Generalized dynamics of three-dimensional vortical singularities (vortons)

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Equations of first order in time are obtained for the trajectories of liquid particles in three-dimensional vortical flows. These equations are used to derive a system of ordinary differential equations, which describes the dynamics of three-dimensional vortical singularities (vortons). An analytic solution of the problem of the interaction of two vortons has made it possible to visualize lucidly the physical mechanism of self-amplification of a three-dimensional vortex field. The collapse of vortons and discrete analogs of vortex filaments and rings are also considered analytically.

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The main difficulty in the investigation of three-dimensional vortical flows is the effect of stretching of the vortex filaments. This effect can give rise to singularities in the vortex field. Such a possibility is indicated by the experimentally observed strongly pronouned intermittency of vortex flows.[1,2] The results of numerical simulation[3,4] offer evidence that the mean squared field of a vortex in three-dimensional flow of an ideal liquid increases in time without limit. The associated possible existence of a finite limit on the rate of energy dissipation as the viscosity tends to zero corresponds to extension of the Kolmogorov-Obukhov "3/2 law"[6,7] to include the region of arbitrarily small scales. In analogy with energy dissipation in shock waves, the energy of three-dimensional vortex flow of an ideal liquid can go off into singularities.

It is very difficult to guess beforehand the character of the singularities that can arise in three-dimensional flow. It is therefore useful to introduce elementary "priming" singularities and a corresponding generalized dynamics, to be able to investigate analytically three-dimensional effects. This calls for a special form of the equations of hydrodynamics.

§1. LAGRANGIAN DESCRIPTION OF VORTEX FLOWS

The equations of an ideal liquid constitute Newton's equations for the coordinates $\mathbf{x}(t, \mathbf{a})$ of the liquid particles ($\mathbf{a}$ is the initial position). The potential of the forces (pressure) in an incompressible liquid is expressed in terms of a quadratic functional of the spatial derivatives of the velocity field. Let us lower the temporal order of these equations, using the theorem on the conservation of the velocity circulation along a liquid contour in differential form[7]:

$$\Omega_i(t, \mathbf{x}) = \Omega_{m0}(\mathbf{a})\,\partial x_i/\partial a_m. \tag{1.1}$$

Here $\mathbf{\Omega}(t, \mathbf{x})$ is the field of the vortex, zero marks the initial motion, and we sum from 1 to 3 over the repeated indices. Differentiation of (1.1) with respect to time yields the Helmholtz equation for the vortex field.

We express the velocity field in terms of an integral of the vortex field in unbounded space, replacing the integration with respect to $\mathbf{x}$ by integration with respect to $\mathbf{a}$ (by virtue of the incompressibility, the Jacobian of the transformation is equal to unity). Using (1.1), we obtain the sought equation of first order in time for the trajectories of the liquid particles:

$$\dot{x}_i = -\frac{1}{4\pi}\varepsilon_{ijk}\int \frac{x_j - x_j'}{|\mathbf{x}-\mathbf{x}'|^3}\frac{\partial x_k'}{\partial a_m'}\Omega_{m0}'\,d^3a'. \tag{1.2}$$

A dot denotes here a derivative with respect to time, ε_{ijk} is a unit antisymmetric tensor, and primes denote quantities taken at the point $\mathbf{a}'$.

In two-dimensional flows there is only one vortex component, and Eq. (1.1) takes the form $\Omega(t, \mathbf{x}) = \Omega_0(\mathbf{a})$. The equations for the liquid-particle trajectories are Hamiltonian:

$$\Omega_0 \dot{x}_i = \varepsilon_{ij}\delta H/\delta x_j,$$

where the indices i and j now run through the values 1 and 2, and $\delta/\delta x_j$ is a variational derivative. The expression for the Hamiltonian can be written down directly in the presence, alongside the continuous field of the vortex, of δ-singularities having intensities $\varkappa_\alpha$ and located at the points $\mathbf{x}^{(\alpha)}(t)$:

$$H = -\frac{1}{4\pi}\int \Omega_0\Omega_0' \ln|\mathbf{x}-\mathbf{x}'|\,d^2a\,d^2a'$$
$$-\frac{1}{2\pi}\sum_\alpha \varkappa_\alpha \int \Omega_0 \ln|\mathbf{x}-\mathbf{x}^{(\alpha)}|\,d^2a - \frac{1}{2\pi}\sum_{\alpha<\beta}\varkappa_\alpha\varkappa_\beta \ln r_{\alpha\beta}.$$

