Effective conductivity of two-dimensional anisotropic media

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The conductivity of two-dimensional anisotropic randomly inhomogeneous media is considered. It is shown that the Dykhne transformation makes it possible to single out systems for which the transformed media is macroscopically equivalent to the initial one. To this end, there should exist a constant c such that the tensor conductivity fields σ and $c\sigma/\det\sigma$ have identical statistical distributions. In this case, a relation $\det\sigma_* = c$ is established. If the system is macroisotropic, then $\sigma_* = c^{1/2}$. A number of inhomogeneous anisotropic structures is considered for which the exact $\det\sigma_*$ or σ_* have been calculated. It is shown that in a number of cases the effective conductivities determined by the self-consistency method either satisfy the exact relations or are exact.

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Many physical processes in inhomogeneous media (electric conductivity, thermal conductivity, filtering of liquids and gases, and others) are characterized by a mathematically equivalent problem of determining the effective (macroscopic) conductivity of the medium from information on the structure of a random local-conductivity field.

The prolonged and intensive study of this problem notwithstanding, there are at present not enough general and the same time constructive methods of determining the effective conductivity. The known approximate solutions obtained using various modifications of the methods of perturbation theory, of the self-consistent field, and others are not always satisfactory in the case of strong fluctuations of the conductivity field. Notice of the lack of effective estimates of the errors of the approximate solutions is now in order. This is why the few cases when the problem of determining the effective conductivity can be exactly solved and the result is of sufficiently simple form are of such great importance. Obviously, such solutions can be obtained only for relatively simple model cases, inasmuch as for only such structures does the effective conductivity depend on a small number of parameters of the random field of the local conductivity.

It is known that the tensor of the effective conductivity of arbitrary layered, i.e., in fact one-dimensional, structures can be determined exactly, as well as the effective conductivities of several planar systems, discovered by Dykhne,[1] with a special type of random-field symmetry.

It will be shown below that the Dykhne method can be transferred also to certain anisotropic planar systems for which the effective conductivities were determined exactly or for which exact relations were established. It is noted that in a number of cases self-consistent effective conductivities satisfy exact relations.

1. Thus let the vector-flux $\mathbf{v}$ and the field $\mathbf{h}$ be connected by the system of equations

$$\mathbf{v}=\sigma\mathbf{h},\qquad \operatorname{div}\mathbf{v}=0,\qquad \operatorname{rot}\mathbf{h}=0, \tag{1}$$

where the local conductivity σ is a random tensor that depends on the coordinates x and y.

We introduce the flux and field averaged over the area:

$$\mathbf{V}=\frac{1}{\Omega}\int\limits_{\Omega}\mathbf{v}\,d\omega,\qquad \mathbf{H}=\frac{1}{\Omega}\int\limits_{\Omega}\mathbf{h}\,d\omega, \tag{2}$$

and define the effective-conductivity tensor σ_* of the system by the relation

$$\mathbf{V}=\sigma_*\mathbf{H}. \tag{3}$$

The characteristic linear scale of the averaging region Ω, for example its diameter, should be considerably larger than the correlation scales of the fields $\mathbf{v}$ and $\mathbf{h}$ in the xy plane, and at the same time much smaller than the characteristic dimensions of the planar structure as a whole. We assume also that the averaging (2) is equivalent to averaging over an ensemble of media.

Following Dykhne,[1] we introduce the fields $\mathbf{v}'$ and $\mathbf{h}'$ as general, for the entire two-dimensional space, linear transformations of $\mathbf{v}$ and $\mathbf{h}$:

$$\mathbf{v}'=\alpha M_1\mathbf{h}+\gamma M_2\mathbf{v},\qquad \mathbf{h}'=\beta M_3\mathbf{v}+\delta M_4\mathbf{h}. \tag{4}$$

Here α, β, γ, and δ are arbitrary constants, M_i are matrices of rotation through a constant angle φ_i

$$M_i=\begin{pmatrix}\cos\varphi_i & \sin\varphi_i\\ -\sin\varphi_i & \cos\varphi_i\end{pmatrix}.$$

For the field $\mathbf{v}'$ to have no sources and for the field $\mathbf{h}'$ to be potential, it suffices to equate $\varphi_1=\pm\pi/2$, $\varphi_2=0$, $\varphi_3=\pm\pi/2$, $\varphi_4=0$, and then we obtain for the primed fields the system of equations

$$\mathbf{v}'=\sigma'\mathbf{h}',\qquad \operatorname{div}\mathbf{v}'=0,\qquad \operatorname{rot}\mathbf{h}'=0, \tag{5}$$

$$\sigma'=(\alpha M+\gamma\sigma)(\beta M\sigma+\delta E)^{-1}, \tag{6}$$

$$M=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\quad E=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

It is natural to stipulate that the tensor σ', just as the tensor σ, be symmetric. To this end it suffices to set $\gamma=\delta=0$, and then it follows from (6) that

$$\sigma'=\frac{\alpha}{\beta}\frac{\sigma}{\det\sigma},\qquad \frac{\alpha}{\beta}>0. \tag{7}$$

