltonian of a bubble located in the field of the wave:

$$\mathscr{H}_0(q,\xi)+4\pi\xi a^2\delta p(\mathbf{r},t), \tag{2.5}$$

where $\mathscr{H}_0$ is the Hamiltonian of the free bubble, q denotes the momentum canonically conjugate to the coordinate ξ, and the vector $\mathbf{r}$ gives the position of the considered bubble. It is convenient to transform from the variables q and ξ to the action variables—the angle I and Φ of the unperturbed Hamiltonian $\mathscr{H}_0$, at the same time, expressing ξ in terms of I and Φ in the second term in (2.5), it suffices to use an approximate relation valid for the linear oscillator:

$$\xi=(2I/m\omega_0)^{1/2}\cos(\Phi+\omega_0 t), \tag{2.6}$$

the role of the mass m for the gas bubble is played by the quantity $4\pi\rho a^3$. In the new variables,

$$\mathscr{H}=\mathscr{H}_0(I)-\omega_0 I+4\pi a^2(2I/m\omega_0)^{1/2}\delta p_0\cos(\mathbf{kr}-\omega t+\theta)\times\cos(\omega_0 t+\Phi).$$

The procedure of averaging now consists in the discarding of the radially oscillating term, which is proportional to $\cos((\omega+\omega_0)t-\mathbf{kr}+\Phi-\theta)$.[10] Simultaneously, using the small nonlinearity of the oscillator, we can expand $\mathscr{H}_0$ in a series in I, keeping only the first two terms;

$$\mathscr{H}_0(I)=\omega_0 I-(6\gamma^2-3\gamma-2)(16ma^2)^{-1}I^2.$$

It is convenient to introduce the dimensionless variables

$$\begin{aligned}&\varphi=\Phi+(\omega_0-\omega)t+\mathbf{kr},\quad \tau=t/T,\\&T=4\omega_0^{-1}(6\gamma^2-3\gamma-2)^{-1}(3\gamma p_0/\delta p_*)^{2/3},\\&J=I(3\gamma p_0/\delta p_*)^{2/3}/4m\omega_0a^2,\\&\Omega=4(1-\omega/\omega_0)(6\gamma^2-3\gamma-2)^{-1}(3\gamma p_0/\delta p_*)^{2/3},\\&P=(6\gamma^2-3\gamma-2)^{-1}\delta p_0/\delta p_*,\end{aligned} \tag{2.7}$$

where δp_* is some characteristic value of the perturbation. Then the Hamiltonian

$$\mathscr{H}'=\Omega J-J^2+P(2J)^{1/2}\cos(\varphi+\theta) \tag{2.8}$$

and the equations of motion

$$\begin{aligned}&dJ/d\tau=-\partial\mathscr{H}'/\partial\varphi=P(2J)^{1/2}\sin(\varphi+\theta),\\&d\varphi/d\tau=\partial\mathscr{H}'/\partial J=\Omega-2J+P(2J)^{-1/2}\cos(\varphi+\theta)\end{aligned} \tag{2.9}$$

look especially simple. We emphasize here that the Hamiltonian (2.8) is not equal to the bubble energy e which (with sufficient accuracy in the linear approximation) is proportional to the action I:

$$e=m\omega_0^2\xi_0^2/2=\omega_0 I=16\pi\rho a^5\omega_0^2(\delta p_*/3\gamma p_0)^{2/3}J. \tag{2.10}$$

We shall express the total energy of the gas bubbles per unit volume, E, in terms of the energy of a single bubble, introducing the distribution function f on the phase plane J, φ:

$$E=\int_0^\infty da\int_0^\infty dJ\int_0^{2\pi}d\varphi\, e(J,a)f(J,\varphi,a;\mathbf{r},t). \tag{2.11}$$

The function f satisfies the normalization condition

$$\int_0^\infty dJ\int_0^{2\pi}d\varphi\, f=g(a). \tag{2.12}$$

For what follows, it is convenient to introduce the mean value of the action for bubbles of a given radius:

$$\bar{J}=(g(a))^{-1}\int_0^\infty dJ\int_0^{2\pi} Jf\,d\varphi. \tag{2.13}$$

Here the energy density (2.11) is obtained by integration of $\bar{J}g(a)$ over the variable a. If, instead of a we use Ω as the variable of integration, then the connection between E and J takes the following form:

$$E=4\pi\rho(6\gamma^2-3\gamma-2)\omega^{-1}(\delta p_0/3\gamma p_0)^{4/3}\int d\Omega\, a^6\omega_0^3 g(a)\bar{J}. \tag{2.14}$$

At the end of this section we carry out an averaging of the wave equation, assuming that the scale and time of change of the envelope of $\delta p_0(\mathbf{r},t)$ and phase $\theta(\mathbf{r},t)$ are large in comparison with k^{-1} and ω^{-1}. After substitution of (1.2) in (2.1) and discarding of small second derivatives of δp_0 and θ, the wave equation takes the following form:

$$\left[\left(k^2-\frac{\omega^2}{c^2}\right)+2\left(\frac{\omega}{c^2}\frac{\partial\theta}{\partial t}+\mathbf{k}\nabla\theta\right)\right]\delta p_0\cos(\mathbf{kr}-\omega t+\theta)$$
$$+2\left[\frac{\omega}{c^2}\frac{\partial\delta p_0}{\partial t}+\mathbf{k}\nabla\delta p_0\right]\sin(\mathbf{kr}-\omega t+\theta)=4\pi\rho\int da\, g(a)\,a^2\frac{\partial^2\xi}{\partial t^2}. \tag{2.15}$$