Account is taken here of the fact that a discrete vortex does not act on itself; $r_{\alpha\beta}$ is the distance between vortices. For the trajectories of the discrete vortices we have the equations

$$\varkappa_\alpha \dot{x}_i^{(\alpha)} = \varepsilon_{ij}\partial H/\partial x_j^{(\alpha)}.$$

From the invariance of H to shifts and rotations of the coordinate frames we obtain the integrals of motion

$$\int \Omega_0 x_i d^2a + \sum_\alpha \varkappa_\alpha x_i^{(\alpha)},$$

$$\int \Omega_0 x_i^2 d^2a + \sum_\alpha \varkappa_\alpha (x_i^{(\alpha)})^2.$$

If the vortex field has no continuous component, we

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return to the usual system of equations for discrete vortices (an exposition of the latest results on the interaction between straight vortex filaments can be found in Ref. 8). The situation is just as simple with introduction of vortical singularities in two-dimensional flows when there is no stretching of the vortex filaments.

§2. VORTON SYSTEM

It is desirable to choose the three-dimensional vortex singularities such that they do not change their structure upon interaction and that it be possible to use them to approximate sufficiently general vortex fields. Vortex rings, for example, are not very suitable for this purpose. They are characterized by two scales, have intrinsic velocities, and their shapes become distorted by short-range interaction. In addition, only vortex fields of very special kind can be formed out of small vortex rings.

The simplext elementary singularity that might have the properties indicated above is a three-dimensional δ-singularity of the vortex field. At the point where this singularity is located the vortex field is no longer solenoidal, making it no longer a hydrodynamic object. In the classical hydrodynamic description, by virtue of (1.1) and of the incompressibility, this violation of the solenoidal behavior of the vortex field persists at all instants of time. Precisely such a three-dimensional δ-singularity, for which the vortex field is non-solenoidal at all time, was introduced in a recent paper,[9] where the name "vorton"was proposed and it was noted specially that so far there are no results whatever.

It is possible, however, to introduce three-dimensional vortex singularities and still deal with hydrodynamic objects. Equation (1.2) presents a closed description of vortex flows in which the vortex field is reconstructed from the velocity field (1.2) and is always solenoidal for any field $\mathbf{\Omega}_0(\mathbf{a})$. If $\mathbf{\Omega}_0$ is solenoidal everywhere and has no singularities, the solution of (1.2) coincides with the solution of the hydrodynamics equation in classical form. If $\mathbf{\Omega}_0$ ceases to be solenoidal even in one point, the use of classical hydrodynamics equations is generally speaking not valid. We note that the vortex field can cease indeed to be solenoidal at a definite instant of time as a result of external action or under the influence of internal physical factors (e.g., in a superfluid liquid).

We shall use the term "vorton" hereafter for a vortical singularity corresponding to a three-dimensional δ-singularity of the field $\mathbf{\Omega}_0$. For a vorton located at the origin, the fields of the velocity and of the vortex take according to (1.2) the form

$$v_i = -(4\pi x^3)^{-1}\varepsilon_{ijk}x_j\gamma_k,$$

$$\Omega_i = \gamma_i\delta(\mathbf{x}) + (4\pi x^5)^{-1}(3x_ix_k\gamma_k - x^2\gamma_i),$$

where γ is the vorton intensity. In this case the liquid rotates at an angular velocity $\gamma(4\pi x^3)^{-1}$. The vortex field is now solenoidal everywhere (including at the origin). With the aid of vortons it is easy to construct discrete analogs of vortex filaments and rings (§4), whose behavior conforms to the classical solutions. We proceed now to study the dynamics of vortons.