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Introducing in accord with (2) the averaged primed fields $\mathbf{V}'$ and $\mathbf{H}'$ and recognizing that they are linearly connected:

$$\mathbf{V}'=\sigma_*'\mathbf{H}', \tag{8}$$

we obtain for the effective conductivity tensor of the primed system

$$\sigma_*'=\frac{\alpha}{\beta}\frac{\sigma_*}{\det\sigma_*}. \tag{9}$$

Thus, Eqs. (7) and (9) determine, at a fixed constant factor α/β, the transformation of the tensors of the local and effective conductivities of the initial and primed systems. If one of them is known, the other can be determined from (9). We note that certain consequences of (7) and (9) are considered in Refs. 2 and 3.

If there exist a positive constant α/β for which the locally anisotropic fiels σ and σ' are equivalent with respect to the statistical distributions, the effective conductivities of the initial and primed systems are identical, and from (9) we have the exact equality

$$\det\sigma_*=\alpha/\beta. \tag{10}$$

If, however, the effective conductivities are isotropic, it is possible to determine exactly

$$\sigma_*=(\alpha/\beta)^{1/2}. \tag{11}$$

2. We now consider concrete systems.

1) Let a planar system contain homogeneous inclusions of two types with conductivity tensors σ_1 and $\sigma_2=n\sigma_1$, where n is an arbitrary positive number. Choosing $\alpha/\beta=n\det\sigma_1$, we find that in the initial and primed systems the inclusions have interchanged place. A corresponding interchange takes place also in the inclusion density. If the densities are equal ($p=\frac{1}{2}$) and the system is such that the effective conductivities of the initial and primed systems coincide, it follows from (10) that

$$\det\sigma_*=n\det\sigma_1, \tag{12}$$

or in other words

$$\det\sigma_*=(\det\sigma_1\det\sigma_2)^{1/2}. \tag{13}$$

If $p\neq\frac{1}{2}$, we have in the primed system $1-p$ for the share of the conductivity σ_1, and for a system in which the geometry of the inclusions does not depend on their conductivity we obtain from (9)

$$\sigma_*(1-p)\det\sigma_*(p)=\sigma_*(p)\,(\det\sigma_1\det\sigma_2)^{1/2}. \tag{14}$$

If the inclusions are isotropic ($\sigma_{x1}=\sigma_{y1}=\sigma_1, n\sigma_1=\sigma_2$), we also have from (13) and (14) for macroisotropic systems the Dykhne formulas[1]:

$$\sigma_*(^1/_2)=(\sigma_1\sigma_2)^{1/2}, \tag{15}$$

$$\sigma_*(p)\sigma_*(1-p)=\sigma_1\sigma_2. \tag{16}$$

It is appropriate to note that self-consistent effective conductivities of fields with the indicated properties, as established in Refs. 3 and 4, satisfy exactly the relations (15) and (16). It is easy to show that if isotropic inclusions with conductivities σ_1 and σ_2 form a macroscopically anisotropic medium, the systems of equations for the self-consistent components of the tensors of the effective conductivities of the initial field with density p and of the primed field with density $1-p$ are compatible with the exact equation (14). For example, substitution of (14) in one of the two equations for the initial medium transforms it into the second equation of the system for the primed field, and vice versa.

A case close to the considered examples is the following: Assume that inclusions of two types are present in a homogeneous anisotropic planar system with principal conductivities $\bar\sigma_x$ and $\bar\sigma_y$. The principal conductivities of the first are σ_x and σ_y, and those of the second $\bar\sigma_x\bar\sigma_y/\bar\sigma_y$ and $\bar\sigma_x\bar\sigma_y\sigma_x$. The volume fractions of the inclusions are equal, and their arrangements are statistically equivalent. Assuming $\alpha/\beta=\bar\sigma_x\bar\sigma_y$, we find from (7) that in the primed system, compared with the initial one, the inclusions have changed place, while the matrix remained unchanged. Obviously, the effective conductivities of the initial and primed systems are identical, and it follows from (9) that

$$\det\sigma_*=\bar\sigma_x\bar\sigma_y. \tag{17}$$

If the matrix is isotropic: $\bar\sigma_x=\bar\sigma_y=\bar\sigma$ and the inclusions and the system as a whole are isotropic, we have from (17)

$$\sigma_*=\bar\sigma. \tag{18}$$

Thus, introduction into a homogeneous planar system of arbitrary and equal fractions of geometrically equivalent inclusions of the indicated structure does not change its effective conductivity σ_* in the case of a macroisotropic system and does not change $\det\sigma_*$ in the general case. In particular, at $\sigma_x=\sigma_y=0$ it follows from (17) and (18) that nonconducting and ideally conducting inclusions make mutually compensating contributions to $\det\sigma_*$ and σ_*.