Taking (2.6) into account, and multiplying (2.15) first by cos $(\mathbf{k}\cdot\mathbf{r}-\omega t+\theta)$ and then by sin $(\mathbf{k}\cdot\mathbf{r}-\omega t+\theta)$ and averaging, we obtain

$$\frac{\omega}{c^2}\frac{\partial\delta p_0}{\partial t}+\mathbf{k}\nabla\delta p_0=-2\pi\rho\int da\, g(a)\,a^2\omega_0^2\left(\frac{2I}{m\omega_0}\right)^{1/2}\sin(\varphi+\theta), \tag{2.16}$$

$$\delta p_0\left(\frac{\omega}{c^2}\frac{\partial\theta}{\partial t}+\mathbf{k}\nabla\theta\right)$$
$$=\frac{\omega}{c^2}(\omega-kc)\delta p_0-2\pi\rho\int da\, g(a)\,a^2\omega_0^2\left(\frac{2I}{m\omega_0}\right)^{1/2}\cos(\varphi+\theta). \tag{2.17}$$

After simple transformations, with account of (2.9), (2.11), the law of conservation of energy is obtained from (2.16). This law connects the change in the energy density of the bubbles and the mean energy density of the wave $W=(\delta p_0)^2/2\rho c^2$ with the energy flux $c\mathbf{n}W$ in the wave ($\mathbf{n}=\mathbf{k}/k$ is the unit vector in the direction of propagation of the wave):

$$\frac{\partial}{\partial t}(W+E)+c\,\mathrm{div}(\mathbf{n}W)=0. \tag{2.18}$$

So far as Eq. (2.17) (the equation describing the change in phase of the wave) is concerned, we must remark that the choice of phase θ and frequency of the wave ω is not unique. This manifests itself, in particular, in the invariance of Eqs. (2.9), (2.16), and (2.17) relative to the substitutions $\omega\to\omega+\delta\omega$, $\theta\to\theta+\delta\omega t$, $\varphi\to\varphi-\delta\omega t$. (Of course, ω must be close to the sound frequency in the pure liquid, kc; in the opposite case the averaging procedure based on the slowness of the change of δp_0 and θ, loses meaning). The quantity ω is assumed hereafter to be equal to the sound frequency in the liquid with gas bubbles, which is found in linear approximation:[1]

$$\omega=kc\left[1+2\pi c^2\int_0^\infty\frac{a g(a)\,da}{(kc)^2-\omega_0^2}\right]. \tag{2.19}$$

With such a choice of ω in the integrals in Eqs. (2.16) and (2.17), we can isolate the contribution of the resonance region and, assuming that the distribution function $g(a)$ changes slightly over the interval of resonant radii of the bubbles, take it outside the integral sign. Then (2.16) and (2.17) take the following form in dimensionless coordinates:

$$\frac{\partial P}{\partial\tau}+cT\frac{\partial P}{\partial x}=-\frac{\gamma_\pi T}{\pi}\int_{-\infty}^{\infty}d\Omega(2J)^{1/2}\sin(\varphi+\theta), \tag{2.20}$$

$$P\left(\frac{\partial\theta}{\partial\tau}+cT\frac{\partial\theta}{\partial x}\right)=-\frac{\gamma_\pi T}{\pi}\int_{-\infty}^{\infty}d\Omega(2J)^{1/2}\cos(\varphi+\theta).$$

3. RELAXATION OF AN INITIAL FINITE-AMPLITUDE PERTURBATION

As was noted in the Introduction, according to the linear theory the monochromatic wave is damped out completely, transferring all its energy to resonant bubbles whose vibrational frequency lies in an interval of width $\gamma_{\rm lin}$ near the wave frequency ω. Allowance for the nonlinear frequency shift (2.4) leads to the result that the frequency of the resonant bubbles changes with increase in the energy of the oscillation and, if the amplitude of the wave is sufficiently great, $\Delta\omega_n$ can exceed $\gamma_{\rm lin}$. The bubbles then go off resonance, the exchange of energy between them and the wave ceases because of phase mixing, while the amplitude of the wave settles at some level and no longer changes with time. The qualitative effect is completely analogous to the nonlinear Mazitov-O'Neil damping of a Langmuir wave in a plasma.[6,7]

Before proceeding to a description of nonlinear damping, we first find the limits of applicability of the linear theory. If the wave is completely damped, all its energy per unit volume $(\delta p_0)^2/\rho c^2$ transforms into energy of vibration of the resonant bubbles, the number of which per unit volume is $(\alpha/a^3)(\gamma_{\rm lin}/\omega_0)\sim\alpha^2/\varepsilon a^3$ in order of magnitude. Finding the energy per bubble, and calculating the corresponding nonlinear frequency shift with the help of (2.4) and (2.10), we find that $\Delta\omega_n\sim\omega(\delta p_0/p_0)^{2/3}$. It follows then that the condition of applicability of the linear theory $\Delta\omega_n\ll\gamma_{\rm lin}$ has the form

$$(\delta p_0/p_0)^{2/3}\ll\alpha/\varepsilon. \tag{3.1}$$

In the opposite limiting case, the amplitude of the monochromatic wave will change insignificantly in the damping process.