Substituting in (1.2)

$$\Omega_{i0}(\mathbf{a}) = \sum_{\beta=1}^{N}\gamma_{i0}^{(\beta)}\delta(\mathbf{a}-\mathbf{a}^{(\beta)}),$$

we obtain for the velocity field an expression which is also an equation for the trajectories $\mathbf{x}(t, \mathbf{a})$ of the liquid particles:

$$\dot{x}_i = -\frac{1}{4\pi}\varepsilon_{ijk}\sum_{\beta=1}^{N}|\mathbf{x}-\mathbf{x}^{(\beta)}|^{-3}(x_j - x_j^{(\beta)})\gamma_k^{(\beta)}. \qquad (2.1)$$

This includes the coordinates and intensities of the vortons:

$$x_i^{(\alpha)}(t) \equiv x_i(t, \mathbf{a}^{(\alpha)}), \quad \gamma_i^{(\alpha)}(t) \equiv \left.\frac{\partial x_i}{\partial a_m}\right|_{\mathbf{a}=\mathbf{a}^{(\alpha)}}\gamma_{m0}^{(\alpha)}.$$

The equation for $x_i^{(\alpha)}$ is obtained by substituting $\mathbf{a} = \mathbf{a}^{(\alpha)}$ in (2.1). It must be taken into account here that the vorton itself does not move, since the velocity it induces at its location has no direction (in analogy with the two dimensional) case in §1). We obtain

$$\dot{x}_i^{(\alpha)} = -\frac{1}{4\pi}\varepsilon_{ijk}{\sum_{\beta=1}^{N}}' r_{\alpha\beta}^{-3}r_j^{(\alpha,\beta)}\gamma_k^{(\beta)},$$

$$r_i^{(\alpha,\beta)} = x_i^{(\alpha)} - x_i^{(\beta)}, \quad r_{\alpha\beta} = |\mathbf{r}^{(\alpha,\beta)}|, \qquad (2.2)$$

where the prime at the summation sign means omission of the term with $\beta = \alpha$. We obtain an equation for $\gamma_i^{(\alpha)}$ by differentiating (2.1) with respect to a_m and then substituting $\mathbf{a} = \mathbf{a}^{(\alpha)}$:

$$\dot{\gamma}_i^{(\alpha)} = -\frac{1}{4\pi}\varepsilon_{ijk} \times {\sum_{\beta=1}^{N}}'(r_{\alpha\beta}^{-3}\gamma_j^{(\alpha)} - 3r_{\alpha\beta}^{-5}r_j^{(\alpha,\beta)}r_m^{(\alpha,\beta)}\gamma_m^{(\alpha)})\gamma_k^{(\beta)}. \qquad (2.3)$$

It is likewise recognized here that the vorton does not act on itself, since it produces neither tension nor compression along the rotation axis.

We have thus obtained a closed system (2.2), (2.3) of ordinary differential equations, which describes the evolution of three-dimensional singular flows of an ideal liquid. This system allows to investigate analytically and numerically various three-dimensional effects and, in particular, illustrate clearly the physical mechanism of self-amplification of a three-dimensional vortex field. We present below examples of analytic solutions.

§3. INTERACTION OF TWO VORTONS

Following the method developed in Ref. 10 for linear vortices, we consider the dynamics of the relative configuration of two vortons ($N = 2$). From (2.2) we have

$$\dot{r}_i = \varepsilon_{ijk}\sigma_j r_k, \quad r_i \equiv r_i^{(1,2)}, \quad \sigma_i = \sigma_i^{(1)} + \sigma_i^{(2)}, \quad \sigma_i^{(\alpha)} = (4\pi r^3)^{-1}\gamma_i^{(\alpha)}. \qquad (3.1)$$

The vector $\mathbf{r}$ thus rotates without changing its modulus, $r = \text{const}$.