Considering an equation for the self-consistent effective conductivity of an isotropic plane with isotropic inclusions of circular shape:

$$p_i\frac{\sigma_*-\sigma_i}{\sigma_*+\sigma_i}=0, \tag{19}$$

$$\sigma_1=\bar\sigma,\quad \sigma_2=\sigma,\quad \sigma_3=\bar\sigma^2/\sigma,\quad p_3=p_2=p,\quad p_1=1-2p,$$

it is easy to verify that the exact solution of (18) transforms it into an identity.

If $p_2\neq p_1$, it follows from (7) and (9) under the assumptions made concerning the geometry of the inclusions

$$\sigma_*(p_3,\ p_2)\det\sigma_*(p_2,\ p_3)=\sigma_*(p_2,\ p_3)\bar\sigma_x\bar\sigma_y \tag{20}$$

and, if the inclusion matrix and the system as a whole are isotropic

$$\sigma_*(p_2,\ p_3)\sigma_*(p_3,\ p_2)=\bar\sigma^2. \tag{21}$$

2) Let the plane be filled by inclusions for whose local-conductivity tensors σ_i satisfy the condition

$$\det\sigma_i=c=\text{const}. \tag{22}$$

Then setting $\alpha/\beta=c$ we get from (7) $\sigma'=\sigma$, i.e., the primed and initial systems are identical. We have therefore from (10)

$$\det\sigma_*=c, \tag{23}$$

and if the effective conductivity is isotropic, it follows from (23) that

$$\sigma_*=c^{1/2}. \tag{24}$$

In particular (24) leads to the Dykhne formula[1] for the effective conductivity of a two-dimensional polycrystal consisting of randomly oriented anisotropic crystallites.

Obviously, relations (23) and (24) retain their meaning also for continuous media satisfying the condition $\det \sigma = c$. Such media can be easily "constructed." In fact, putting $\sigma_x(\mathbf{r}) = c^{1/2} f(\mathbf{r})$, where $f(\mathbf{r})$ is an arbitrary non-negative stochastically homogeneous random field, and $\sigma_y(\mathbf{r}) = c^{1/2} f^{-1}(\mathbf{r})$, we obtain a medium with the sought properties.

3) Consider now a plane random field whose local conductivity tensor is a continuous function fo the coordinates. Let the principal axes of the local conductivity tensor be independent of the coordinates and coincide with the axes x and y. Let also the components of the conductivity tensor $\sigma(\mathbf{r})$, namely the scalar random fields $\sigma_x(\mathbf{r})$ and $\sigma_y(\mathbf{r})$, have identical multipoint distribution functions. Choosing

$$\alpha/\beta = \exp(2\langle \ln \sigma_x \rangle) = \exp(2\langle \ln \sigma_y \rangle),$$

we find from (7) that the mean values of the logarithms of the components of the tensors σ and σ' are identical, while the vector

$$\varkappa = (\ln \sigma_x - \langle \ln \sigma_x \rangle,\ \ln \sigma_y - \langle \ln \sigma_y \rangle) \tag{25}$$

corresponds for the primed field to the vector

$$\varkappa' = (\langle \ln \sigma_y \rangle - \ln \sigma_y,\ \langle \ln \sigma_x \rangle - \ln \sigma_x). \tag{26}$$

If it is assumed that the random structure is such that the multipoint distribution function of the field $\varkappa$ is an even function of $\varkappa$, then it follows from the identity of the distributions σ_x and σ_y that the vector fields $\varkappa$ and $\varkappa'$ have equal distributions, and consequently the primed field has the same effective conductivity as the initial one. Therefore

$$\det \sigma_* = \exp(2\langle \ln \sigma_x \rangle) = \exp(2\langle \ln \sigma_y \rangle). \tag{27}$$

If the fields σ_x and σ_y are not correlated, the effective tensor σ_* is isotropic and

$$\sigma_* = \exp\langle \ln \sigma_x \rangle = \exp\langle \ln \sigma_y \rangle. \tag{28}$$

The situation will be exactly the same in the case of complete correlation of the fields $\sigma_x = \sigma_y$, i.e., in the case of isotropy of the local field. From (28) we get the Dykhne formula[1]

$$\sigma_* = \exp\langle \ln \sigma \rangle. \tag{29}$$

We show now that the self-consistent effective conductivity for isotropic continuous fields, whose centered logarithm has an even distribution, is exactly equal to (29).

In fact, substituting in the equation for the self-consistent effective conductivity of the isotropic planar system

$$\left\langle \frac{\sigma_* - \sigma}{\sigma_* + \sigma} \right\rangle = 0 \tag{30}$$

the expressions

$$\sigma_* = \exp\langle \ln \sigma \rangle, \qquad \sigma = \exp[\varkappa + \langle \ln \sigma \rangle],$$

we obtain the equality

$$\left\langle \frac{1 - e^{\varkappa}}{1 + e^{\varkappa}} \right\rangle = 0, \tag{31}$$

which is an identity, since the averaged function is odd in $\varkappa$, and the distribution density of $\varkappa$ is even.

Therefore in this case, too, the exact result satisfies the equation for the self-consistent effective conductivity.

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