It is not difficult to estimate the energy E which the bubbles absorb at $(\delta p_0/p_0)^{2/3}\gg\alpha/\varepsilon$. The amplitude of their steady-state oscillations, ξ_0, is now determined from the condition of nonlinear resonance:[9]

$$\omega_0\Delta\omega_n\xi_0\sim\delta p_0/\rho a,$$

which, together with (2.4) gives $\xi_0\sim a(\delta p_0/p_0)^{1/3}$. Multiplying the energy of a single resonant bubble by the number per unit volume, which is equal to $(\alpha/a^3)(\Delta\omega_n/\omega)$, we find

$$E\sim\frac{\alpha}{\varepsilon}\left(\frac{\delta p_0}{p_0}\right)^{-2/3}W\sim\gamma_{\rm lin}TW.$$

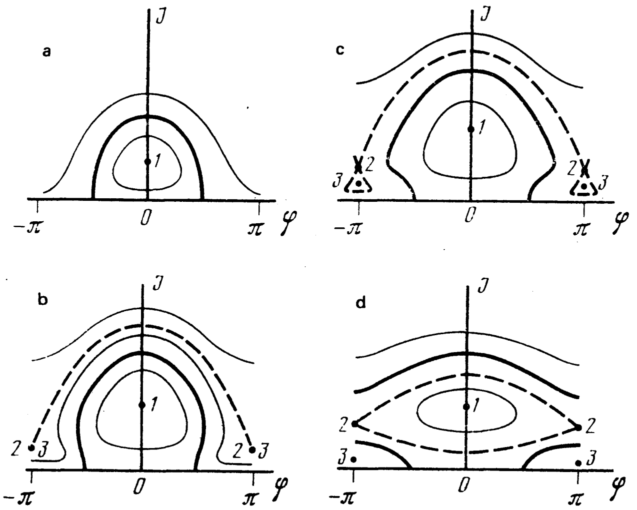

FIG. 1. Shape of the trajectories on the J, φ plane for different values of detuning Ω [see (2.8)]: a—$\Omega<0$, b—$\Omega=3(P/2)^{2/3}$, c—$3(P/2)^{2/3}<\Omega<3P^{2/3}/2^{1/3}$, d—$\Omega>3P^{2/3}/2^{1/3}$. The heavy lines indicate trajectories corresponding to the value $\mathscr{H}'=0$. The numbers 1, 2, and 3 designate the stationary points.

The presence of bubbles in the liquid leads to dispersion of the sound velocity; therefore, the phase θ of the wave changes along with its amplitude, i.e., the instantaneous frequency of the sound is $\omega+\partial\theta/\partial t$. The change of phase θ due to the nonlinearity of the oscillations of the resonant bubbles can be estimated with the help of the second of Eqs. (2.20):

$$\partial\theta/\partial t\sim\omega_0\alpha/\varepsilon\sim\gamma_{\text{lin}}. \tag{3.2}$$

In the nonlinear regime, the frequency shift (3.2) is significantly less than the resonance width $\Delta\omega_n$. We can then assume the phase θ of the wave to be constant and set it equal to zero for definiteness. Then the evolution of the wave is completely described by the law of energy conservation (2.18).

We proceed to an accurate calculation of the energy E. In the solution of the equation of bubble oscillations, we can assume the amplitude of the wave δp_0 to be constant. In such a case, the Hamiltonian (2.8) is an integral of the motion.

We choose the quantity $(6\gamma^2-3\gamma-2)^{-1}\delta p_0$ as the normalizing fact δp_* in (2.7). Then the motion of the bubbles is described by Eq. (2.9) with $P=1$. The qualitative form of the phase trajectories on the J, φ plane for different values of the dimensionless detuning Ω is shown in Fig. 1. At the initial instant, all the bubbles lie in a narrow band near the axis $J=0$, which corresponds to the small (noise) amplitude of the initial oscillation and to a random phase distribution. Since the Hamiltonian (2.8) is equal to zero, in what follows the motion will take place along trajectories that are determined by the equation $\mathscr{H}'=0$ (in Fig. 1, these are delineated by the heavy lines). The period of the motion over the phase trajectories depends on the radius a of the bubble or, in other words, on the magnitude of the detuning Ω. This leads to the result that the smooth distribution function $f(J,\varphi,a)$ becomes strongly ragged as a function of a even after several periods of the phase oscillations. Averaging over small intervals δa allows us to regard the function f ergodic, $f=f(\mathscr{H}',a)$. It is easy to write down the explicit form for f, taking into account the normalization (2.12):

$$f=\delta(\mathscr{H}')g(a)\left[\int\delta(\mathscr{H}')\,dJ\,d\varphi\right]^{-1}. \tag{3.3}$$

Using this formula, we need only keep it in mind that at $\Omega>3/2^{1/3}$, when there are two $\mathscr{H}'=0$ curves separated by the separatrix (the dashed line in Fig. 1), the dissipation function differs from zero only on the one that touches the axis $J=0$ (see Fig. 1d).