We write the equations for the intensities (2.3) in a reference frame that rotates together with **r** at an angular velcoty $\boldsymbol{\sigma}$:

$$d\sigma_i^{(\alpha)}/dt=3\sigma_r^{(\alpha)}\varepsilon_{ijk}n_j\sigma_k^{(\alpha+1)},\quad n_i=r^{-1}r_i,\quad \boldsymbol{\sigma}^{(3)}\equiv\boldsymbol{\sigma}^{(1)},\quad \sigma_r^{(\alpha)}=n_i\sigma_i^{(\alpha)}. \tag{3.2}$$

Here d/dt is a derivative with respect to time in the rotating frame. Since **r** does not change in the rotating frame, $\sigma_r^{(\alpha)}$ are according to (3.2) integrals of the motion. Equations (3.2) have to additional integrals:

$$\sigma_i^{(1)}\sigma_i^{(2)}=\text{const},\qquad (\sigma_i^{(1)})^2\sigma_r^{(2)}+(\sigma_i^{(2)})^2\sigma_r^{(1)}=\text{const}.$$

Differentating (3.2) with respect to time we obtain

$$d^2\tilde{\sigma}_i^{(\alpha)}/dt^2=-9\sigma_r^{(1)}\sigma_r^{(2)}\tilde{\sigma}_i^{(\alpha)},\quad \tilde{\sigma}_i^{(\alpha)}=\sigma_i^{(\alpha)}-n_i\sigma_r^{(\alpha)}. \tag{3.3}$$

Depending on the vorton configuration (Fig. 1), Eq. (3.3) offers three possibilities: At $s\equiv 9\sigma_r^{(1)}$ and $\sigma_r^{(2)}=0$ the time-independent component of $\boldsymbol{\sigma}^{(\alpha)}$ is the one for which $\sigma_r^{(\alpha)}\neq 0$. At $s<0$ we have sinusoidal oscillations with frequency $\omega=s^{1/2}$. At $s<0$ we obtain an exponential growth with exponent $\lambda=(-s).^{1/2}$

Thus, within the framework of the generalized dynamics, the vorticity in three-dimensional flow can increase without limit even in the case of two vortons. The physical meaning of the condition for exponential growth of the vorticity ($s<0$) consists of a suitable orientation of vortons at which they amplify each other on account of the deformation of the liquid elements in which they are situated. That the vorticity in three-dimensional flows of an ideal liquid increases without limit is attested by numerical experiments.[4]

We consider now absolute motion of two vortons. As $\tilde{\boldsymbol{\sigma}}^{(\alpha)}\equiv 0$ the vortons are immobile. This applies also to a system with an arbitrary number of vortons if all lie on one straight line. The drift of the center of gravity of two vortons is determined from (2.2):

$$\dot{x}_i=\varepsilon_{ijk}r_j\tilde{\sigma}_k,\quad x_i={}^1/_2(x_i^{(1)}+x_i^{(2)}),\quad \tilde{\sigma}_i=\sigma_i^{(1)}-\sigma_i^{(2)}. \tag{3.4}$$

A "pair" of vortons ($\tilde{\boldsymbol{\sigma}}^{(1)}=-\tilde{\boldsymbol{\sigma}}^{(2)}$, $\sigma_r^{(\alpha)}\equiv 0$) moves with constant velocity (3.4) perpendicular to the line that joins them. At $\sigma_r^{(1)}=\sigma_r^{(2)}=\sigma_r/2$, as can be easily shown, the vector $\tilde{\sigma}_i$ precesses around the vector $p_i=3\sigma_r n_i/2$ with an angular velcotity determined by the magnitude of this vector. The vector p_i, in turn, rotates (together with σ_i and n_i) around the vector $q_i=\sigma_i+3\sigma_r n_i/2$, which is an integral of the motion. The two identical vortons rotates with an angular-velocity vector q_i around the immobile center of gravity.