The sought energy E is now calculated by Eq. (2.11) which, after changing the integration variable

$$d\varphi=d\mathscr{H}'(\partial\mathscr{H}'/\partial\varphi)^{-1}=-d\mathscr{H}'(dJ/d\tau)^{-1}$$

and taking into account (3.3), takes the following form:

$$E=\int_0^\infty da\,g(a)\left[\oint dJ\,e(J,a)\,(dJ/d\tau)^{-1}\Big/\oint dJ\,(dJ/d\tau)^{-1}\right]. \tag{3.4}$$

Assuming that the scale of change of the function $g(a)$ is large in comparison with range $\Delta a\sim a(\Delta\omega_n/\omega)$ of the values of the radius at which the second factor in the integrand of (3.4) is appreciably different from zero, we can take $g(a)$ outside the integral sign, setting the value of a in $g(a)$ to correspond to the resonance value $\omega_0(a)=\omega$. As a result we get

$$E=4\pi g(a)a^4(\delta p_0)^{4/3}(6\gamma^2-3\gamma-2)^{-1/3}(3\gamma p_0)^{-1/3}\int_{-\infty}^{\infty}d\Omega F(\Omega), \tag{3.5}$$

where the function $F(\Omega)$ is given by the equation

$$F(\Omega)=\frac{\int_0^{y_m}y^{1/2}[1-y(y-\Omega)^2/2]^{-1/2}dy}{\int_0^{y_m}y^{-1/2}[1-y(y-\Omega)^2/2]^{-1/2}dy}, \tag{3.6}$$

while y_m is determined from the condition of vanishing of the expression under the radical. The graph of the function $F(\Omega)$ is given in Fig. 2. Numerical integration of (3.6) yields

$$\int_{-\infty}^{\infty}F(\Omega)\,d\Omega=3.56.$$

Equation (3.5) solves the stated problem—it allows us to find the energy transferred to wave by the bubbles as a result of nonlinear damping.

With the help of (3.5) and the energy conservation law,

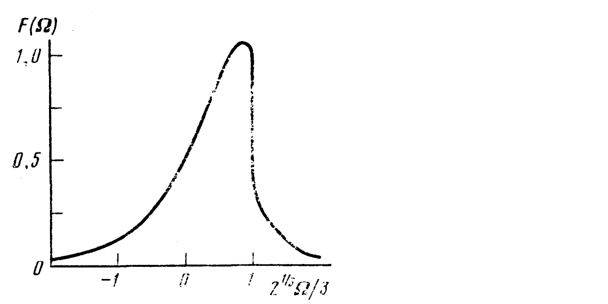

FIG. 2. Plot of $F(\Omega)$. At the point $\Omega=3/2^{1/3}(P=1)$, the arbitrary function $F(\Omega)$ becomes infinite.

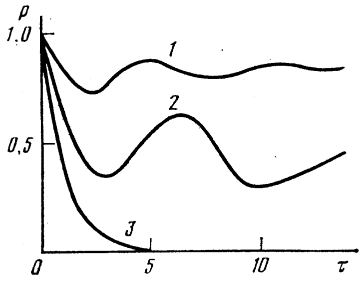

FIG. 3. Dependence of the amplitude of a monochromatic wave on time. The curve 1 corresponds to the value $\gamma_{\rm lin} T = 0.157$; 2—$\gamma_{\rm lin} T = 0.47$; 3—$\gamma_{\rm lin} T = 0.94$

we can easily calculate how much the amplitude of the wave is changed by phase mixing:

$$\Delta P(\tau=\infty)=\frac{44.7\rho c^2 a^4 g(a)(\delta p_0)^{-2/3}}{(6\gamma^2-3\gamma-2)^{1/6}(3\gamma p_0)^{1/6}}=\frac{3.56}{\pi}\gamma_{\rm lin}\,T. \quad (3.7)$$

The $P(\tau)$ dependence is itself of interest. It describes how the wave amplitude changes during the damping process. The corresponding curves, obtained as a result of numerical solution of Eqs. (2.9) and (2.20) (in which it was assumed that P and θ do not depend on x) are given in Fig. 3. Since the time of phase oscillations T (2.7) is identical in order of magnitude with the reciprocal of the nonlinear correction to the frequency $\Delta\omega_n^{-1}$, the value $\gamma_{\rm lin} T = 0.157$ represents the nonlinear regime with excellent accuracy; the limit towards which ΔP tends with large τ agrees to within 10% with the calculation according to Eq. (3.7) (we note that (3.7) gives the first term of the expansion of ΔP in $\gamma_{\rm lin} T$ and is therefore accurate to within 10% at $\gamma_{\rm lin} T \sim 0.1$). The curve $\gamma_{\rm lin} T = 0.94$ demonstrates the exponential linear damping of the wave. The value $\gamma_{\rm lin} T = 0.47$ illustrates the intermediate case.

To conclude this section we note that since the characteristic damping time in the linear regime is of the order of $\Delta\omega_n^{-1} \sim T$, the assumed condition of sudden application of the wave actually means that the initial perturbation should be generated more rapidly than the time of phase mixing T.

4. NONLINEAR EVOLUTION OF A WAVE PACKET

In this section we consider the evolution of a quasimonochromatic wave packet, the equation of which is given by Eq. (1.2). As will be shown, in contrast with the monochromatic wave, the localized perturbation is always completely damped out, since, migrating in space at the group velocity, it sets up oscillations of new bubbles and transfers its energy to them.

At low amplitude δp_0, when the inequality (3.1) is valid, the amplitude of the packet falls off, in correspondence with the linear theory, in proportion to $\exp(-\gamma_{\rm lin} t)$. We shall be interested here in the case in which an inequality inverse to (3.1) is satisfied. Moreover, we shall assume that the size of the leading edge of the packet L is sufficiently great that the time of its passage past the bubble, L/c, is much greater than the time of phase mixing[2)] $\Delta\omega_n^{-1}$, and we limit ourselves only to the leading edge of the packet, where the amplitude δp_0 is a monotonically decreasing function of x.