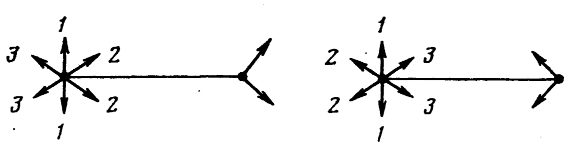

FIG. 1. Configuration of two vortons.

§4. COLLAPSE OF VORTONS, VORTON FILAMENTS AND RINGS

We consider a system of vortons located at the initial instant of time in a single plane ($x_3=$ const). If the initial intensities of the vortons are perpendicular to this plane, the vortons remain according to (2.2) and (2.3) the vortons remain in the same plane, and the intensities do not change. Equations (2.1) are written in Hamiltonian form:

$$\gamma^{(\alpha)}\dot{x}_i^{(\alpha)}=\varepsilon_{ij}\frac{\partial H}{\partial x_j^{(\alpha)}},\quad H=\frac{1}{4\pi}\sum_{\alpha<\beta}\frac{\gamma^{(\alpha)}\gamma^{(\beta)}}{r_{\alpha\beta}},\quad \gamma^{(\alpha)}\equiv\gamma_3^{(\alpha)}. \tag{4.1}$$

A similar system is obtained also in a description of localized vortices in a rapidly rotating stratified liquid (the atmosphere of a planet). Such a problem was recently considered by V. M. Gryanik. From the invariance of **H** to shifts and rotations in the plane $x_3=$ const follow integrals of motion of the same type as in two-dimensional flow:

$$\sum_\alpha\gamma^{(\alpha)}x_i^{(\alpha)},\quad \sum_\alpha\gamma^{(\alpha)}(x_i^{(\alpha)})^2,\quad \sum_{\alpha,\beta}\gamma^{(\alpha)}\gamma^{(\beta)}r_{\alpha\beta}^2.$$

The last integral is a combination of the first two. When the conditions

$$\sum_{\alpha<\beta}\gamma^{(\alpha)}\gamma^{(\beta)}/r_{\alpha\beta}(0)=0,\qquad \sum_{\alpha,\beta}\gamma^{(\alpha)}\gamma^{(\beta)}r_{\alpha\beta}^2(0)=0 \tag{4.2}$$

are satisfied, homogeneous collapse of the vortons is possible:

$$r_{\alpha\beta}^3(t)=r_{\alpha\beta}^3(0)\,(1-t/t_*). \tag{4.3}$$

Here t_* is the collapse time, and the case $t_*<0$ corresponds to the vortons moving apart. The value of t_* at an arbitrary number of vortons is expressed in terms of the intensities and the initial distances, in analogy with the procedure used for two-dimensional flows.[11] Equation (4.3), as well as a corresponding formula with $r_{\alpha\beta}^3$ replaced by $r_{\alpha\beta}^2$ in the two-dimensional case,[12,11] is a consequence of the scale invariance of the equations. Conditions (4.2) are necessary and sufficient for uniform collapse (dispersal) in the case of three vortons or of symmetrical configurations of four or five vortons (see Ref. 11). This can be verified by direct substitution of (4.3) in the equations written in terms of the relative motion (for $r_{\alpha\beta}$). At an arbitrary number of vortons, the necessary and sufficient collapse conditions are expressed in terms of a Hamiltonian in analogy with the two-dimensional case.[11]

The transition to two-dimensional hydrodynamics is ef-

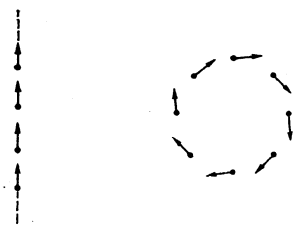

FIG. 2. Vorton filament and vorton ring.

fected with the aid of vorton filaments. A vorton filament is an infinite sequence of single vortons (Fig. 2) having intensities γ and placed along a straight line with equal spacing d. Several such parallel filaments will drift mutually, according to (2.2) and (2.3), in a plane perpendicular to them without changing the vorton intensities. One filament rotates the other, located at a distance r, with velocity $v = (\varkappa/2\pi r) f(d/r)$, where $\varkappa = \gamma d^{-1}$ is the specific intensity of the filament. Without writing down here the expressions for f, we indicate that $f(0) = 1$. This corresponds to the classical Helmholtz formula for interacting vortex filaments.