The dependence of the dimensionless amplitude P of the wave and the phase θ on time leads to the result that the Hamiltonian (2.8) is no longer an integral of the motion, while the form of the phase trajectories (now defined at fixed τ as the lines of constant $\mathscr{H}'$) changes with time. The slowness of the change of P and θ allows us, however, to assume the distribution function at all time to be ergodic and use the conservation of a new adiabatic invariant K, equal to the area under the trajectories on the J, φ plane

$$K=\oint J d\varphi. \quad (4.1)$$

Since the frequency shift of the wave is small $\partial\theta/\partial t \sim \gamma_{\rm lin} \ll \Delta\omega_n$ [see Eq. (3.2)] and, moveover, as is shown, does not depend on time, its account leads simply to a small shift in the resonance as a whole relative to the wave frequency ω. In view of this, the phase θ, as also in the previous section, can be assumed to be equal to zero.

Since all the bubbles are quiescent prior to the arrival of the wave, it follows that J, and with it K, is equal to zero. The equality $K=0$ in the succeeding instants of time, when $P \neq 0$, means that the state of the oscillator corresponds to a stationary point J_c, φ_c whose coordinates are determined from the conditions

$$\frac{\partial\mathscr{H}'}{\partial\varphi}=0, \quad \frac{\partial\mathscr{H}'}{\partial J}=0. \quad (4.2)$$

The dependence of J_c on Ω, which is computed with the help of (2.9), coincides with the resonance curve of the nonlinear oscillator:[9]

$$J_c=\frac{P^{2/3}}{2^{5/3}}\left[\left(1+\left(1-\frac{4\Omega^3}{27P^2}\right)^{1/2}\right)^{1/3}+\left(1-\left(1-\frac{4\Omega^3}{27P^2}\right)^{1/2}\right)^{1/3}\right]^2,$$
$$\Omega<3\left(\frac{P}{2}\right)^{2/3};$$
$$J_c=\frac{2\Omega}{3}\cos^2\left[\frac{1}{3}\,{\rm arctg}\left(\frac{4\Omega^3}{27P^2}-1\right)^{1/2}+\frac{\pi n}{3}\right],$$
$$n=0,\pm1, \quad \Omega>3\left(\frac{P}{2}\right)^{2/3}, \quad (4.3)$$

and is shown in Fig. 4 by the solid line.

It is obvious that at small amplitudes of the wave, $P\to 0$, the bubbles fall on branch 1 (at $\Omega<0$) and 3 (at $\Omega>0$) of the resonance curve, to which correspond the centers 1 on Fig. 1a and 3 on Fig. 1d. With increase in P, J_c increases and these centers move upwards on the phase plane. For oscillators

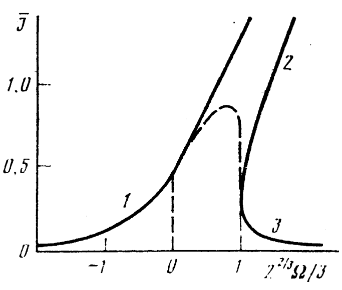

FIG. 4. Dependence of $\bar J(\Omega, 1)$. The numbered branches (1, 2 and 3) correspond to designation of stationary points on Fig. 1 (the portions 2 and 3 are separated by a point with a vertical tangent).

with $\Omega > 0$ a rearrangement of the structure of the phase plane occurs at $P = P_c = 2(\Omega/3)^{3/2}$: the center 3 merges with the saddle point 2, to form a complex singular point (Fig. 1b) which vanishes upon further increase in P together with the separatrix. Upon approach of the singular points to one another, the period of small oscillations around the center 3 tends to infinity (in proportion to $(P - P_c)^{-1/4}$); therefore, in the course of the interval in which it becomes equal to and then greater than the characteristic time of growth of the pressure L/c, the adiabatic invariant K ceases to be conserved. In order of magnitude, this interval turns out to be equal to

$$\Delta\omega_n^{-1}(\Delta\omega_n L/c)^{-3} \ll \Delta\omega_n^{-1},$$

therefore, the Hamiltonian $\mathscr{H}'$ of the bubble changes insignificantly during this interval and can be assumed to be equal to its value at the separatrix

$$\mathscr{H}' = \mathscr{H}_c \equiv -\Omega_c^2(P)/12, \text{ where } \Omega_c(P) = 3(P/2)^{2/3}.$$

Next, after passage of the separatrix, the invariant K again becomes a conserved quantity, but now it is equal not to zero but to the area under the separatrix, when bifurcation occurs: $K = K_{\text{sep}}$.

We now turn to the calculation of the energy of the bubbles as a function of P. It is clear from what was said above that at a given P the distribution function of bubbles with $\Omega < 0$ and $\Omega > \Omega_c(P)$ is given by the equation

$$f = g(a)\,\delta(J - J_c)\,\delta(\varphi - \varphi_c), \tag{4.4}$$

where at $\Omega > \Omega_c(P)$, the branch 3 in Fig. 4 corresponds to the values J_c, φ_c. For bubbles from the interval $0 < \Omega < \Omega_c(P)$ for which the adiabatic invariant is violated, the distribution function can be regarded as ergodic:

$$f = g(a)\,\delta(\mathscr{H}' - \mathscr{H}_*) \Big/ \int \delta(\mathscr{H}' - \mathscr{H}_*)\,dJ\,d\varphi. \tag{4.5}$$

The phase trajectory on which $f \neq 0$, i.e., $\mathscr{H}_*$ as a function of Ω and P, is given by the value of the invariant $K(\mathscr{H}_*, \Omega, P) = K_{\text{sep}}$. It is easy to see from (2.8) that the form of the integral line $J(\varphi; \mathscr{H}', \Omega, P)$ is a homogeneous function

$$J(\varphi;\, \mathscr{H}',\, \Omega,\, P) = \nu J(\varphi;\, \nu^{-2}\mathscr{H}',\, \nu^{-1}\Omega,\, \nu^{-3/2}P), \tag{4.6}$$

therefore the equation of the separatrix for the bubble Ω (corresponding to values of the parameters $\mathscr{H}' = -\Omega^2/12$, $P = 2(\Omega/3)^{3/2}$ can be written down as

$$J_{\text{sep}}(\varphi;\, \Omega) = \Omega J(\varphi;\, -1/12,\, 1,\, 2/3^{3/2})$$

and, consequently, $K_{\text{sep}}(\Omega) = \beta\Omega$. The constant β was found by numerical integration and is equal to 6.16.

Next, the known value of $K_{\text{sep}}(\Omega)$ at fixed amplitude $P = 1$ was used to find numerically the corresponding value of $\mathscr{H}_*$, and consequently also the distribution function (4.5). The result of calculation with the help of this distribution function of the averaged action $\bar{J}(\Omega, P = 1)$ (2.13) is shown in Fig. 4. At $\Omega < 0$ and $\Omega > \Omega_c(1)$ this function obviously coincides with the sections 1 and 3 of the resonance curve; it is shown by the dashed line at $0 < \Omega < \Omega_c(1)$. At an arbitrary pressure P, the averaged action $\bar{J}(\Omega, P)$ is connected with the known $\bar{J}(\Omega, 1)$ dependence by the simple relation

$$\bar{J}(\Omega,\, P) = P^{2/3}\bar{J}(\Omega P^{-2/3},\, 1), \tag{4.7}$$

which follows from (4.6).

Now, using the Eq. (2.14) and (4.7), we can easily find the energy density of the bubbles as a function of the amplitude δp_0 on the wavefront

$$E = 4\pi g(a)\,a^4(\delta p_0)^{4/3}(6\gamma^2 - 3\gamma - 2)^{-1/3}(3\gamma p_0)^{-1/3}\int_{-\infty}^{\infty}\bar{J}(y, 1)\,dy. \tag{4.8}$$

As in (3.5), the value of the radius a corresponds to the resonance frequency $\omega_0(a) = \omega$. Numerical integration of the function $\bar{J}$ yields

$$\int_{-\infty}^{\infty}\bar{J}(y, 1)\,dy = 2.49.$$

In order to obtain an equation describing the evolution of the shape of the packet, we must transform in the energy conservation law (2.18) to a system of coordinates moving with the velocity c: $\mathbf{r}' = \mathbf{r} - \mathbf{n}ct$. Using Eq. (4.8) and introducing the variable $v = 3.32c\gamma_{\text{lin}}TP^{-2/3}\pi$, we get

$$\partial v/\partial t = v\mathbf{n}\,\partial v/\partial \mathbf{r}'. \tag{4.9}$$

It is then seen that each point of the envelope moves with a constant velocity v inversely proportional to $(\delta p_0)^{2/3}$. This means that a steepening of the wavefront takes place with time, up to the scale $c/\Delta\omega_n$. After the width of the front becomes comparable with $c/\Delta\omega_n$, the adiabatic approximation is violated and further evolution of the envelope will no longer be described by Eq. (4.9). However, the picture is preserved qualitatively; the discontinuity formed at the leading edge of the front, moving relative to the packet, will continually erode it. This erosion of a packet of sound waves is analogous to the damping of a packet of Langmuir waves in a plasma.[11]

In conclusion, we consider the effect, on the evolution of the envelope, of dispersion effects which were not taken into account above. For this purpose, we equate the rate of dispersion spreading $\Delta k \partial^2\omega/\partial k^2$ to the calculated rate of deformation of the packet as a result of nonlinear damping $c\gamma_{\text{lin}}/\Delta\omega_n$. Recognizing that the scatter of the wave vectors in the packet is $\Delta k \sim L^{-1} \sim \Delta\omega_n/c$ and using the expression for the dispersion increment in a liquid with gas bubbles (2.19), we find that $\Delta k \partial^2\omega/\partial k^2 \sim \gamma_{\text{lin}}\Delta\omega_n/ck^2$. Thus, as a consequence of the dispersion, the packet spreads out at least $(\omega/\Delta\omega_n)^2$ times more slowly than it is deformed under the action of nonlinear effects, so that the dispersion spreading can actually be neglected.

5. PENETRATION OF THE WAVE INTO A LIQUID ACROSS THE BOUNDARY

We now consider the problem of the penetration, into a liquid containing gas bubbles, of a sound wave excited on the boundary by a harmonic source of frequency ω. We shall assume that a source with pressure-oscillations amplitude δp^0 is switched on at the instant of time $t = 0$. For the quantity δp_* in this section, we choose $\delta p^0(6\gamma^2 - 3\gamma - 2)^{-1}$.