A discrete analog of a vortex ring is a vorton ring: a system of identical vortons placed tangent to a circle at the vertices of a regular polygon (see Fig. 2). In such a configuration the vorton intensities, according to (2.2) and (2.3), do not change, and the ring itself drifts perpendicular to its plane. Simple calculation yields an expression for the drift velocity:

$$v=\frac{\varkappa}{2\pi r}c_n, \quad \varkappa=\frac{\gamma n}{2\pi r},$$

$$c_n=\frac{\pi}{n}\sum_{k=1}^{[(n-1)/2]}\operatorname{cosec}\frac{k\pi}{n}+\frac{\pi[1+(-1)^n]}{4n}. \qquad (4.4)$$

Here r is the radius of the ring, n is the number of vortons, and $\varkappa$ is the specific intensity. In the case of a continuous toroidal vortex ring, when the cross-section radius ρ is small compared with r, there is a known[13] asymptotic formula $n \approx (\varkappa/4\pi r)\ln(r/\rho)$. From (4.4) as $n \to \infty$ we obtain $c_n \approx \ln n$. The vorton approximates a segment of a continuous vortex ring with volume of the order of ρ^3. For thick rings ($\rho \sim r$) an analytic description is difficult. Equation (4.4) allows us to estimate the velocity of such rings.

5. CONCLUSION

The procedure proposed above of introducing three-dimensional singularities, and the method of investigating their dynamics are sufficiently universal and can be used for other equations of physics. The direct observation of objects such as vortons (in an ordinary or superfluid liquid, in a plasma) is not excluded. In a stratified liquid, in particular, the vortex tubes can be broken up into pieces.

On the other hand, vortons present a natural approximation of continuous fields that substantially simplify the analysis of three-dimensional effects. It is of interest to investigate configurations that consist of three and more vortons and lead to an unbounded growth of the vorticity within a finite time (we compare the results of the numerical experimental[4]). It can be shown that in this case the flow energy will go over into the vortex singularities in accordance with the remark made above. We can hope hereafter to construct configurations corresponding to the "3/2 law."

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[1]E. A. Novikov and R. W. Stewart, Izv. AN SSSR, ser. geofiz. No. 3, 408 (1964).
[2]E. A. Novikov, Prikl. Mat. Mekh. **35**, 266 (1971).
[3]R. H. Morf, S. A. Orszag, and U. Frisch. Phys. Rev. Lett. **44**, 572 (1980).
[4]A. J. Chorin, Commun. Math. Phys. **83**, 517 (1982).
[5]L. D. Landau and E. M. Lifshitz, Fluid Mechanics, Pergamon, 1959.
[6]A. S. Monin and A. M. Yaglom, Statisticheskaya gidromekhanikia (Statistical Hydromechanics), Part 2, Nauka, 1967.
[7]J. Serrin, Mathematical Principles of Classical Fluid Mechanics (Russ. transl.), IIL, 1963, p. 52.
[8]E. A. Novikov, Ann. N. J. Acad. Scien. **357**, 47 (1981).
[9]P. G. Saffman, in: Transition and Turbulence, Academic, 1981, p. 149.
[10]E. A. Novikov, Zh. Eksp. Teor. Fiz. **68**, 1868 (1975) [Sov. Phys. JETP **41**, 937 (1975)].
[11]E. A. Novikov and Yu. B. Sedov, Zh. Eksp. Teor. Fiz. **77**, 588 (1979) [Sov. Phys. JETP **50**, 297 (1979)].
[12]G. K. Batchelor, and H. Aref, Phys. Fluids **23**, 393 (1979).
[13]Introduction to Fluid Dynamics (Russ. transl.), Mir, 1973.

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