We first assume that the amplitude δp^0 is so small that it satisfies the condition of applicability of the linear theory

(3.1). The frequency of the wave is set by the source, and its wave vector should be found from the dispersion relation and will contain an imaginary part $k = \omega/c + i\gamma_{\text{lin}}/c$. This means that the wave decays exponentially into the interior of the liquid, and its energy is absorbed by the bubbles in a surface layer of thickness c/γ_{lin}. It is obvious that within a time interval $\sim\Delta\omega_n^{-1}$, the amplitude of oscillations of these bubbles increases so that nonlinear effects come into play and, as described in Sec. 3, exchange of energy between the wave and the bubbles ceases. Then the perturbation of the pressure can still penetrate a distance c/γ_{lin} and the entire process repeats itself. As a result, a wavefront of width $\sim c/\gamma_{\text{lin}}$ travels inside the liquid with a speed V determined from the condition that the wave travel a distance c/γ_{lin} within a time $\Delta\omega_n^{-1}$: $V\sim c\Delta\omega_n/\gamma_{\text{lin}}$. Behind the front, in the initially unperturbed liquid, oscillations of amplitude δp^0 are established. Thus we see that the penetration of the harmonic wave in the liquid, even of arbitrarily small amplitude, has a nonlinear character (and not only upon satisfaction of an inequality inverse to (3.1)).

After the front moves away from the boundary to a distance that is much greater than its width, the shape and velocity of the front remain practically unchanged. This means that the solution of the problem tends to become self-similar and dependent only on a single variable $\xi = x/VT - \tau$.

For a quantitative description of the structure of the front, the adiabatic approximation, developed in the previous section, is inapplicable, since the time of growth of the amplitude at the front is comparable with the period of phase oscillations $T\sim\Delta\omega_n^{-1}$. For the same reason, it is necessary to take into account the change in phase θ of the wave. Therefore we use the initial set of equations (2.9) and (2.20), which we rewrite by introducing the variable ξ in the following form:

$$\frac{dJ}{d\xi}=-P(2J)^{1/2}\sin(\theta+\varphi),$$

$$\frac{d\varphi}{d\xi}=-\Omega+2J-P(2J)^{-1/2}\cos(\theta+\varphi); \tag{5.1}$$

$$\frac{dP}{d\xi}=-\frac{\gamma_{\text{lin}}T}{\pi}\frac{V}{c-V}\int_{-\infty}^{\infty}(2J)^{1/2}\sin(\theta+\varphi)\,d\Omega, \tag{5.2}$$

$$P\frac{d\theta}{d\xi}=-\frac{\gamma_{\text{lin}}T}{\pi}\frac{V}{c-V}\int_{-\infty}^{\infty}(2J)^{1/2}\cos(\varphi+\theta)\,d\Omega.$$

Ahead of the front the bubbles are at rest and the pressure is equal to its equilibrium value

$$J=0,\quad P=0\quad \text{at}\quad \xi=+\infty, \tag{5.3}$$

and behind the front the pressure perturbation tends to δp^0 or, in dimensionless coordinates,

$$P=1\quad \text{at}\quad \xi=-\infty. \tag{5.4}$$

Equations (5.3) and (5.4) are boundary conditions for the set (5.1) and (5.2), the solution of which should determine both the shape of the envelope $P(\xi)$ and the value of the velocity V.

The problem of the determination of the eigenvalue V can be reduced to an initial-condition problem. For this, we make the substitution

$$P=\nu P',\quad \xi=\nu^{-1/3}\xi',\quad \Omega=\nu^{+2/3}\Omega',\quad J=\nu^{2/3}J',\quad \varphi=\varphi',\quad \theta=\theta'. \tag{5.5}$$

where $\nu = [\gamma_{\text{lin}}TV/\pi(c-V)]^{3/2}$. Then Eqs. (5.1) remain without change, while in (5.2) the factors in front of the integral on the right side vanish:

$$\frac{dP'}{d\xi'}=-\int_{-\infty}^{\infty}(2J')^{1/2}\sin(\theta'+\varphi')\,d\Omega',$$

$$P'\frac{d\theta'}{d\xi'}=-\int_{-\infty}^{\infty}(2J')^{1/2}\cos(\theta'+\varphi')\,d\Omega'. \tag{5.6}$$

The substitution (5.5) preserves the boundary condition (5.3) and in place of (5.4), we obtain

$$P'(-\infty)=[\gamma_{\text{lin}}TV/\pi(c-V)]^{-3/2}. \tag{5.7}$$

Thus, if we solve the set (5.1) and (5.6) with the initial conditions (5.3) and find the limit to which P tends as $\xi\to-\infty$, then V is found

$$V=c(1+A\gamma_{\text{lin}}T)^{-1}, \tag{5.8}$$

where $A = [P'(-\infty)]^{2/3}/\pi$. The pressure in the wave is expressed in terms of the solution of the boundary problem (5.1)–(5.4) in the following fashion:

$$\delta p(x,t)=\delta p^0P(\xi)\cos(kx-\omega t+\theta(\xi)),\quad \xi=x/VT-\tau. \tag{5.9}$$

Since the function $P(\xi)$ increases from zero to a value of the order of unity in the interval $\Delta\xi\sim1$, the width of the front L is equal to VT apart from an insignificant numerical factor. We can easily obtain the asymptote of the solution at $\xi\gg1$, where $J,P\to0$, by neglecting the nonlinearity of the oscillations of the bubbles:

$$P(\xi)=\exp(-\gamma_{\text{lin}}TV(\xi-\xi_0)/(c-V)),\quad \theta(\xi)=\text{const},$$
$$J(\xi)=\tfrac{1}{2}P^2(\xi)\,[(\gamma_{\text{lin}}TV/(c-V))^2+\Omega^2]^{-1}, \tag{5.10}$$
$$\varphi(\xi)=\operatorname{arctg}\left[\left(\frac{\gamma_{\text{lin}}TV}{c-V}\operatorname{tg}\theta+\Omega\right)\Big/\left(\frac{\gamma_{\text{lin}}TV}{c-V}-\Omega\operatorname{tg}\theta\right)\right],$$

where ξ_0 is a constant.

In the region $\xi\sim1$, the solution is obtained as a result of numerical computer integration of the initial, non-self-similar equations (2.9) and (2.20), in which we have made the transition to the set of coordinates moving with speed V: $\xi = x/VT - \tau$. The integration was carried out by a difference scheme of second-order approximation[12] on the finite section $0\leqslant\xi\leqslant l$. We specified arbitrary initial values (P,θ,J,φ) inside this section at $\tau = 0$ the amplitude $P(\xi = 0,\tau) = 1$ and the phase of the wave θ $(\xi = 0,\ \tau) = 0$ at the left boundary. On the right we specified the bubble distribution function $f(J,\varphi,a)$, according to Eq. (5.10), in which we must set $\xi = l$ and choose the constant ξ_0 such that $P(\xi = l)\ll1$. The phase θ of the wave at $\xi = l$ (unlike $P(l)$) could change in the boundary condition (5.10) and during each time step it was set equal to its value at $\xi = l$ obtained in the preceding step. Within a time several times longer than $\max[l/V, l/(c-V)]$, with accuracy to within 15%, the stationary form of the envelope $P(\xi)$ and of the phase $\theta(\xi)$ of the wave was set. Here, of course, the pressure at $\xi = l$ depends on the parameters V

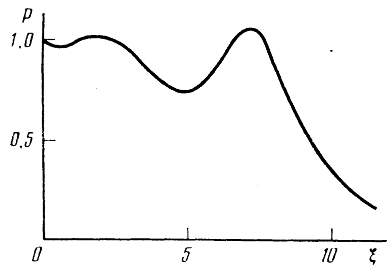

FIG. 5. Amplitude of the pressure $P(\xi)$ on the front of the wave.

and $\gamma_{\text{lin}} T$ and does not necessarily coincide with the value $P(l)$ used in the calculation of the bubble distribution function (5.10) on the right boundary. The family of parameters V and $\gamma_{\text{lin}} T$, which assure this coincidence, give the desired solution of the boundary-value problem (5.1)–(5.4). A graph of the function $P(\xi)$ is given in Fig. 5. The constant A in (5.8) is approximately equal to 2.2.

The described solution is valid if the inequalities $\alpha \ll \varepsilon$ and $\delta p^0 \ll p_0$ are satisfied, while the parameter $\gamma_{\text{lin}} T$ which usually characterizes the role of the nonlinearity of oscillations of the bubbles can be either greater than or smaller than unity.

6. CONCLUSION

In conclusion, we discuss the conditions under which we can neglect the dissipative processes that were not taken into account above. The dissipation does not play a significant role if the time of the bubble oscillations damping due to dissipation (viscosity, thermal conductivity) and radiation losses (radiation of sound waves) is much greater than the characteristic time of the problem, such as the value of $\Delta\omega_n^{-1}$ in the problem of the damping of a monochromatic wave, the time of complete erosion of the packet, in the problem of the evolution of a packet, and so on. Under the assumptions made in this work, all these times are greater than the period of free oscillations $2\pi\omega_0^{-1}$.

The decrement of dissipative damping increases with increase in frequency and for an air bubble in water at atmospheric pressure it becomes comparable with the frequency of linear oscillations at $\omega_0 \sim 10^6$–10^7 s^{-1} ($a \sim 10^{-4}$ cm).[13] Therefore we cannot neglect dissipation at the higher frequencies. In the range of low frequencies, the radiation losses (at $\omega < 10^3$ s^{-1}) and the losses due to heat conduction (at $\omega > 10^3$ s^{-1}) are dominant. The decrements $\gamma_{\text{lin}} \sim \omega_0 \varepsilon^{1/2}$ of the radiative and $\gamma_t \sim (\omega_0 \chi)^{1/2}/a$ of the thermal damping (χ is the coefficient of thermal diffusivity of air) is significantly smaller than ω_0. Therefore at low frequencies, a range of parameters $(\alpha, \delta p_0)$ exists in which the dissipative effects are unimportant.

In the damping process, by virtue of the law of momentum conservation, the momentum of the wave is transferred to the bubbles, which are therefore accelerated in the direction of propagation of the wave. The velocity v to which the bubbles are accelerated can be estimated by dividing the momentum lost by the wave (W/c in the linear and $\gamma_{\text{lin}} TW/c$ in the nonlinear regimes) by the number of resonant bubbles $((\alpha/a^3)(\gamma_{\text{lin}}/\omega_0)$ and $(\alpha/a^3)(\Delta\omega_n/\omega_0))$ respectively and by the effective mass of the bubble[3] $4\pi\rho a^3$[3)]. It is easy to estimate that the Doppler shift of frequency $\omega v/c$ turns out to be at least ε^{-1} times smaller than the width of the resonance (γ_{lin} in the linear and $\Delta\omega_n$ in the nonlinear regimes, respectively); therefore, the forward motion has no effect on the interaction of the bubble with the wave.

In conclusion, the authors express their thanks to D. D. Ryutov for useful discussion.

[1)]For water at atmospheric pressure, $\varepsilon = 5 \times 10^{-5}$.

[2)]If $L/c \ll \Delta\omega_n^{-1}$, the packet decays linearly even upon satisfaction of the inequality $(\delta p_0/p_0)^{2/3} \gg \alpha/\varepsilon$.

[3)]Under the action of the sound perturbation, the bubble also executes oscillatory motion with a velocity $\delta p/\rho c$ lower than v.

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